21-131 Analysis I – PRACTICE FOR TEST 2
Test 1– 11:30 October 20, 2004
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ON ALL PROBLEMS, EXPLAIN HOW YOU REACH YOUR CONCLUSIONS

MATERIAL COVERED: CHAPTER I UP TO – AND EXCLUDING – Section 2.5

1 Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. Prove that
\[
\bar{I}(f)[a, b] = \bar{I}(f)[a, c] + \bar{I}(f)[c, b]
\]
for all \( c \in (a, b) \).

Note: Here we are using the notation
\[
\bar{I}(f)([a, b]) := \inf \left\{ \int_a^b t : t \geq f \text{ on } [a, b], \quad t \text{ is a step function} \right\}.
\]

2 Let \( g : [-2, 0] \to \mathbb{R} \) be a bounded function such that
\[
g(x) = -g(-x - 2)
\]
for all \( x \in (-2, 1) \). Prove that \( \bar{I}(g)[-2, 0] = 0 \).

Note: The translation \( f(x) := g(x - 1) \) defined on \([-1, 1]\) is odd, i.e. \( f(x) = -f(-x) \) for all \( x \in (-1, 0) \).

3 Prove that if \( f \) is a Lipschitz function on \([0, b]\) satisfying \( |f(x) - f(y)| \leq L|x - y| \) for all \( x, y \in [0, b] \) and for some \( L > 0 \) then
\[
\int_0^b f(x) \, dx \leq f(0)b + \frac{b^2}{2}L.
\]

4 Let \( f : [-1, 4] \to \mathbb{R} \) be defined by
\[
f(x) := \begin{cases} 
x^2 + 1 & -1 \leq x \leq 0, \\
3 & 0 < x < 1, \\
x^3 & 1 \leq x \leq 4.
\end{cases}
\]
Prove that \( f \) is integrable and find \( \int_{-1}^{1} f(x) \, dx \).

5 Let \( f : [a, b] \to [0, +\infty) \) be integrable. Prove that \( f^2 \) is also integrable.

6 Assume that for all \( n \in \mathbb{N} \) the following hold:
\[
\sum_{i=1}^{n} \sqrt{i} \leq \frac{2}{3} n \sqrt{n} + C \sqrt{n}, \quad \sum_{i=1}^{n-1} \sqrt{i} \geq \frac{2}{3} n \sqrt{n} - C \sqrt{2n}
\]
for some constant \( c \in \mathbb{R} \). Show that \( x \mapsto \sqrt{x} \) is integrable on \([0, 2]\) and that \( \int_{0}^{2} \sqrt{x} \, dx = \frac{4 \sqrt{2}}{3} \).

7 Assume that for all \( n \in \mathbb{N} \) the following hold:
\[
\sum_{i=n}^{2n-1} \frac{1}{i^2} \leq \frac{1}{2n} + \frac{3}{n^2}, \quad \sum_{i=n+1}^{2n} \frac{1}{i^2} \geq \frac{1}{2n} - \frac{1}{n^2}.
\]
Prove that \( x \mapsto x^{-2} \) is integrable on \([1, 2]\) and that \( \int_{1}^{2} x^{-2} \, dx = \frac{1}{2} \).

8 Prove that
\[
\frac{1}{4} \leq \int_{0}^{1} \frac{x}{1 + x^4} \, dx \leq \frac{1}{2}
\]

9 Let \( f : [0, 2] \to \mathbb{R} \) be defined by
\[
f(x) := \begin{cases} x & x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}
\]
Is \( f \) integrable on \([0, 2]\)?

10 Find the area of the region between the curves \( y = x^3 \) and \( y = \sqrt{x} \) on \([0, 2]\).