ANOTHER LOOK INTO THE WONG–ZAKAI THEOREM FOR STOCHASTIC HEAT EQUATION

YU GU, LI-CHENG TSAI

ABSTRACT. Consider the heat equation driven by a smooth, Gaussian random potential:
\[ \partial_t u_\varepsilon = \frac{1}{2} \Delta u_\varepsilon + u_\varepsilon (\xi_\varepsilon - c_\varepsilon), \quad t > 0, x \in \mathbb{R}, \]
where \( \xi_\varepsilon \) converges to a spacetime white noise, and \( c_\varepsilon \) is a diverging constant chosen properly. For any \( n \geq 1 \), we prove that \( u_\varepsilon \) converges in \( L^n \) to the solution of the stochastic heat equation. Our proof is probabilistic, hence provides another perspective of the general result of Hairer and Pardoux [HP15], for the special case of the stochastic heat equation.

1. INTRODUCTION AND MAIN RESULT

The study of stochastic PDEs has witnessed significant progress in recent years. Several theories have been developed to make sense of singular equations with multiplication of distributions, see [Hai13, Hai14, GIP15, Kup16, OW16] (and the references therein). One example is the Wong–Zakai theorem for stochastic PDEs [CS17, HL15, HP15], which is an infinite dimensional analogue of [WZ65a, WZ65b, SV72]. In this article, we revisit this problem for a special case: the Stochastic Heat Equation (SHE) in one space dimension:
\[ \partial_t U = \frac{1}{2} \partial_{xx} U + U \xi, \quad t > 0, x \in \mathbb{R}, \quad (1.1) \]
where \( \xi \) is a spacetime white noise, built on an underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

The SHE (1.1) has played an important role in the study of directed polymers and random growth phenomena. On the one hand, the solution of (1.1) gives the partition function of a directed polymer in a noisy environment. On the other hand, via the inverse Hopf–Cole transform, the equation (1.1) yields the physical phenomena described by SHE goes to higher dimensions, we focus on one dimension here. In fact, \( d = 2 \) and \( d > 2 \) corresponds to the so-called critical and supercritical cases, and sit beyond our existing theory. There, however, has been works on the cases (in \( d \geq 2 \)) where the noise is tuned down to zero suitably with a scaling parameter. See [Fen12, Fen16, CSZ17, GRZ17, MU17].

Throughout this article, we fix a bounded continuous initial condition \( u_0(x) \in C_b(\mathbb{R}) \), and a mollifier \( \phi \in C_c(\mathbb{R}^2; \mathbb{R}_+) \) with a unit total mass \( \int \phi \, dt dx = 1 \). Using this mollifier, we construct the mollified noise as
\[ \xi_\varepsilon(t, x) = \int_{\mathbb{R}^2} \phi_\varepsilon(t - s, x - y) \xi(s, y) \, dyds, \quad \phi_\varepsilon(t, x) = \frac{1}{\varepsilon^2} \phi\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right). \quad (1.2) \]
Given the smooth function \( \xi_\varepsilon \), consider the equation
\[ \partial_t u_\varepsilon = \frac{1}{2} \partial_{xx} u_\varepsilon + u_\varepsilon (\xi_\varepsilon - c_\varepsilon), \quad t > 0, x \in \mathbb{R}. \quad (1.3) \]
For the analogous equation where \( \xi_\varepsilon \) is white-in-time, regularized in space, and interpreted in the Itô’s sense, Bertini and Cancrini [BC95] showed that, for \( c_\varepsilon = 0 \), the solution \( u_\varepsilon \) converges to the solution of the SHE.

When the noise is regularized in both space and time, a non-zero, divergent constant \( c_\varepsilon \to \infty \) arises. Our main result states that, for a suitable and explicit choice of \( c_\varepsilon \), which depends explicitly on \( \phi \), the solution \( u_\varepsilon \) of (1.3) converges pointwise in \( L^n(\Omega) \) to the solution of SHE (1.1), for any \( n \geq 1 \). It is a classical result that (1.1) admits a unique (weak and mild) solution starting from \( U(0, x) = u_0(x) \). Also, for fixed \( \varepsilon > 0 \) and for almost every realization of \( \xi_\varepsilon \), it is standard (by Feynman–Kac formula) to show that the PDE (1.3) admits a unique classical solution.

**Theorem 1.1.** Let \( c_\varepsilon = c_\varepsilon \varepsilon^{-1} + \frac{1}{2} \sigma^2 \) with \( c_\varepsilon, \sigma_\varepsilon \) given by (2.7) and (2.10). Let \( u_\varepsilon \) and \( U \) denote the respective solutions of (1.3) and (1.1), both with initial condition \( u_0(x) \). Then, for any \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \) and \( n \geq 1 \), the...
random variable \( u_\varepsilon(t, x) \) converges in \( L^n(\Omega) \) to \( U(t, x) \), i.e.,
\[
\mathbb{E}[|u_\varepsilon(t, x) - U(t, x)|^n] \to 0, \quad \text{as} \ \varepsilon \to 0.
\] (1.4)

As mentioned earlier, analogs of Theorem 1.1 have already been established in different settings. Hairer and Pardoux [HP15] established the Wong–Zakai theorem for a general class of semi-linear equations on the torus, with the SHE being a special case. This result was later extended to non-Gaussian noise by Chandra and Shen [CS17], and the problem on the whole line \( \mathbb{R} \) was studied by Hairer and Labbé [HL15]. In a related direction, Bailleul, Bernicot, and Frey [BBF17] have studied similar stochastic PDEs via paracontrolled calculus.

All the aforementioned works build on the recently developed theory of regularity structure and paracontrolled calculus [Hai14, GIP15]. In this article, we present a more probabilistic proof of Theorem 1.1. The proof is short, entirely contained within the scope of classical stochastic analysis. This offers a different perspective of the Wong–Zakai theorem for the SHE. For example, the renormalizing constant \( c_\varepsilon \) is identified in terms of the first and second moments of certain additive functionals of Brownian motions. See the discussion in Section 2.

Outline and conventions. In Section 2, we use the Feynman–Kac formula to analyze moments of \( u_\varepsilon(t, x) \). These formula are expressed in terms of functionals of some auxiliary Brownian motions. We then establish various properties of these functionals. In Section 3, we prove Theorem 1.1 using the results established in Section 2. This is done by first showing that \( u_\varepsilon(t, x) \) is a Cauchy sequence in \( L^2(\Omega) \), and then identifying the limit via a Wiener chaos expansion.

Throughout the paper, we denote the Fourier transform of \( f \) by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx.
\]
We use \( C(a_1, \ldots) \) to denote a generic, deterministic, finite constant that may change from line to line, but depends only on the designated variables \( a_1, \ldots \). This is not to be confused with the renormalization constant \( c_\varepsilon \). Also, we use \( r_\varepsilon = (r_\varepsilon(t))_{t \geq 0} \) to denote a generic (random) process, that uniformly converges to zero, i.e.,
\[
\sup_{\varepsilon \in \mathbb{R}_+} |r_\varepsilon(t)| \leq h_\varepsilon \to 0, \quad \text{for some deterministic } h_\varepsilon.
\] (1.5)

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2. Feynman–Kac Formula and Brownian Functionals

A main tool in this article is the Feynman–Kac formula, which expresses the solution of the PDE (1.3) as
\[
u_\varepsilon(t, x) = \mathbb{E}_B \left[ u_0(x + B(t)) \exp \left( \int_0^t \xi_\varepsilon(t - s, x + B(s)) ds - c_\varepsilon t \right) \right]. \] (2.1)
Here \( B(t) \) is a standard Brownian motion starting from the origin, independent of the driving noise \( \xi \). In fact, we will be considering several independent Brownian motions. We expand the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) to a larger one \((\Omega \times \Sigma, \mathcal{F} \times \mathcal{F}_B, \mathbb{P} \otimes \mathbb{P}_B)\) to include several independent Brownian motions \( B, B_1, B_2, \ldots, W, W_1, W_2, \ldots \), independent of \( \xi \). We will use \( \mathbb{E}_B \) to denote the expectation on \( \Sigma \). Also, we will often work with the marginal probability space \((\Omega, \mathcal{F}, \mathbb{P})\) or \((\Sigma, \mathcal{F}_B, \mathbb{P}_B)\).

