SOLUTIONS TO HW 1 WITH COMMENTS

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I.1 (Not actually on the homework but useful for it). Let $x^n = 0$ and let $y = 1 - x + \ldots x^{n-1}$. Then $(1 + x)y = 1 - x^n = 1$ and $1 + x$ is a unit. Now if $u$ is a unit and $x$ is nilpotent then $u + x = u(1 + u^{-1}x)$. $u^{-1}x$ is nilpotent, so $1 + u^{-1}x$ is a unit and hence $u + x$ is a unit.

I.2 My comments are in boldface

(1) Let $a_0$ be a unit and the $a_i$ for $i > 0$ be nilpotent in $A$. Then $a_1x^i$ is nilpotent in $A[x]$ and so $a_1x + \ldots a_nx^n$ is nilpotent in $A[x]$. Now $a_0$ is a unit in $A[x]$ and so by Ex I.1 $f$ is a unit.

We already know that the nilpotent elements form an ideal, so there is no need to reinvent the wheel and show by hand that a linear combination of nilpotent elements is nilpotent.

Conversely let $g = b_0 + \ldots b_mx^m$ with $fg = 1$, so that $a_0b_0 = 1$ and for $k > 0$ we have $\sum_{i+j=k} a_ib_j = 0$. We assume that $m, n > 0$ otherwise the problem is trivial (why?) Note that $a_0$ and $b_0$ are both units.

Now $a_nb_m = 0$ and also $a_nb_{m-1} + a_{n-1}b_m = 0$. Multiplying these equations by $a_n$, $a_n^2b_m = 0$ and $a_n^2b_{m-1} = 0$. Repeating in the obvious way, we conclude that $a_n^{m+1}b_0 = 0$. So $a_n$ is nilpotent.

Now $a_nx^n$ is nilpotent so (by Ex I.1 again) $f - a_nx^n$ is a unit. Repeating the argument all of $a_1, \ldots a_n$ are nilpotent.

(2) If $f$ is nilpotent then $1 + xf$ is nilpotent, so by the previous part all the coefficients $a_i$ are nilpotent. Conversely if all the $a_i$ are nilpotent, then since the nilpotent elements form an ideal $f$ is nilpotent.

(3) Let $g = b_0 + \ldots b_mx^m$ be nonzero of minimal degree with $fg = 0$. Note that $b_m \neq 0$. Now $a_nb_m = 0$ and and $f(a_ng) = 0$, so by minimality of the degree of $g$ we have $a_ng = 0$. But now $a_nb_n = 0$ because $(f - a_nx^n)g = 0$, and repeating the argument we finally get that $a_ng = 0$ for all $i$. In particular $a_nb_n = 0$ for all $i$, so that $fb_n = 0$. The other direction is easy.

(4) No one got this one completely right. It is a generalisation of a fact about integer polynomials called the Gauss lemma. In the context of $\mathbb{Z}[x]$ a polynomial is called primitive if the gcd of its coefficients is one, and the Gauss’ lemma says the product of two such things is primitive. The classical proof of the hard direction of Gauss’ lemma goes by saying that if $fg$ is not primitive, then some prime $p$ divides all the coefficients of $fg$; we then argue that $p$ divides $f$ or $p$ divides $g$. This will work in a UFD but not in a general ring, as will often be the case in this course we need to replace prime elements with prime ideals.
If $f$ is not primitive let $I$ be the proper ideal generated by the coefficients of $f$. It is easy to see that all the coefficients of $fg$ lie in $I$ so that $fg$ is not primitive.

Now suppose that $fg$ is not primitive. Find a maximal ideal $M$ containing all the coefficients of $fg$ and note $M$ is prime. Form the quotient ring $B = A/M$ and let $f', g'$ be the images of $f, g$ respectively in $B[x]$. $f'g' = 0$ in the ID $B[x]$ so WLOG $f' = 0$. This means the coefficients of $f$ all lie in $M$ and so $f$ is not primitive.

I.3 As many of you pointed out this one is quite boring. The reason it is boring is that $A[x_1, \ldots, x_{n+1}]$ is isomorphic to $A[x_1, \ldots, x_n][x_{n+1}]$ so we can use induction.

For simplicity just look at $A[x, y] = A[x][y]$. Let $f = \sum_{i,j} a_{ij} x^i y^j$ be a typical element. Let $f = \sum_{i} f_i y^i$, where $f_i = \sum_{j} a_{ij} x^i \in A[x]$.

1) $f$ is a unit iff $f_0$ is a unit and the $f_i$ for $i > 0$ are nilpotent iff $a_{00}$ is a unit and the remaining $a_{ij}$ are nilpotent.

2) $f$ is nilpotent iff all the $f_i$ are nilpotent iff all the $a_{ij}$ are nilpotent.

3) This needs a bit of work, I think. By the last exercise $f$ is a zero-divisor iff there exists $h \in A[x]$ not equal to zero such that $f_i h = 0$ for all $i$. Let $h = b_0 + \ldots b_m x^m$ be of minimal degree such that $f_i h = 0$ for all $i$. Now repeat the argument of the last exercise to show that $b_m f_i = 0$ for all $i$, and thus $b_m f = 0$.

4) Exactly the same argument as in the previous exercise should work to show $fg$ primitive iff both $f$ and $g$ are.

I.4 Trivially the nilradical is contained in the Jacobson radical. Let $f$ be in the Jacobson radical, then $1 + xf$ is a unit and so by Ex I.2 all the coefficients of $f$ are nilpotent. It follows that $f$ is in the nilradical.

III.6 This one caused a few problems. The tricky bit is to see that the complement of a maximal element of $\Sigma$ is an ideal.

$\Sigma$ has maximal elements by Zorn, because (check it!) $\Sigma$ is closed under unions of chains. It is easy to see that the complement of any prime ideal is in $\Sigma$.

The next step is typical for Zorn’s lemma, we use maximality to figure out what distinguishes things which are not in the maximal set.

Suppose now that $S$ is maximal. We claim that $a \notin S$ iff there is $n \geq 0$ and $s \in S$ with $a^n s = 0$. This is easy once we note that $\{a^n s : n \geq 0, a \in S\}$ is a multiplicatively closed set containing $S$ and $a$.

Now if $a^n s = 0$ then $(ra)^n s = 0$, so $a \notin S$ implies $ra \notin S$. Also if $a^n s_1 = b^n s_2 = 0$ then $(a + b)^{n+n}(s_1, s_2) = 0$, so the complement of $S$ is closed under addition. Therefore the complement of $S$ is an ideal, which is obviously prime. It must be minimal because a smaller prime ideal would give a larger element of $\Sigma$.

Conversely let $p$ be minimal prime and let $S$ be the complement. $S \in \Sigma$ and if it is not maximal then $S \subseteq T$ for $T$ maximal, which is impossible since then the complement of $T$ is prime and properly contained in the minimal $p$. 