Hull & White model for interest rates

Standard form:
\[ d\kappa_t = (\theta_t - \lambda \kappa_t) \, dt \]
\[ + \alpha dW_t \]

\( \kappa_t \): short-term rate
\( \lambda \): mean-reversion rate
\( \alpha \): volatility of short-term rate
\( (\theta_t) \): some deterministic function (to calibrate any discount curve)
Time-dependent version:
\[ d\kappa_t = (\theta_t - \lambda_t \kappa_t) \, dt + \sigma \kappa_t \, dW_t \]

Calibration:

(\theta_t): calibrates discount curve
(\lambda_t) k (\kappa_t): implied volatility curve.
Notations:

\( B(s,t) \): discount factor computed at \( s \) for maturity \( t \)

\( \kappa(s,t) \): continuously compounded yield

\[ B(s,t) = e^{-(t-s)\kappa(s,t)} \]

\[ \kappa(s,t) = -\frac{1}{t-s} \ln B(s,t) \]
Hull & White model in Black (HJM) methodology.

Input parameters:

\((B(0, t))_{t \geq 0}\): initial discount curve
\(t\): time
\(m\): maturity of zero-coupon bond

\((A(t))_{t \geq 0}\): initial shape curve for changes in discount curve

Convention:
\(A(0) = 0\)  \(A'(0) = 1\)
\[ n(z) \uparrow \quad 1 \text{ b.p. (basic point)} \]

\[ n(z, t) \uparrow \quad \frac{A(t)}{t} \quad \text{b.p.} \]

\[ \frac{\delta B(z, t)}{B(z, t)} \downarrow \quad A(t) \quad \text{b.p.} \]
In practice
1. discount factors with longer maturities move "faster" \( \iff \) \( A \) is increasing \( (A' > 0) \)
2. yields with longer maturities move "slower" \( \iff \)
\[
\left( \frac{A(t)}{t} \right) \text{ is decreasing}
\]
\[
( A'(t) \leq \frac{A(t)}{t} )
\]
It is reasonable to assume that \( A' \) is decreasing \((A'' < 0)\).

Hence,
\[
A'(t) = \exp \left( -\int_0^t \lambda(u) \, du \right)
\]
where \((\lambda_t)\) is the mean-reversion rate \((\lambda_t \geq 0)\).
\[(\Sigma^t(t))_{t \geq 0}^{\text{normalized}}\] volatility curve

\[\Psi(0, s, t) = (A(t) - A(s))\Sigma^s(s)\] implied volatility

0: initial time
s: maturity of option
t: maturity of underlying zero-coupon bond

\[\Sigma^t(t) = \sqrt{\frac{1}{t} \int_0^t \sigma^2(u)\,du}\]
\[ \Delta \varepsilon(t) = \Lambda'(t) \varepsilon'(t) : \text{ volatility of short-term rate} \]
\[ F(s, t, u) : \text{forward price} \]
\[ s: \text{current time} \]
\[ t: \text{delivery time for forward} \]
\[ u: \text{maturity of underlying zero-coupon bond} \]

Hull & White model:

\[
dF(s, t, u) = F(s, t, u) \left( A(u) - A(t) \right) \sigma_s \, dW^s_t
\]

\[ W^t: \text{Brownian motion for forward measure } P^t. \]
Output for Hull & White model.

We compute the value of the option as the function of

\[ x = \kappa(0) - \kappa(x) \]

initial short-term rate

perturbed short-term rate
Output:

\[ V_0 = \left( V_0(x) \right) - \frac{A}{2} \leq x \leq \frac{A}{2} \]

\[ \Delta : \text{interval for changes in } V_0(x) \]

\[ \text{scenario: rate } \downarrow 1\% \]

\[ \text{scenario: rate } \uparrow 2\% \]

\[ \text{initial discount curve} \]