Several functionals of the Brownian motions enter our analysis via (2.1). More precisely, the \( n \)-th moment of \( u_\varepsilon(t, x) \) is expressed in terms of functionals of Brownian motions. We begin with the first moment. To this end, define the covariance function
\[
R(t, x) := \int_{\mathbb{R}^3} \phi(t - s, x - y) \phi(-s, -y) ds dy, \quad R_\varepsilon(t, x) := \frac{1}{\varepsilon^2} R(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}) = \mathbb{E}[\xi_\varepsilon(t, x) \xi_\varepsilon(0, 0)].
\]
Recall that \( \phi \) is compactly supported. Without loss of generality, throughout this article we assume that \( \phi \) is supported in \((-\frac{1}{2}, \frac{1}{2})\) in \( t \), i.e., \( \phi(t, \cdot) = 0, \ |t| \geq \frac{1}{2} \), and hence \( R(t, \cdot) = 0, \ |t| \geq 1 \). With \( \xi_\varepsilon(t, x) \) being a
Gaussian process, averaging over ξ in (2.1) gives
\[ E[u_ξ(t, x)] = E_B \left[ u_0(x + B(t)) \exp \left( \frac{1}{2} E \left[ \left( \int_0^t \xi_ξ(t - s, x + B(s))ds \right)^2 \right] - c_ξ t \right) \right] \]
\[ = E_B \left[ u_0(x + B(t)) \exp \left( \int_0^t \int_0^s \R(u_ξ(s - u, B(s) - B(u))duds - c_ξ t \right) \right]. \tag{2.2} \]

Before progressing to the formula for higher moments, let us use (2.2) to explain how the renormalizing constant \( c_ξ \) comes into play. To this end, take \( u_0(x) = 1 \) for simplicity. In this case the solution \( U \) of the limiting SHE (1.1) satisfies \( E[U(t, x)] = 1 \). For the convergence in (1.4) to hold, we must choose \( c_ξ \) so that
\[ \lim_{\varepsilon \to 0} \left( \int_0^t \int_0^s \R(u_ξ(s - u, B(s) - B(u))duds - c_ξ t \right) \to 1 \]
with \( \varepsilon \to 1 \).

To this end, consider the centered double-integral process
\[ X_ξ(t) := \int_0^t \int_0^s \R(u_ξ(s - u, B(s) - B(u))duds - \int_0^t \int_0^s E_B[R(s - u, B(s) - B(u))]duds. \tag{2.3} \]

It is more convenient to express \( X_ξ \) in ‘microscopic’ coordinates. That is, we use the scaling property
\( (\varepsilon^{-1}B(\varepsilon^2t))_{t \geq 0} \to (B(t))_{t \geq 0} \) to write
\[ X_ξ(t) = X_ξ^m(t) := \varepsilon \int_{\varepsilon^{-2}t}^{\varepsilon^{-2}t} \int_0^s \R(u_ξ(s - u, B(s) - B(u))duds - \varepsilon \int_{\varepsilon^{-2}t}^{\varepsilon^{-2}t} \int_0^s E_B[R(s - u, B(s) - B(u))]duds \]
\[ = \varepsilon \int_{\varepsilon^{-2}t}^{\varepsilon^{-2}t} \left( \int_0^s (R(u_ξ(s - u, B(s) - B(u)) - E_B[R(u_ξ(s - u, B(s) - B(u)))]ds \right). \tag{2.4} \]

Since \( R(u_ξ) = 0 \) whenever \( |u| \geq 1 \), the \( u \)-integral in (2.3) goes over \( u \in [0, 1] \) for all \( s \geq 1 \). Dropping those values of \( s < 1 \) in the integral gives
\[ X_ξ^m(t) = \varepsilon \int_{1}^{\varepsilon^{-2}t} \X(s)ds + r_ξ(t), \tag{2.5} \]
where, recall that, \( r_ξ(t) \) denotes a generic process satisfying (1.5),
\[ \X(s) := \int_0^1 R(u_ξ(B(s) - B(s - u))du - c_1, \tag{2.6} \]
\[ c_1 := E_B \left[ \int_0^\infty R(u_ξ(B(u))du \right] = E_B \left[ \int_0^1 R(u_ξ(B(s) - B(s - u))du \right], \tag{2.7} \]
and we take \( B \) to be a two-sided Brownian motion in (2.6)–(2.7) so that the resulting expression is defined for all \( s \geq 0 \) (including \( s \in [0, 1] \)). It is straightforward to verify that
\[ \{\X(s)\}_{s \geq 0} \text{ is stationary in } s, \text{ bounded, } E_B[\X(s)] = 0, \tag{2.8} \]
with \( \{\X(s)\}_{s \geq s_0} \) being independent whenever \( s_0 - s'_0 \geq 1 \). \tag{2.9} \]

Thus it is natural to expect \( X_ξ(t) \) to converges to \( \sigma_ξ W(t) \), with \( W \) being a standard Brownian motion, and
\[ \sigma_ξ^2 := E_B \left[ \int_0^\infty \X(s)\X'(0)ds \right]. \tag{2.10} \]

In light of these discussions, we find that \( c_ξ := c_ξ\varepsilon^{-1} + 1/2\sigma_ξ^2 \) (as in Theorem 1.1) is the reasonable choice in order for \( u_ξ \) to converge to the solution \( U \) to the SHE.

With \( X_ξ(t) \) and \( \sigma_ξ \) defined in the preceding, we rewrite the formula (2.2) in a more compact form as
\[ E[u_ξ(t, x)] = E_B \left[ u_0(x + B(t)) \exp \left( X_ξ(t) - \frac{1}{2}\sigma_ξ^2 t + r_ξ(t) \right) \right]. \tag{2.11} \]

Similar calculations give formulas of higher moments:
\[ E[u_ξ(t, x)^n] = E_B \left[ \prod_{j=1}^n u_0(x + B_j(t)) \exp \left( \sum_{j=1}^n X_{j,ξ}(t) - \frac{1}{2}\sigma_ξ^2 t + r_ξ(t) \right) \right]. \tag{2.12} \]
Here $B_1, \ldots, B_n$ are independent Brownian motions; the process $X_{j, \varepsilon}(t)$ is obtained by replacing $B$ with $B_j$ in (2.3); and $Y_{i,j,\varepsilon}(t)$ is given by

$$Y_{i,j,\varepsilon}(t) := \int_0^t \int_0^t R_{\varepsilon}(s-u, B_i(s) - B_j(u)) ds du.$$  

(2.13)

Indeed, $(X_{j,\varepsilon}(t))_{t \geq 0} \overset{\text{law}}{=} (X_{\varepsilon}(t))_{t \geq 0}$. Likewise, writing $Y_{\varepsilon}(t) := Y_{1,2,\varepsilon}(t)$, we have $(Y_{i,j,\varepsilon}(t))_{t \geq 0} \overset{\text{law}}{=} (Y_{\varepsilon}(t))_{t \geq 0}$, for all $i < j$.

We will also need to consider $E[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)]$, i.e., the second moment calculated at different values of $\varepsilon$. A similar calculation gives the formula

$$E[u_{\varepsilon_1}(t,x)u_{\varepsilon_2}(t,x)] = E_B \left[ \prod_{j=1}^{2} u_0(x+B_j(t)) \exp \left( Y_{\varepsilon_1,\varepsilon_2}(t) + \sum_{j=1}^{2} \left( X_{j,\varepsilon}(t) - \frac{1}{2}\sigma_j^2 t + r_{\varepsilon_j}(t) \right) \right) \right],$$

(2.14)

where

$$Y_{\varepsilon_1,\varepsilon_2}(t) := \int_0^t \int_0^t R_{\varepsilon_1,\varepsilon_2}(s-u, B_1(s) - B_2(u)) ds du,$$

(2.15)

$$R_{\varepsilon_1,\varepsilon_2}(t,x) := \int_{\mathbb{R}^2} \phi_{\varepsilon_1}(t-s, x-y) \phi_{\varepsilon_2}(-s, -y) ds dy = E[\xi_{\varepsilon_1}(t,x)\xi_{\varepsilon_2}(0,0)].$$

2.1. Exponential moments. We first establish bounds on exponential moments of $X_{\varepsilon}(t)$ and $Y_{\varepsilon_1,\varepsilon_2}(t)$.

