CONTINUUM MECHANICS
Notes by Walter Noll (1991)

1 BODIES AND INTERACTIONS

1.11 Material Systems.

By a material system we mean a set Ω, endowed with a mathematical structure by the prescription of a relation ≺ in Ω. The elements of Ω are to be interpreted as the subbodies of a physical body under investigation and the statement \( \mathcal{P} ≺ \mathcal{Q} \), for any given \( \mathcal{P}, \mathcal{Q} ∈ \Omega \), should be interpreted to mean that \( \mathcal{P} \) is a part of \( \mathcal{Q} \).

Remark 1: The use of the terms “body” and “part” may be somewhat misleading for the intended applications. These terms have the connotation of connectedness. This connotation should be disregarded. Thus, we do not want to exclude the possibility that a body or a part thereof has several pieces that are not in any way connected.

It is assumed that the structured set Ω satisfies the axioms (M1)–(M6) below. These axioms merely reflect our common sense concerning bodies and their parts.

(M1) ≺ is transitive, i.e.,

\[
(P ≺ Q \text{ and } Q ≺ R) \implies P ≺ R
\]

for all \( P, Q, R ∈ \Omega \).

(M2) ≺ is antisymmetric and reflexive, i.e.,

\[
(P ≺ Q \text{ and } Q ≺ P) ⇔ P = Q
\]

for all \( P, Q ∈ \Omega \).

These first two axioms state that ≺ is a (partial) order on Ω. Hence all the concepts and results of the theory of ordered sets apply. In particular, a subset of Ω may or may not have a supremum or infimum. If \( \mathcal{P}, \mathcal{Q} ∈ \Omega \), then the supremum [infimum] of \( \{\mathcal{P}, \mathcal{Q}\} \), if it exists, will be denoted by \( \mathcal{P} \lor \mathcal{Q} [\mathcal{P} \land \mathcal{Q}] \) and called the join [meet] of \( \mathcal{P} \) and \( \mathcal{Q} \).

(M3) Ω has a maximum, denoted by \( \mathcal{B} \), and a minimum, denoted by \( \emptyset \), so that

\[\emptyset ≺ \mathcal{P} ≺ \mathcal{B} \text{ for all } \mathcal{P} ∈ \Omega.\]
The element $B$ of $\Omega$ represents the whole body being investigated. The element $\emptyset$ of $\Omega$ has no physical interpretation. It is useful to have it available in order to simplify the mathematics. We call $\emptyset$ the material nothing.

If $P, Q$ have only $\emptyset$ in common, i.e., if $P \land Q = \emptyset$, we say that $P$ and $Q$ are separate subbodies. We denote the set of all separate pairs of subbodies by

$$(\Omega \times \Omega)_{\text{sep}} := \{(P, Q) \in \Omega \times \Omega \mid P \land Q = \emptyset\}$$

(M4) For every $P \in \Omega$ there is exactly one $Q \in \Omega$ such that $Q \land P = \emptyset$ and $Q \lor P = B$. We denote this $Q$ by $P^b := Q$ and call it the exterior in $B$ of $P$.

The subbody $P^b$ is to be interpreted as the remainder after the subbody $P$ has been removed from $B$.

(M5) For every $P, Q \in \Omega$, if $P \land Q^b = \emptyset$ then $P \prec Q$.

Axiom (M5) is immediately justified by common sense.

(M6) Any subbodies $P, Q \in \Omega$ have a meet $P \land Q$.

We now assume that a material system $\Omega$ is given. The following results are easy consequence of the axioms (M1)–(M6):

**Proposition 1:** Any subbodies $P, Q \in \Omega$ have a join $P \lor Q$, and this join is given by

$$P \lor Q = (P^b \land Q^b)^b.$$  \hspace{1cm} (2)

**Proposition 2:** For every $P \in \Omega$, we have

$$P \land P^b = \emptyset, \quad P \lor P^b = B, \quad \text{and} \quad (P^b)^b = P.$$  \hspace{1cm} (3)

**Proposition 3:** Let $(P, Q) \in (\Omega \times \Omega)_{\text{sep}}$ be given and put $R := P \lor Q$. Then

$$R^b \land P = R^b \land Q = \emptyset$$  \hspace{1cm} (4)

and

$$P^b = R^b \lor Q, \quad Q^b = R^b \lor P.$$  \hspace{1cm} (5)

**Remark 2:** One can prove from (M1)–(M6) that the operations $\land$ and $\lor$ endow $B$ with the structure of a Boolean algebra. Thus we could have simply assumed
that a material system $\Omega$ is a Boolean algebra. However, the relation $\prec$ is closer to physical intuition that the operations $\land$ and $\lor$ and the axioms (M1)–(M6) are more easily justified by common sense than the axioms of a Boolean algebra. Moreover, there may be circumstances in which one might want to replace (M6) by a weaker axiom. □

