Homework 4

You should be able to do problems 1–12 on p. 295.
Supplementary Problem 1*, 2*, 3*.

Supplementary problems

**Problem 1.** Suppose that \( f \) is twice continuously differentiable on an open interval containing \([0, 1]\). Suppose that \( f(0) = 0, f'(0) = 0 \), and for some \( M \in \mathbb{R} \) we have \( |f''(x)| \leq M|f(x)| \) for all \( x \in [0, 1] \). Show that if \( M < 2 \), then \( f(x) = 0 \) for all \( x \in [0, 1] \).

**For analysis enthusiasts:** Show that \( f(x) = 0 \) for all \( x \in [0, 1] \) even if \( M \geq 2 \).

**Problem 2.** Suppose that \( f \) is three times continuously differentiable on \( \mathbb{R} \) and has bounded second and third derivatives. Let \( M_2 := \sup\{|f''(x)| \mid x \in \mathbb{R}\}, \quad M_3 := \sup\{|f'''(x)| \mid x \in \mathbb{R}\} \).

Consider the difference-quotient operators \( D^h \) and \( D_h \) defined for \( h \neq 0 \) by

\[
D^h f(x) := \frac{f(x + h) - f(x)}{2h},
\]
\[
D_h f(x) := \frac{f(x + h) - f(x)}{h}.
\]

(a) Show that \( |D^h f(x) - f'(x)| \leq \frac{M_3}{6} h^2 \) for all \( x \in \mathbb{R}, h \neq 0 \).

(b) Obtain the estimate for \( |D_h f(x) - f'(x)| \).

(c) Which of the quotients \( D^h f(x), D_h f(x) \) converges to \( f'(x) \) faster as \( h \to 0 \)?

(d) **For analysis enthusiasts:** Design a better difference-quotient than \( D^h, D_h \). Which additional assumptions do you need to make?

**Problem 3.** Let \( n \) be a positive integer, \( a, b \in \mathbb{R} \) with \( a < b \). Let \( x_0, \ldots, x_n \in [a, b] \) such that \( x_0 < \cdots < x_n \) be given. Suppose that \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and \((n + 1)\)-times differentiable on \((a, b)\). Let \( P(x) \) be a polynomial of degree \( \leq n \) such that \( P(x_i) = f(x_i) \) for \( i = 0, \ldots, n \).

Show that for every \( x \in [a, b] \) there is \( \xi \in (a, b) \) such that

\[
f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i).
\]

Hint: For \( x = x_i, i = 0, \ldots, n \), the result is immediate. Take \( x \in [a, b] \setminus \{x_0, \ldots, x_n\} \) and define

\[
w(t) := \prod_{i=0}^{n} (t - x_i); \quad q := \frac{f(x) - P(x)}{w(x)}.
\]

Apply Rolle’s theorem repeatedly to the function \( \varphi(t) := f(t) - P(t) - qw(t), t \in [a, b] \).