**Proposition 2.1.** For any $\lambda, t > 0$, we have

$$\sup_{\varepsilon \in (0,1)} E_B[e^{\lambda X_{\varepsilon}(t)}] + \sup_{\varepsilon_1,\varepsilon_2 \in (0,1)} E_B[e^{\lambda Y_{\varepsilon_1,\varepsilon_2}(t)}] < \infty.$$

**Proof.** For $X_{\varepsilon}(t)$, we appeal to the microscopic coordinates, using (2.4)–(2.5) to write

$$X_{\varepsilon}(t) \overset{\text{law}}{=} X_{\varepsilon_{\text{mi}}}(t) = \varepsilon \int_1^{e^{-2t}} \mathcal{X}(s) ds + r_{\varepsilon}(t) = \varepsilon \int_1^{e^{-2t}} \mathcal{X}(s) ds + r_{\varepsilon}(t).$$

In view of the finite range property (2.9) of $X$, we decompose

$$X_{\varepsilon_{\text{mi}}}(t) = \varepsilon \sum_{k \in I_{\text{even}}} \tilde{X}_k + \varepsilon \sum_{k \in I_{\text{odd}}} \tilde{X}_k,$$

where $\tilde{X}_k := \int_{k+1}^{k+1} \mathcal{X}(s) ds$, and $I_{\text{even}} := \{1 \leq k \leq [\varepsilon^2 t] - 1, \text{ even}\}$, and $I_{\text{odd}} := \{1 \leq k \leq [\varepsilon^2 t] - 1, \text{ odd}\}$. This gives

$$E_B[e^{\lambda X_{\varepsilon}(t)}] = E_B \left[ e^{\lambda r_{\varepsilon}(t)} \exp \left( \lambda \varepsilon \sum_{k \in I_{\text{even}}} \tilde{X}_k \right) \exp \left( \lambda \varepsilon \sum_{k \in I_{\text{odd}}} \tilde{X}_k \right) \right]$$

$$\leq C(\lambda, t) \sqrt{E_B \left[ \exp \left( 2\lambda \varepsilon \sum_{k \in I_{\text{even}}} \tilde{X}_k \right) \right] E_B \left[ \exp \left( 2\lambda \varepsilon \sum_{k \in I_{\text{odd}}} \tilde{X}_k \right) \right]} = C(\lambda, t) \sqrt{\prod_{k \in [t/\varepsilon^2]-1} E_B[e^{2\lambda \varepsilon \tilde{X}_k}].}$$

By (2.8), we know that $\tilde{X}_k$ is stationary with zero mean, so

$$\prod_{k \in [t/\varepsilon^2]-1} E_B[e^{2\lambda \varepsilon \tilde{X}_k}] = \left( E_B[e^{2\lambda \varepsilon \tilde{X}_1}] \right)^{[t/\varepsilon^2]-1} \leq C(\lambda, t).$$

From this we conclude the desired exponential moment bound on $X_{\varepsilon}(t)$:

$$\sup_{\varepsilon \in (0,1)} E_B[e^{\lambda X_{\varepsilon}(t)}] \leq C(\lambda, t) < \infty.$$
From this we calculate the \( n \)-th moment of \( Y_{\varepsilon_1,\varepsilon_2}(t) \) as

\[
E_B[Y_{\varepsilon_1,\varepsilon_2}(t)^n] = \int_{[0,t]^2\times\mathbb{R}^n} \prod_{j=1}^n \left( (2\pi)^{-1} \hat{R}_{\varepsilon_1,\varepsilon_2}(s_j - u_j, \xi_j) \right) \times E_B \left[ \prod_{j=1}^n e^{i \xi_j B_1(s_j)} \right] E_B \left[ \prod_{j=1}^n e^{-i \xi_j B_2(u_j)} \right] dsdudx.
\]

(2.16)

Let us first focus on the integral over \( u \in [0, t]^n \). We write

\[
\int_{[0,t]^n} \prod_{j=1}^n \hat{R}_{\varepsilon_1,\varepsilon_2}(s_j - u_j, \xi_j) E_B \left[ \prod_{j=1}^n e^{-i \xi_j B_2(u_j)} \right] du = \int_{[0,t]^n} \prod_{j=1}^n R_{\varepsilon_1,\varepsilon_2}(s_j - u_j, x_j) e^{-i \xi_j x_j} E_B \left[ \prod_{j=1}^n e^{-i \xi_j B_2(u_j)} \right] du dx.
\]

The exponents are purely imaginary. We hence bound those exponentials by 1 in absolute value, and use

\[
0 \leq \int_{[0,t]^n} R_{\varepsilon_1,\varepsilon_2}(s - u, x) dsdx \leq 1
\]

to get

\[
\left| \int_{[0,t]^n} \prod_{j=1}^n \hat{R}_{\varepsilon_1,\varepsilon_2}(u_j - s_j, \xi_j) E_B \left[ \prod_{j=1}^n e^{-i \xi_j B_2(u_j)} \right] du \right| \leq 1.
\]

Inserting this into (2.16) gives

\[
E_B[Y_{\varepsilon_1,\varepsilon_2}(t)^n] \leq (2\pi)^n \int_{[0,t]^n} \prod_{j=1}^n e^{i \xi_j B_1(s_j)} dsdx.
\]

(2.17)

The last integral in (2.17) is in fact the \( n \)-th moment of Brownian localtime at the origin. More precisely, let \( L(t, x; B) \) denote the localtime process of a Brownian motion \( B \), it is a standard result that

\[
(2\pi)^n \int_{[0,t]^n} \prod_{j=1}^n e^{i \xi_j B_1(s_j)} dsdx = E_B \left[ (L(t, 0; B))^n \right].
\]

(2.18)

Informally speaking, this formula is obtained by interpreting \( L(t, 0; B_1) \) as \( \int_0^t \delta(B_1(s))ds \), where \( \delta(\cdot) \) denotes the Dirac function, and taking Fourier transform, similarly to the preceding. The prescribed informal procedure is rigorously implemented by taking a sequence approximating the Dirac function. We omit the details here as the argument is standard.

Now, combine (2.17)–(2.18), and sum over \( n \geq 0 \). We arrive at \( E_B[e^{\lambda \varepsilon_1 \varepsilon_2(t)}] \leq E_B[e^{\lambda L(t,0;B_1)}] \). As the Brownian localtime has finite exponential moments, i.e., \( E[e^{\lambda \varepsilon(t)}] < \infty \) for any \( \lambda, t > 0 \), we obtain the desired exponential moment bound on \( Y_{\varepsilon_1,\varepsilon_2}(t) \):

\[
\sup_{\varepsilon_1,\varepsilon_2\in(0,1)} E_B[e^{\lambda Y_{\varepsilon_1,\varepsilon_2}(t)}] \leq C(\lambda, t) < \infty.
\]

This completes the proof. \( \square \)

2.2. Weak convergence. In this section, we derive the distributional limit of \( X_{\varepsilon,t} \) and \( Y_{\varepsilon_1,\varepsilon_2} \). First, since the covariance function \( R_{\varepsilon_1,\varepsilon_2}(t, x) \) converges to Dirac function \( \delta(\cdot) \delta(x) \), we expect the process \( Y_{\varepsilon_1,\varepsilon_2}(t) \) (defined in (2.15)) to converge to the mutual intersection localtime of \( B_1 \) and \( B_2 \). More precisely, recalling that \( L(t, x; B) \) denote the localtime process of a Brownian motion \( B \), we define the mutual intersection localtime of \( B_1 \) and \( B_2 \) as

\[
\ell(t) := L(t, 0; B_1 - B_2).
\]

(2.19)

Proposition 2.2. For any fixed \( t > 0 \), \( Y_{\varepsilon_1,\varepsilon_2}(t) \to \ell(t) \) in \( L^2(\Sigma) \), as \( \varepsilon_1, \varepsilon_2 \to 0 \).

