I.15 Notice that this is different from the \( V(E) \) operation from class. After the Nullstellensatz we can see that in case \( k \) is algebraically closed, the \( V(E) \) from class corresponds to the set of maximal ideals containing \( E \). Notice also that there are a couple of places in this where we are using the primeness of the ideals.

1. Clearly \( V(E) \supseteq V(a) \supseteq V(r(a)) \). On the other hand let \( P \in V(E) \), so that \( a \subseteq P \). Since \( P \) is a radical ideal \( r(a) \subseteq r(P) = P \), so that \( P \in V(r(a)) \).

2. No prime ideal contains 1 and every prime ideal contains 0.

3. \( P \in V(\bigcup_i E_i) \) iff \( \bigcup_i E_i \subseteq P \) iff \( E_i \subseteq P \) for all \( i \) iff \( P \in \bigcap_i V(E_i) \).

4. \( ab \subseteq a \cap b \subseteq a \), so \( V(a) \subseteq V(a \cap b) \subseteq V(ab) \). So easily (by symmetry) \( V(a) \cup V(b) \subseteq V(a \cap b) \subseteq V(ab) \). If \( P \) is prime and \( P \in V(ab) \) then by primeness either \( a \subseteq P \) or \( b \subseteq P \), so \( V(ab) \subseteq V(a) \cup V(b) \).

II.14 Nothing to do.

II.15 This one is pretty straightforward, the moral from both parts is that elements are defined using finitely many coordinates, and by directedness we can go up to a coordinate which is above a given finite set of coordinates.

If \( m \in M \) then \( m = \mu(c) \) for some \( c \in C \). Let \( c \) be the sum of \( m_i \) for \( i \) in some finite subset of \( I_0 \) of the index set \( I \). By directedness we find \( j \in I \) such that \( I_0 \subseteq j \), and let \( n_i = \mu_{ij}(m_i) \) for all \( i \in I_0 \) and \( d \) be the sum of the \( n_i \). Now \( c - d \in D \) so \( m = \mu(c) = \mu(d) \).

Suppose that \( \mu_j(x_i) = 0 \) in \( M \). Then \( x_i \in D \) so \( x_i \) can be written as a finite sum of elements of form \( x_n = \mu_{a_kb_n}(x_n) \) for various \( a_n \leq b_n \) and \( x_n \in M_{a_n} \). Using directedness, fix \( k \) above \( i \) and all the \( a_n \) and \( b_n \). For each \( j \) let \( X_j = \sum_{a_n=j} x_n \) and \( Y_j = \sum_{b_n=j} \mu_{a_n,b_n}(x_n) \).

Since \( C \) is the direct sum of the \( M_j \), we have \( X_j - Y_i = x_i \) and \( X_j - Y_j = 0 \) for all other \( j \). So the sum \( \sum_j \mu_{jk}(X_j - Y_j) = \mu_{ij}(x_i) \). But for each \( n \) there is a contribution \( \mu_{a_n,k}(x_n) = \mu_{b_n,k} \circ \mu_{a_n,b_n}(x_n) = 0 \) to this sum, so it is zero.

VI.3 Many of you reinvented the wheel here. It can be done quickly using isomorphism theorems and some facts about Noetherian modules.

It is easy to see that \( \frac{N_1 + N_2}{N_1} \simeq \frac{N_2}{N_1 \cap N_2} \) and \( \frac{M}{N_2} \simeq \frac{M}{N_1} \) (just check the standard isomorphisms from group theory are module isomorphisms). Now \( \frac{N_1 + N_2}{N_1} \) is a submodule of the Noetherian \( \frac{M}{N_1} \) so is Noetherian, and by hypothesis \( \frac{M}{N_2} \) is Noetherian. So \( \frac{M}{N_1 \cap N_2} \) is Noetherian.