Homework 6

Problems 34∗, 35a∗, c∗ on p. 383.
Supplementary Problems 1∗, 2∗.
Problems 1∗, 2, 3, 4∗, 5∗ on p. 391.
Supplementary Problem 3∗.

Supplementary problems

Problem 1. Let $T : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

1. there is exactly one number $r \in \mathbb{R}$ such that $r = T(r)$ ($r$ is called a fixed point of $T$);
2. for all $x > r$ we have $x > T(x) > r$.

Given a number $\alpha > r$, consider the sequence defined recursively by

$$a_1 := \alpha, \quad a_{n+1} = T(a_n) \text{ for all } n \in \mathbb{N}.$$ 

Prove that $\lim_{n \to \infty} a_n = r$.

Problem 2. Prove that the following sequences converge, find the limit, and justify that the number you have found is in fact the limit:

(a) $x_1 := \sqrt{2}$, $x_2 := \sqrt{2 + \sqrt{2}}$, $x_n := \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}$, $\ldots$;

(b) $x_1 := 101$, $x_2 := 101 \cdot \frac{102}{3}$, $x_3 := 101 \cdot \frac{102}{3} \cdot \frac{103}{4}$, $\ldots$.

Problem 3. Either prove or provide a counterexample to each of the following statements:

(a) Suppose that $\{a_n\}_{n \in \mathbb{N}}$ converges and $\{b_n\}_{n \in \mathbb{N}}$ diverges. Then the sequence $c_n := a_n + b_n$ diverges; the sequence $d_n := a_n b_n$ diverges.

(b) Suppose that the sequences $\{a_n\}_{n \in \mathbb{N}}$ and $\{b_n\}_{n \in \mathbb{N}}$ both diverge. Then the sum of these sequences diverges; the product diverges.

(c) Suppose that the sequences $\{a_n\}_{n \in \mathbb{N}}$, $\{b_n\}_{n \in \mathbb{N}}$ are such that $\lim_{n \to \infty} a_n b_n = 0$. Then either $\lim_{n \to \infty} a_n = 0$ or $\lim_{n \to \infty} b_n = 0$ (or maybe both).