Proof. Instead of directly proving the convergence of \( Y_{\varepsilon_1,\varepsilon_2}(t) \), let us first consider a modified process \( \tilde{Y}_{\varepsilon_1,\varepsilon_2}(t) \) where \( B_1 \) and \( B_2 \) are evaluated at the same time:

\[
\tilde{Y}_{\varepsilon_1,\varepsilon_2}(t) := \int_0^t \int_0^t R_{\varepsilon_1,\varepsilon_2}(s - u, B_1(s) - B_2(s)) dsdu,
\]

and show that \( \tilde{Y}_{\varepsilon_1,\varepsilon_2}(t) \) converges to \( \ell(t) \) in \( L^2(\Sigma) \). To this end, set

\[
F_{\varepsilon_1,\varepsilon_2}(s) := \int_0^s R_{\varepsilon_1,\varepsilon_2}(s - u, B_1(s) - B_2(s)) du, \quad s \in [0, t].
\]
In the preceding double-integral expression of $\mathcal{Y}_{\epsilon_1,\epsilon_2}(t)$, divide the range of integration over $s$ into subintervals depending on its distance from 0 to $t$. We rewrite the expression as

$$\mathcal{Y}_{\epsilon_1,\epsilon_2}(t) = \int_0^t F_{\epsilon_1,\epsilon_2}(s) \left( I_{(\epsilon_1^2+\epsilon_2^2,t-\epsilon_1^2-\epsilon_2^2)}(s) + I_{[0,\epsilon_1^2+\epsilon_2^2]}(s) + I_{[t-\epsilon_1^2-\epsilon_2^2,t]}(s) \right) ds.$$  

Recall that $\phi(t,\cdot) = 0$, $|t| \geq \frac{1}{2}$. This gives $R_{\epsilon_1,\epsilon_2}(s-u,\cdot) = 0$ for all $|s-u| \geq \epsilon_1^2 + \epsilon_2^2$. Consequently, for $s \in (\epsilon_1^2 + \epsilon_2^2, t-\epsilon_1^2-\epsilon_2^2)$ we have

$$F_{\epsilon_1,\epsilon_2}(s) = \int_\mathbb{R} R_{\epsilon_1,\epsilon_2}(s-u, B_1(s) - B_2(s)) du.$$  

Further setting $\Phi(x) := \int_\mathbb{R} \phi(t,x) dx$, $\Phi_x(x) := \varepsilon^{-1} \Phi(\varepsilon^{-1} x)$, and $\Phi_{\epsilon_1,\epsilon_2}(x) := \int \Phi_{\epsilon_1}(x-y) \Phi_{\epsilon_2}(-y) dy$, we rewrite the last expression as $F_{\epsilon_1,\epsilon_2}(s) = \Phi_{\epsilon_1,\epsilon_2}(B_1(s) - B_2(s))$. On the other hand, we also have $|F_{\epsilon_1,\epsilon_2}(s)| \leq \Phi_{\epsilon_1,\epsilon_2}(B_1(s) - B_2(s))$, for all $s \in [0,t]$. This takes into account those values of $s \not\in (\epsilon_1^2 + \epsilon_2^2, t-\epsilon_1^2-\epsilon_2^2)$, thereby giving

$$\mathcal{Y}_{\epsilon_1,\epsilon_2}(t) = \int_{\epsilon_1^2 + \epsilon_2^2}^{t-\epsilon_1^2-\epsilon_2^2} \Phi_{\epsilon_1,\epsilon_2}(B_1(s) - B_2(s)) ds + r_{\epsilon_1,\epsilon_2}(t),$$  

where $r_{\epsilon_1,\epsilon_2}(t)$ is a remainder term satisfying

$$|r_{\epsilon_1,\epsilon_2}(t)| \leq \int_0^t \Phi_{\epsilon_1,\epsilon_2}(B_1(s) - B_2(s)) \left( I_{[0,\epsilon_1^2+\epsilon_2^2]}(s) + I_{[t-\epsilon_1^2-\epsilon_2^2,t]}(s) \right) ds.$$

Now, for any interval $[a, b] \subset [0,\infty)$, by the definition of the localtime,

$$\int_a^b \Phi_{\epsilon_1,\epsilon_2}(B_1(s) - B_2(s)) ds = \int_\mathbb{R} \Phi_{\epsilon_1,\epsilon_2}(x) \left( L(b, x; B_1 - B_2) - L(a, x; B_1 - B_2) \right) dx$$

$$= \int_\mathbb{R} \Phi(x) \Phi(-y) \left( L(b, \epsilon_1 x + \epsilon_2 y; B_1 - B_2) - L(a, \epsilon_1 x + \epsilon_2 y; B_1 - B_2) \right) dxdy.$$  

For almost every realization of $B_1 - B_2$, the function $x \mapsto L(t, x; B_1 - B_2)$ is continuous and compactly supported, and the function $t \mapsto L(t, x; B_1 - B_2)$ is increasing and continuous. Thus, from (2.20)–(2.21) and the fact that $\int \Phi dx = 1$, we conclude that $\mathcal{Y}_{\epsilon_1,\epsilon_2}(t) \rightarrow \ell(t) = L(t, 0; B_1 - B_2)$ almost surely as $\epsilon_1,\epsilon_2 \rightarrow 0$. Further, the same calculations (via Fourier transform) as in the proof Proposition 2.1 yields that

$$\sup_{\epsilon_1,\epsilon_2} \mathbb{E}_B \left[ e^{\lambda \mathcal{Y}_{\epsilon_1,\epsilon_2}(t)} \right] \leq \mathbb{E}_B \left[ e^{\lambda \ell(t); B_1} \right] < \infty.$$  

This property leverages the preceding almost sure convergence into a convergence in $L^2(\Sigma)$:

$$\mathbb{E}_B [ (\mathcal{Y}_{\epsilon_1,\epsilon_2}(t) - \ell(t))^2 ] \rightarrow 0 \quad \text{as } \epsilon_1,\epsilon_2 \rightarrow 0. \quad (2.22)$$

Given (2.22), it remains to show that $Y_{\epsilon_1,\epsilon_2}(t) - \mathcal{Y}_{\epsilon_1,\epsilon_2}(t) \rightarrow 0$ in $L^2(\Sigma)$. We will actually prove the following result: for any choice of $Z_1, Z_2 \in \{ Y_{\epsilon_1,\epsilon_2}(t), \mathcal{Y}_{\epsilon_1,\epsilon_2}(t) \}$, as $\epsilon_1,\epsilon_2 \rightarrow 0$,

$$\mathbb{E}_B [Z_1 Z_2] \rightarrow (2\pi)^{-2} \int_{[0,t]^2} \int_{\mathbb{R}^2} \mathbb{E}_B \left[ e^{i \xi (B_1(s) - B_2(s)) - e^{i \xi' (B_1(s') - B_2(s'))} } \right] d\xi d\xi' ds ds'.$$

Once this is done, expanding $\mathbb{E}[(Y_{\epsilon_1,\epsilon_2}(t) - \mathcal{Y}_{\epsilon_1,\epsilon_2}(t))^2]$ into four terms, and passing to the limit complete the proof.

The proof for all cases of $Z_1, Z_2 \in \{ Y_{\epsilon_1,\epsilon_2}(t), \mathcal{Y}_{\epsilon_1,\epsilon_2}(t) \}$ is the same, and we take $Z_1 = Z_2 = Y_{\epsilon_1,\epsilon_2}(t)$ as an example. As in the proof of Proposition 2.1, we express $Y_{\epsilon_1,\epsilon_2}(t)$ via Fourier transform as

$$Y_{\epsilon_1,\epsilon_2}(t) = \int_{[0,t]^2} \left( \int_\mathbb{R} (2\pi)^{-2} R_{\epsilon_1,\epsilon_2}(s-u,\xi) e^{i \xi (B_1(s) - B_2(u))} d\xi \right) duds$$

$$= \int_{[0,t]^2} \left( \int_\mathbb{R} (2\pi)^{-2} \tilde{R}_{\epsilon_1,\epsilon_2}(s-u,\xi) e^{i \xi (B_1(s) - B_2(u))} d\xi \right) duds,$$

where

$$\tilde{R}_{\epsilon_1,\epsilon_2}(s-u,\xi) = \int_\mathbb{R} (2\pi)^{-2} \tilde{R}_{\epsilon_1,\epsilon_2}(s-u,\xi) e^{i \xi (B_1(s) - B_2(u))} d\xi.$$
where \( \hat{\phi} \) denotes the Fourier transform of \( \phi \) in the \( x \)-variable. Squaring the last expression and taking expectation gives

\[
E_B[Y_{t,s}(t)^2] = \int_{[0,t]^2} G_{t,s}(s,s')dsds',
\]

where

\[
G_{t,s}(s,s') := \int_{R^4} (2\pi \varepsilon_1^2 \varepsilon_2^2 )^{-\frac{1}{2}} \hat{\phi}(\frac{s-u-w}{\varepsilon_1},\varepsilon_1 \xi) \hat{\phi}(\frac{-w}{\varepsilon_2},-\varepsilon_2 \xi) \hat{\phi}(\frac{s-u'-w'}{\varepsilon_2},\varepsilon_1 \xi') \hat{\phi}(\frac{-w'}{\varepsilon_2},-\varepsilon_2 \xi')
\times \mathbb{E}_B \left[ e^{iK(B_1(s)-B_2(u)))e^{iK(B_1(s')-B_2(u'))} \right] d\xi d\xi' d\varepsilon d\varepsilon'.
\]

