Let \( R \) be a ring.

**Definition 1.** The nilradical of \( R \) is \( \sqrt{0} \).

**Claim 1.** \( \sqrt{0} = \bigcap \{ P : P \text{ prime} \} \)

**Proof.** We already saw that \( \sqrt{0} \subseteq \bigcap \{ P : P \text{ prime} \} \). Suppose \( a \in R \) is not nilpotent, and let \( S = \{ 1, a, a^2, \ldots \} \). Note that \( 0 \notin S \) and \( S \) is multiplicatively closed, so any maximal element in \( \{ I : I \cap S = \emptyset, I \text{ ideal} \} \) is a prime ideal not containing \( a \). \( \square \)

If \( a \) is a unit, then \( (a) = R \) so \( a \) is not in any maximal ideal. If \( a \) is a nonunit, then \( (a) \neq R \), so \( (a) \) can be extended to a maximal ideal of \( R \).

From this, we can conclude that \( \bigcup \{ M : M \text{ maximal ideal} \} \) is the set of nonunits.

**Definition 2.** We say \( R \) is local iff \( R \) has exactly one maximal ideal.

**Claim 2.** \( R \) is local iff the set of nonunits in \( R \) forms an ideal.

**Definition 3.** The Jacobson radical of \( R \) (denoted \( J \)) is the intersection of all maximal ideals.

**Claim 3.** Let \( M \) be a maximal ideal, and let \( r \in R \). Then, \( r \notin M \) iff there is \( s \in R \) such that \( rs - 1 \in M \).

**Proof.** Note that \( R/M \) is a field as \( M \) is maximal, so \( r \notin M \) iff \( r + M \neq 0 \) in \( R/M \) iff \( r + M \) unit. So this is true iff there is \( s \in R \) such that \( 1 + M = (r + M)(s + M) = rs + M \), or \( rs - 1 \in M \).

Now, \( r \notin J \) iff there is \( M \) maximal such that \( r \notin M \). From the previous lemma, \( r \notin M \) iff there is \( s \in R \) such that \( rs - 1 \in M \). Taking the contrapositive, we see that \( r \in J \) iff for all \( M \) maximal and all \( s \in R \), \( rs - 1 \notin M \). This is equivalent to saying that for all \( s \in R \), \( rs - 1 \) is a unit. For cosmetic reasons we we rewrite the conclusion as \( r \in J \) iff \( 1 + rs \) is a unit for all \( s \in R \).

**Definition 4.** We say that \( M \) is an \( R \)-module iff

1. \((M, +)\) is an abelian group
2. There is a map \( R \times M \to M \) that maps \((r, m) \mapsto rm\) such that
   - \( r(m_1 + m_2) = rm_1 + rm_2 \)
   - \( (rs)m = r(sm) \)
   - \( (r_1 + r_2)m = r_1m + r_2m \)
   - \( 1m = m, 0m = 0 \)

**Example:** Let \( \phi : R \to S \) be a ring HM. Define scalar multiplication \( R \times S \to S \) by \((r, s) \mapsto \phi(r)s\). NOTE: As in this course we are assuming \( \phi(1_R) = 1_S \), this makes \( S \) into an \( R \)-module.

**Definition 5.** An \( R \)-algebra is a ring \( S \) together with a ring HM \( R \to S \).
Note: If $R$ is a ring, then $R$ is an $R$-module.

**Definition 6.** Let $M$ be an $R$-module. Then, $N \subseteq M$ is a submodule of $M$ (we write $N \leq M$) iff

1. $(N, +) \leq (M, +)$
2. $\forall r \in R, \forall n \in N, rn \in N$.

Note: The $R$-submodules of $R$ are the ideals.

If $N \leq M$, then $M/N$ has a module structure by $r(m + N) = rm + N$. This is well-defined since $m_1 + N = m_2 + N$ iff $m_1 - m_2 \in N$, so $r(m_1 - m_2) \in N$.

**Definition 7.** If $M, N$ are $R$-modules, then $\phi : M \rightarrow N$ is a module homomorphism (HM) iff

1. $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$
2. $\phi(rm) = r\phi(m)$.

First IM theorem: $\text{im}(\phi) \cong M/\ker(\phi)$.

**Claim 4.** Let $M$ be an $R$-module, $X \subseteq M$. The least submodule of $M$ containing $X$ is

$$ (X)_{R} = \{ \sum_{\text{finite}} r_i x_i : r_i \in R, x_i \in X \} $$

**Definition 8.** We say $M$ is finitely generated (fg) iff there is $X \subseteq M$ finite such that $(X)_{R} = M$.

Fact: There is an integral domain $R$ and a fg $R$-module $M$ such that not all submodules of $M$ are fg.

Example: Let $R = \mathbb{Z}[x_1, x_2, \ldots] = \bigcup_{i \in \mathbb{N}} \mathbb{Z}[x_1, \ldots, x_i]$. Let $M = R$, and let $N = (x_1, x_2, \ldots)_{R}$. Note that $M = (1)_{R}$, so $M$ is f.g. However, $N$ is not f.g. Suppose that $N = (f_1, \ldots, f_k)_{R}$. Choose $m$ so large that all variables appearing in the $f_i$s are $x_j$ for some $j < m$. As $x_m \in N$, we have $x_m = \sum g_i f_i$ for some $g_i \in R$. Set $x_j = 0$ for $j < m$ and $x_m = 1$ to get a contradiction, as all polynomials in $N$ have no constant term.