Fix \((s,s') \in (0,t)^2\). Recall that \( \hat{\phi}(t,\cdot) = 0 \) whenever \(|t| \geq \frac{1}{2}\), and note that, with \((s,s') \in (0,t)^2\) being fixed, the conditions \((t-s),(t-s'),s,s' \geq \frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2)\) holds for all small enough \( \varepsilon_1,\varepsilon_2 \). Consequently, in the last expression of \( G_{t,s}(s,s') \), for all \( \varepsilon_1,\varepsilon_2 \) small enough, the integration domain of \( u, u' \) can be (and is) replaced \([0,t]^2 \mapsto \mathbb{R}^2\). A change of variables in this case yields

\[
G_{t,s}(s,s') = \int_{R^6} (2\pi)^{-2} \hat{\phi}(u,\varepsilon_1 \xi) \hat{\phi}(-w,-\varepsilon_2 \xi) \hat{\phi}(u',\varepsilon_1 \xi') \hat{\phi}(-w',-\varepsilon_2 \xi')
\times \mathbb{E}_B \left[ e^{iK(B_1(s)-B_2(s))e^{iK(B_1(s')-B_2(s'))} \right] d\xi d\xi' d\varepsilon d\varepsilon'.
\]

By the dominated convergence theorem and the fact that \( \int \hat{\phi}(u,0)du = \int \phi dx = 1 \), we obtain

\[
G_{t,s}(s,s') \to (2\pi)^{-2} \int_{R^2} \mathbb{E}_B \left[ e^{iK(B_1(s)-B_2(s))e^{iK(B_1(s')-B_2(s'))} \right] d\xi d\xi', \text{ pointwisely in } (0,t)^2.
\]

To achieve (2.23), we need to upgrade the pointwise convergence of (2.24) to convergence in \( L^1([0,t]^2) \). To this end, with \( |\hat{\phi}(\cdot, \xi)| \leq \hat{\phi}(\cdot,0) \), we bound

\[
|G_{t,s}(s,s')| \leq \int_{R^6} (2\pi \varepsilon_1^2 \varepsilon_2^2 )^{-\frac{1}{2}} \hat{\phi}(\frac{s-u-w}{\varepsilon_1},0) \hat{\phi}(\frac{-w}{\varepsilon_2},0) \hat{\phi}(\frac{s-u'-w'}{\varepsilon_2},0) \hat{\phi}(\frac{-w'}{\varepsilon_2},0)
\times \mathbb{E}_B \left[ e^{iK(B_1(s'))e^{iK(B_1(s'))} \right] d\xi d\xi' d\varepsilon d\varepsilon'.
\]

After integrating in \( u, u' \) on the RHS of the last integral, we have

\[
|G_{t,s}(s,s')| \leq (2\pi)^{-2} \int_{R^2} \mathbb{E}_B \left[ e^{iK(B_1(s))e^{iK(B_1(s'))} \right] d\xi d\xi' \leq \frac{C}{\sqrt{(s \wedge s')|s-s'|}} \in L^1([0,t]^2).
\]

Given this, the dominated convergence theorem upgrades (2.24) into a convergence in \( L^1([0,t]^2) \). This gives (2.23) and hence completes the proof.

We next turn to the distributional limit of \( X_{t,x} \) and \( X_{x,t} \) (defined in (2.3)). Hereafter, we use \( \Rightarrow \) to denote the weak convergence of probability laws in a designated space, and endow the space \( C[0,\infty) \) with the topology of uniform convergence over compact subsets of \([0,\infty)\).

**Proposition 2.3.** As \( \varepsilon \to 0 \),

\[
(B_1, B_2, \xi, X_{t,x}, X_{x,t}) \Rightarrow (B_1, B_2, \xi, W_1, \sigma W_2) \quad \text{in } (C[0,\infty))^5.
\]

where \( W_1, W_2 \) are standard Brownian motions independent of \( B_1, B_2 \), and \( \sigma \in (0,\infty) \) is given in (2.10).

**Proof.** The proof consists of two steps.

**Step 1:** Instead of showing (2.25) directly, let us first establish

\[
(B_1, B_2, X_{t,x}, X_{x,t}) \Rightarrow (B_1, B_2, \sigma W_1, \sigma W_2) \quad \text{in } (C[0,\infty))^4.
\]

To this end, we appeal to microscopic coordinates. That is, with \( X_{\xi,t} \) given in (2.4) (with \( B \) replaced by \( B_j \)), we have

\[
(B_1(t), B_2(t), X_{t,x}(t), X_{x,t}(t))_{t \geq 0} \law \Rightarrow (\varepsilon B_1(\varepsilon^{-2}t), \varepsilon B_2(\varepsilon^{-2}t), X_{t,x}^{mi}(t), X_{x,t}^{mi}(t))_{t \geq 0}.
\]
We begin by writing $X_{j,t}^\varepsilon$ in terms of stochastic integrals. Let $D_{j,t}$ denote the Malliavin derivative with respect to $dB_j(t)$ on $(\Sigma, \mathcal{F}_t, \mathbb{P}_B)$, and let $\mathcal{F}_t$ denote the canonical filtration of $B_j$. The Clark–Ocone formula states that (with $X_{j,t}^\varepsilon(t)$ having zero mean)

$$X_{j,t}^\varepsilon(t) = \varepsilon \int_0^{\varepsilon^2 t} Z_{j,t}(r,t)dB_j(r), \quad Z_{j,t}(r,t)(r,t) := \varepsilon^{-1}E[D_{j,t}X_{j,t}^\varepsilon(t)|\mathcal{F}_t(r)].$$

A direct calculation yields

$$D_{j,t}X_{j,t}^\varepsilon(t) = \varepsilon \int_0^{\varepsilon^2 t} \int_0^t \partial_x R(s-u,B_j(s)-B_j(u))duds\mathbf{1}_{\{u \leq r < s\}},$$

so

$$Z_{j,t}(r,t) = \int_{\varepsilon^{-2} t}^{\varepsilon^2 t} \int_0^r E_B[\partial_x R(s-u,B_j(s)-B_j(u))|\mathcal{F}_t(r)]duds.$$  \hspace{1cm} (2.28)

The function $\partial_x R(s,x)$ vanishes for all $|s| \geq 1$ (because $R$ does), so by defining

$$\tilde{Z}_j(r) := \mathbf{1}\{r \geq 1\} \int_{\varepsilon^{-2} t}^{\varepsilon^2 t} \int_0^r E_B[\partial_x R(s-u,B_j(s)-B_j(u))|\mathcal{F}_t(r)]duds,$$  \hspace{1cm} (2.29)

we have $Z_{j,t}(r,t) = \tilde{Z}_j(r)$, for all $r \in [1,\varepsilon^2 t - 1)$. The latter is preferred for our purpose, because it does not depend on $t$. In particular, the analogous integrated process:

$$t \mapsto \tilde{X}_{j,t}^\varepsilon(t) := \varepsilon \int_0^{\varepsilon^2 t} \tilde{Z}_j(r)dB_j(r)$$

is a martingale (unlike $X_{j,t}^\varepsilon(t)$, which is not due to the $t$-dependence of $Z_{j,t}(r,t)$). Also, for each $t \in \mathbb{R}_+$, the $L^2(\Sigma)$-distance between $\tilde{X}_{j,t}^\varepsilon(t)$ and $X_{j,t}^\varepsilon(t)$ vanishes as $\varepsilon \to 0$:

$$E_B[|\tilde{X}_{j,t}^\varepsilon(t) - X_{j,t}^\varepsilon(t)|^2] = \varepsilon^2 \int_0^{\varepsilon^2 t} E_B(|Z_{j,t}(r,t) - \tilde{Z}_j(r)|^2)dr \leq C\varepsilon^2.$$ \hspace{1cm} (2.30)

Given this, let us focus on the modified process $\tilde{X}_{j,t}^\varepsilon(t)$ instead of $X_{j,t}^\varepsilon(t)$.

Now, consider the $C([0,\infty), \mathbb{R}^4)$-valued process

$$M^\varepsilon_t = (M^\varepsilon_{(i,t)}(s))_{s \geq 1} := (\varepsilon B_1(\varepsilon^{-2} t), \varepsilon B_2(\varepsilon^{-2} t), \tilde{X}_{1,t}^\varepsilon(t), \tilde{X}_{2,t}^\varepsilon(t)).$$

It is a continuous martingale, with cross-variance

$$\langle M^\varepsilon_{(i,t)}, M^\varepsilon_{(j,t)} \rangle(t) = \begin{cases} 0 & \text{, for } (i,j) = (1,2), (3,4), (1,4), (2,3), \\
\varepsilon^2 \int_0^{\varepsilon^2 t} \tilde{Z}_i(s) ds & \text{, for } (i,j) = (1,3), (2,4), \\
\varepsilon^2 \int_0^{\varepsilon^2 t} \tilde{Z}_i^2(s) ds & \text{, for } (i,j) = (1,1), (2,2), \\
\varepsilon^2 \int_0^{\varepsilon^2 t} \tilde{Z}_i^2(s) ds & \text{, for } (i,j) = (3,3), (4,4). \end{cases}$$

Further, straightforward calculations from the expression (2.28) gives, with $\partial_x R$ being an odd function in $x$,

$$E_B[\tilde{Z}_i(s)] = 0 \text{ and } E_B[\tilde{Z}_i^2(s)] = (\sigma_i')^2, \text{ for all } s \geq 1,$$

where $\sigma_i' := E_B[\tilde{Z}_i^2(1)] = E_B \left[\int_1^2 \partial_x R(s-u,B_j(s)-B_j(u))duds\mathbf{1}_\{s \geq 1\}\right].$ \hspace{1cm} (2.31)

Calculating the conditional expectation in (2.29) gives

$$\tilde{Z}_j(r) = \mathbf{1}\{r \geq 1\} \int_{\varepsilon^{-2} t}^{\varepsilon^2 t} \int_0^r \tilde{R}(s,r,u,B_j(r) - B_j(u))duds,$$

where $\tilde{R}(s,r,u,x) := \int_0^s q(s-r, x-y)\partial_y R(s-u,y)dy$, and $q(t, x) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ denotes the standard heat kernel. From this expression, it is readily check that $(\tilde{Z}_i(s))_{s \geq 1}$ is bounded, stationary, and has finite range correlation similar to (2.9). Consequently, Ergodic theorem applied to $(\tilde{Z}_i(s))_{s \geq 1}$ and $\{\tilde{Z}_i^2(s)\}_{s \geq 1}$ gives

$$\varepsilon^2 \int_0^{\varepsilon^2 t} \tilde{Z}_i(s)ds \to 0, \quad \varepsilon^2 \int_0^{\varepsilon^2 t} \tilde{Z}_i^2(s)ds \to (\sigma_i')^2 t,$$
almost surely as \( \varepsilon \to 0 \), for any fixed \( t \in \mathbb{R}_+ \). Given these properties, the martingale central limit theorem [EK86, Thm 1.4, p339] yields that \( M \Rightarrow (B_1, B_2, \sigma_1^{}W_1, \sigma_2^{}W_2) \), in \((C[0,\infty))^4\). This together with (2.27) and (2.30) gives

\[
(B_1(t), B_2(t), X_{1,\varepsilon}(t), X_{2,\varepsilon}(t)) \Rightarrow (B_1(t), B_2(t), \sigma_1^{}W_1(t), \sigma_2^{}W_2(t)), \text{ in fdd.} \tag{2.32}
\]

Given (2.32), it now suffices to establish the tightness of \((B_1, B_2, X_{1,\varepsilon}, X_{2,\varepsilon})\) in \((C[0,\infty))^4\), and show that \((\sigma_1^2) = \sigma_2^2\). The first step is to appeal to microscopic coordinates. Using (2.4)–(2.5) we write

\[
\int_1^e X_j(\varepsilon)ds + r_\varepsilon(t).
\]

Given the properties (2.8)–(2.9) of \(X_j\), a classical functional central limit theorem, see, e.g., [Bil99, pp 178–179], asserts that

\[
X_{j,\varepsilon} \Rightarrow \sigma_j W_j, \text{ in } C[0,\infty). \tag{2.33}
\]

This in particular implies the tightness of \((B_1, B_2, X_{1,\varepsilon}, X_{2,\varepsilon})\). Further, comparing (2.32)–(2.33), we see that \(\sigma_2^2 = (\sigma_1^2)^2\). Alternatively, it is possibly to show \(\sigma_2^2 = (\sigma_1^2)^2\) by calculations from the expressions (2.10) and (2.31). We thus conclude (2.26).

**Step 2:** Having established (2.26), our next goal is to extend the convergence result to include the local time process \(\ell\). First, Tanaka’s formula gives

\[
2\ell(t) = |B_1(t) - B_2(t)| - \int_0^t \text{sgn}(B_1(s) - B_2(s))d(B_1 - B_2)(s).
\]

Had it been the case that the RHS were a continuous function of \(B_1 - B_2\), the desired result (2.25) would follow immediately from (2.26). We show in Lemma 2.4 that, in fact, the stochastic integral \(\int_0^t \text{sgn}(B_1(s) - B_2(s))d(B_1 - B_2)(s)\) is well-approximated by a sequence of continuous functions of \(B_1 - B_2\). That is, there exists a sequence \(\{f_n\}_{n \geq 1} \subset C(C[0,\infty); C[0,\infty])\) such that

\[
\left\| \int_0^t \text{sgn}(B_1(s) - B_2(s))d(B_1 - B_2)(s) - f_n(B_1 - B_2) \right\|_{C[0,\infty)} \rightarrow 0 \text{ in probability, as } n \rightarrow \infty. \tag{2.34}
\]

Now, fix arbitrary bounded and continuous \(g : (C[0,\infty))^3 \rightarrow \mathbb{R}\) and \(h : (C[0,\infty))^2 \rightarrow \mathbb{R}\), and consider test functions of the type \(g \otimes h \in (C[0,\infty))^5 \rightarrow \mathbb{R}\). It is known that the linear span of functions of this type is dense in \(C((C[0,\infty))^3; \mathbb{R})\). Hence proving (2.25) amounts to proving

\[
\mathbb{E}_B[g(B_1, B_2, \ell)h(X_{1,\varepsilon}, X_{2,\varepsilon})] \rightarrow \mathbb{E}_B[g(B_1, B_2, \ell)h(\sigma_1 W_1, \sigma_2 W_2)], \text{ as } \varepsilon \to 0. \tag{2.35}
\]

Set \(2\ell_n(t) := |B_1(t) - B_2(t)| - f_n(B)(t)\). Since \(f_n\) is continuous, for each fixed \(n\), from (2.26) we have

\[
\mathbb{E}_B[g(B_1, B_2, \ell_n)h(X_{1,\varepsilon}, X_{2,\varepsilon})] \rightarrow \mathbb{E}_B[g(B_1, B_2, \ell)h(\sigma_1 W_1, \sigma_2 W_2)] \text{ as } \varepsilon \to 0.
\]

On the other hand, with \(g\) being bounded and continuous, by (2.34), we have

\[
\mathbb{E}_B[|g(B_1, B_2, \ell_n) - g(B_1, B_2, \ell)|] \rightarrow 0,
\]

as \(n \rightarrow \infty\). From these the desired result (2.35) follows. The proof is complete. \(\Box\)

**Lemma 2.4.** The claim (2.34) holds for a sequence \(\{f_n\}_{n \geq 1} \subset C(C[0,\infty); C[0,\infty])\).

**Proof.** Set \(B(t) := B_1(t) - B_2(t)\) and \(U(t) := \int_0^t \text{sgn}(B(s))dB(s)\) to simplify the notation. We begin by constructing the continuous function \(f_n\). Set \(\zeta(x) := x\{[x] \leq 1\} + \text{sgn}(x)\{[x] > 1\}\), \(\zeta_n(x) := \zeta(x/n^{1/4})\), and define

\[
f_n(y)(t) := \int_0^t \sum_{k=0}^{\infty} \zeta_n(y(k/n))\mathbf{1}_{[\frac{k}{n}, \frac{k+1}{n})}(s)dy(s) := \sum_{k=0}^{\infty} \zeta_n(y(k/n))(y(k+1/n) \land t) - y(k/n) \land t).
\]
Indeed, \( f_n \) is continuous for each fixed \( n \). Fix an arbitrary \( T > 0 \). Using Doob’s \( L^2 \)-martingale inequality and Itô isometry, we calculate

\[
E_B \left[ \sup_{[0,T]} |U(t) - f_n(B(t))|^2 \right] \leq C \int_0^T E_B \left[ \left| \sum_{k=0}^\infty \zeta_n \left( B \left( \frac{k}{n} \right) \right) 1_{k \frac{n-1}{n}}(t) - \text{sgn}(B(t)) \right|^2 \right] dt
\]

\[
= C \sum_{k=0}^\infty \int_{I_{k,n}(T)} E_B \left| \zeta_n \left( B \left( \frac{k}{n} \right) \right) - \text{sgn}(B(t)) \right|^2 dt,
\]

where \( I_{k,n}(T) := \left[ k \frac{n-1}{n}, k+1 \frac{n-1}{n} \right) \cap [0,T] \). Set \( V_{k,n} := \sup_{s \in \left[ k \frac{n-1}{n}, k+1 \frac{n-1}{n} \right]} |B(s + \frac{k}{n}) - B(\frac{k}{n})| \). On the interval \( t \in I_{k,n}(T) \), we have \( \zeta_n \left( B \left( \frac{k}{n} \right) \right) = \text{sgn}(B(t)) \) whenever \( |B(\frac{k}{n})| > n^{-\frac{1}{4}} \) and \( V_{k,n} < n^{-\frac{1}{4}} \). Hence

\[
E_B \left[ \sup_{[0,T]} |U(t) - f_n(B(t))|^2 \right] \leq C \sum_{k=0}^\infty \left( P_{B}[V_{k,n} \geq n^{-\frac{1}{4}}] + P_{B}[|B(\frac{k}{n})| \leq n^{-\frac{1}{4}}] \right)^T_n.
\]

By the scaling property of Brownian motion and the reflection principle, we have that \( V_{k,n} \overset{\text{law}}{=} \sqrt{2k/n} Z \), where \( Z \) is a standard Gaussian. This gives

\[
E_B \left[ \sup_{[0,T]} |U(t) - f_n(B(t))|^2 \right] \leq C \sum_{k=0}^\infty \left( P_{B} \left[ |Z| \geq 2^{-\frac{1}{4}} n^{\frac{1}{4}} \right] + P_{B} \left[ |Z| \leq \frac{n^{1/4}}{(2k)^{1/4}} \right] \right)^T_n,
\]

with \( \frac{n^{1/4}}{(2k)^{1/4}} := \infty \) when \( k = 0 \). It is now readily verified that the last expression tends to 0 as \( n \to \infty \). From this the desired result follows: \( \sup_{[0,T]} |U(t) - f_n(B(t))| \to 0 \) in probability, as \( n \to \infty \), for each fixed \( T \). \( \square \)

3. **Proof of Theorem 1.1**

Let us first establish the boundedness of moments of \( u_{t}(t,x) \). Set \( \lambda_n := \frac{n(n+1)}{2} \). With the initial condition \( u_0 \) being bounded, applying Hölder’s inequality (with exponents \( (\lambda_n, \ldots, \lambda_n) \)) in the formula (2.12), we have

\[
E[|u_{t}(t,x)|^n] \leq C \left( \prod_{j=1}^n E_B \left[ \exp \left( \frac{1}{2} \sigma_j^2 t + r_j(t) \right) \exp \left( \lambda_n X_j(t) - \frac{1}{2} \sigma_j^2 t + r_j(t) \right) \right] \prod_{1 \leq i < j \leq n} E_B \left[ \exp \left( \frac{1}{2} \sigma_j^2 t + r_j(t) \right) \right] \right)^{1/\lambda_n}
\]

\[
\leq C \left( \prod_{j=1}^n E_B \left[ \exp \left( \frac{1}{2} \sigma_j^2 t + r_j(t) \right) \right] \right)^{n/\lambda_n} \left( \prod_{j=1}^n E_B \left[ \exp \left( \frac{1}{2} \sigma_j^2 t + r_j(t) \right) \right] \right)^{n(n-1)/2\lambda_n}.
\]

Using the exponential moment bounds from Proposition 2.1, we obtain \( \sup_{t \in (0,1)} E[|u_{t}(t,x)|^n] < \infty \). That is, moments of \( u_{t}(t,x) \) are bounded uniformly in \( \varepsilon \). This reduces proving (1.4) for all \( n \geq 1 \) to proving (1.4) for just one \( n \geq 1 \). We henceforward consider \( n = 2 \).

Let us first identify the limit of \( E[\nu_{x_1}(t,x)u_{x_2}(t,x)] \). Recall from (2.19) that \( \ell(t) \) denotes the mutual intersection localtime of \( B_1, B_2 \), and that \( \mathcal{U} \) denotes the solution of the SHE (1.1).

**Proposition 3.1.** We have

\[
\lim_{\varepsilon_1,\varepsilon_2 \to 0} E[\nu_{x_1}(t,x)u_{x_2}(t,x)] = E_B \left[ u_0(x + B_1(t))u_0(x + B_2(t))\exp(\ell(t)) \right].
\]

**Proof.** The starting point of the proof is the formula (2.14):

\[
E[\nu_{x_1}(t,x)u_{x_2}(t,x)] = E_B \left[ \prod_{j=1}^2 u_0(x + B_j(t)) \exp \left( \sum_{j=1}^2 \left( X_{j,\varepsilon_j}(t) - \frac{1}{2} \sigma_j^2 t + r_j(t) \right) \right) \right].
\]

By virtue of Propositions 2.2–2.3, we have

\[
(B_1(t), B_2(t), X_{1,\varepsilon_1}(t), X_{2,\varepsilon_2}(t)) \Rightarrow (B_1(t), B_2(t), \ell(t), \sigma_{\ast} W_1(t), \sigma_{\ast} W_2(t))
\]

in distribution. The proof is complete by invoking Propositions 2.1. \( \square \)

Now, with

\[
E[(\nu_{x_1}(t,x) - u_{x_2}(t,x))^2] = E[\nu_{x_1}(t,x)u_{x_1}(t,x)] - 2E[\nu_{x_1}(t,x)u_{x_2}(t,x)] + E[u_{x_2}(t,x)u_{x_2}(t,x)],
\]

Proposition 3.1 has an immediate corollary:
Corollary 3.2. The sequence \( \{u_\varepsilon(t,x)\}_{\varepsilon \in (0,1)} \) is Cauchy in \( L^2(\Omega) \).

Given this result, it suffices to identify the unique limit of \( u_\varepsilon(t,x) \) in \( L^2(\Omega) \). We achieve this by Wiener chaos expansion. Fix \((t,x) \in \mathbb{R}_+ \times \mathbb{R} \) hereafter, and denote \( \mathcal{H}_k := \{(s_1, \ldots, s_k) \in \mathbb{R}^k : s_1 < \ldots < s_k \} \). Given any \( f \in L^2(\mathcal{H}_k \times \mathbb{R}^k) \), we consider the k-th order multiple stochastic integral

\[
I_k(f) := \int_{\mathcal{H}_k \times \mathbb{R}^k} f(s_1, \ldots, s_k, x_1, \ldots, x_k) \prod_{i=1}^{n} \xi(s_i, y_i) ds_i dy_i.
\]

Let \( q(s, y) = \frac{1}{\sqrt{2\pi s}} e^{-y^2/2s} \) denotes the standard heat kernel, the solution \( \mathcal{U} \) of the SHE permits the chaos expansion (we omit the dependence on \( (t,x) \))

\[
\mathcal{U}(t, x) = \sum_{k=0}^{\infty} I_k(f_k), \quad f_k := \mathbf{1}_{\{0 < s_1 < \ldots < s_k < t\}} \int_{\mathbb{R}} u_0(y_0) \prod_{i=0}^{k} q(s_{i+1} - s_i, y_{i+1} - y_i) dy_0,
\]

under the convention \( y_{k+1} := x, \) \( s_0 := 0, \) and \( s_{k+1} := t \). For \( u_\varepsilon \), a similar expansion also exists: the Stroock formula [Str87, Eqn (7), p3] states that

\[
u_\varepsilon(t, x) = \sum_{k=0}^{\infty} I_k(f_{k, \varepsilon}), \quad f_{k, \varepsilon} := \mathbb{E}[D^k u_\varepsilon(t, x)].
\]

Here \( D \) denotes the Malliavin derivative with respect to \( \xi \) on \((\Omega, \mathcal{F}, \mathbb{P})\). To calculate the chaos coefficient \( f_{k, \varepsilon} \), we set

\[
\Phi_{\varepsilon,B}(r, y) := \int_0^t \phi_{\varepsilon}(t - s, r, x + B(s) - y) ds,
\]

and rewrite the Feynman–Kac formula (2.1) as

\[
u_\varepsilon(t, x) = \mathbb{E}_B\left[ u_0(x + B(t)) \exp \left( \int_{\mathbb{R}^2} \Phi_{\varepsilon,B}(r, y) \xi(r, y) dy dr - \varepsilon t \right) \right].
\]

From this expression we calculate

\[
f_{k, \varepsilon}(r_1, \ldots, r_k, y_1, \ldots, y_k)
= \mathbb{E}_B\left[ u_0(x + B(t)) \exp \left( \int_0^t \int_0^s R_{\varepsilon}(s - u, B(s) - B(u)) du ds - \varepsilon t \right) \prod_{i=1}^{k} \Phi_{\varepsilon,B}(r_i, y_i) \right].
\]

Denoting the \( L^2(\Omega) \)-limit of \( \nu_\varepsilon(t, x) \) by \( \mathcal{W}(t, x) \), and the chaos expansion of \( \mathcal{W}(t, x) \) is written as

\[
\mathcal{W}(t, x) = \sum_{k=0}^{\infty} I_k(\tilde{f}_k).
\]

The following lemma completes the proof of Theorem 1.1.

Lemma 3.3. \( \mathcal{W}(t, x) = \mathcal{U}(t, x) \) in \( L^2(\Omega) \).

Proof. Given the chaos expansions in (3.2) and (3.6), it suffices to show \( \tilde{f}_k = f_k \). Since \( \nu_\varepsilon(t, x) \to \mathcal{W}(t, x) \) in \( L^2(\Omega) \), using the orthogonality of the chaos, i.e.,

\[
\mathbb{E}\left[(\nu_\varepsilon - \mathcal{W})^2\right] = \sum_{k=0}^{\infty} \int_{\mathcal{H}_k \times \mathbb{R}^k} (f_{k, \varepsilon} - \tilde{f}_k)^2 ds dy,
\]

we see that, for each \( k \geq 0 \), \( f_{k, \varepsilon} \to \tilde{f}_k \) in \( L^2(\mathcal{H}_k \times \mathbb{R}^k) \). Fix arbitrary \( g \in C_c(\mathbb{R}^{2n}) \), we consider

\[
\langle f_{k, \varepsilon}, g \rangle := \int_{\mathcal{H}_k \times \mathbb{R}^k} f_{k, \varepsilon}(r_1, \ldots, r_k, y_1, \ldots, y_k) g(r_1, \ldots, r_k, y_1, \ldots, y_k) dr dy.
\]

To prove \( \tilde{f}_k = f_k \), it suffices to show \( \langle f_{k, \varepsilon}, g \rangle \to \langle f_k, g \rangle \) as \( \varepsilon \to 0 \).
The formula (3.5) yields
\[ \langle f_{k,\varepsilon}, g \rangle = \int_{\mathbb{R}^{2k} \times \mathbb{R}^k} E_B \left[ u_0(x + B(t)) e^{\int_0^t R_c(s-u, B(s)-B(u))duds - ct} \right. \]
\[ \left. \left( \int_{[0,t]^k} \prod_{i=1}^k \phi_c(t - s_i - r_i, x + B(s_i) - y_i) ds_i \right) g(r_1, \ldots, r_k, y_1, \ldots, y_k) dr dy. \right] \tag{3.8} \]

With \( \phi_c(\cdot, \cdot) := \varepsilon^{-3} \phi(\varepsilon^{-2} \cdot, \varepsilon^{-1} \cdot) \), we perform a change of variables \( r_i \mapsto \varepsilon^2 r'_i + t - s_i, y_i \mapsto \varepsilon y'_i + x + B(s_i) \) to rewrite the last expression as
\[ \langle f_{k,\varepsilon}, g \rangle = \int_{\mathbb{R}^{3k}} 1_{A_c \cap A'_c}(r', s) \prod_{i=1}^k \phi(-r'_i, -y'_i) E_B \left[ u_0(x + B(t)) e^{\int_0^t R_c(s-u, B(s)-B(u))duds - ct} \right. \]
\[ \left. \times g(\varepsilon^2 r'_1 + t - s_1, \ldots, \varepsilon^2 r'_k + t - s_k, \varepsilon y'_1 + x + B(s_1), \ldots, \varepsilon y'_k + x + B(s_k)) \right] dr' dy' ds, \]
where \( A_c := \{ \varepsilon^2 r'_1 + t - s_1 < \ldots < \varepsilon^2 r'_k + t - s_k \} \) and \( A'_c := \{ (s_1, \ldots, s_k) \in [0, t]^k \} \) translate the constraints on the old variables into the new ones. In order to pass to the limit, we note that, by Proposition 2.3, for any fixed \((s_1, \ldots, s_k) \in [0, t]^k\),
\[ \left( \int_0^t \int_0^s R_c(s-u, B(s)-B(u))duds - ct, B(s_1), \ldots, B(s_k), B(t) \right) \Rightarrow \left( \sigma, W_t - \frac{1}{2} \sigma^2 t, B(s_1), \ldots, B(s_k), B(t) \right) \]

In addition, for any fixed \((r'_1, \ldots, r'_k) \in \mathbb{R}^k, 1_{A_c \cap A'_c}(r', s) \to 1_{0 < s_1 < \ldots < s_k < t}. \) By applying Proposition 2.1 and the dominated convergence theorem, we arrive at
\[ \langle f_{k,\varepsilon}, g \rangle \to \int_{\mathbb{R}^{3k}} 1_{0 < s_1 < \ldots < s_k < t} \prod_{i=1}^k \phi(-r'_i, -y'_i) \]
\[ \times E_B \left[ u_0(x + B(t)) e^{\sigma W_t - \frac{1}{2} \sigma^2 t} g(t - s_1, \ldots, t - s_k, x + B(s_1), \ldots, x + B(s_k)) \right] dr' dy' ds. \]

In the last expression, integrate over \((r_i, y_i)\) using \( \int \phi dr dy = 1 \), and perform a change of variables \( t - s_i \mapsto s_i \).
We see that it equals
\[ \int_{\mathbb{R}^{2k} \cap [0,t]^k} E_B \left[ u_0(x + B(t)) g(s_1, \ldots, s_k, x + B(t - s_1), \ldots, x + B(t - s_k)) \right] ds = \langle f_k, g \rangle. \]

The proof is complete. \( \square \)

**References**


(Yu Gu) DEPARTMENT OF MATHEMATICS, CARNEGIE MELLON UNIVERSITY, PITTSBURGH, PA 15213

(Li-Cheng Tsai) DEPARTMENTS OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY, NEW YORK, NY 10027