For any given subbody $\mathcal{P} \in \Omega$, we denote the set of a parts of $\mathcal{P}$ by

$$\Omega_{\mathcal{P}} := \{ Q \in \Omega \mid Q \prec \mathcal{P} \}.$$ 

It is easily seen that $\Omega_{\mathcal{P}}$ has itself the natural structure of a material universe. The “is a part of”-relation in $\Omega_{\mathcal{P}}$ is simply the restriction of $\prec$ to $\Omega_{\mathcal{P}}$. The whole body of $\Omega_{\mathcal{P}}$ (axiom (M3)) is $\mathcal{P}$ rather than $\mathcal{B}$ and the exterior in $\mathcal{P}$ (axiom (M4)) of a subbody $\mathcal{R} \in \Omega_{\mathcal{P}}$ is $\mathcal{R}^b \land \mathcal{P}$ rather than $\mathcal{R}^b$. The set $\Omega_{\mathcal{P}}$ is also the set of all subbodies that are separate from $\mathcal{P}^b$, i.e.,

$$\Omega_{\mathcal{P}} = \left\{ \mathcal{R} \in \Omega \mid (\mathcal{R}, \mathcal{P}^b) \in (\Omega \times \Omega)_{\text{sep}} \right\}. \quad (6)$$

Examples:

(1) Let a set $\mathcal{B}$ be given and put $\Omega := \text{Sub}\mathcal{B}$ and $\prec := \subset$ (set inclusion). Then the axioms (M1)–(M6) are all satisfied. The material nothing is the empty set, the whole body is $\mathcal{B}$, and for every $\mathcal{P} \in \Omega$ we have $\mathcal{P}^b = \mathcal{B}\setminus\mathcal{P}$. For all $\mathcal{P}, \mathcal{Q} \in \Omega$, we have $\mathcal{P} \land \mathcal{Q} = \mathcal{P} \cap \mathcal{Q}$ and $\mathcal{P} \lor \mathcal{Q} = \mathcal{P} \cup \mathcal{Q}$. If the set $\mathcal{B}$ is finite, then the material system just described is appropriate for classical particle physics.

(2) Let a topological space $\mathcal{B}$ be given and put

$$\Omega := \{ \mathcal{P} \in \text{Sub} \mathcal{B} \mid \mathcal{B} = \text{Int} \text{ Clo} \mathcal{B} \}. \quad (7)$$

(The members of $\Omega$ are called the regularly open subsets of $\mathcal{B}$.) Again, the axioms (M1)–(M6) are all satisfied, the material nothing is the empty set and the whole body is $\mathcal{B}$. For every $\mathcal{P} \in \Omega$, we have

$$\mathcal{P}^b = \text{Int}(\mathcal{B}\setminus\mathcal{P}). \quad (8)$$

For all $\mathcal{P}, \mathcal{Q} \in \Omega$, we have

$$\mathcal{P} \land \mathcal{Q} = \mathcal{P} \cap \mathcal{Q} \quad (9)$$

$$\mathcal{P} \lor \mathcal{Q} = \text{Int} \text{ Clo}(\mathcal{P} \cup \mathcal{Q}). \quad (10)$$
Although there seems to be no physical situation for which the material system just described is appropriate, the system has some features in common with the system appropriate for continuum mechanics to be described in Section 1.4.

\[ \square \]

### 1.12 Interactions

In this section, we assume that a material system \( \Omega \) and a linear space \( \mathcal{U} \) are given. We say that a mapping \( F : \Omega \rightarrow \mathcal{U} \) is additive if

\[
F(\mathcal{P} \vee \mathcal{Q}) = F(\mathcal{P}) + F(\mathcal{Q}) \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega \times \Omega)_{\text{sep}}.
\]  

(11)

A mapping \( I : (\Omega \times \Omega)_{\text{sep}} \rightarrow \mathcal{U} \) is called an interaction in \( \Omega \) if, for every \( \mathcal{P} \in \Omega \), the mappings

\[
I(\cdot, \mathcal{P}^c) : \Omega \rightarrow \mathcal{U},
\]

and

\[
I(\mathcal{P}^c, \cdot) : \Omega \rightarrow \mathcal{U}
\]

are both additive.

We say that the given interaction \( I : (\Omega \times \Omega)_{\text{sep}} \rightarrow \mathcal{U} \) is balanced if

\[
I(\mathcal{P}, \mathcal{P}^b) = 0 \quad \text{for all } \mathcal{P} \in \Omega
\]  

(12)

we say that \( I \) is skew if

\[
I(\mathcal{Q}, \mathcal{P}) = -I(\mathcal{P}, \mathcal{Q}) \quad \text{for all } (\mathcal{P}, \mathcal{Q}) \in (\Omega \times \Omega)_{\text{sep}}.
\]  

(13)

(The assertion that \( I \) is skew is often expressed by saying that \( I \) satisfies the “law of action and reaction”.)

**Theorem:** An interaction \( I : (\Omega \times \Omega)_{\text{sep}} \rightarrow \mathcal{U} \) is skew if and only if there is an additive mapping \( F : \Omega \rightarrow \mathcal{U} \) such that

\[
I(\mathcal{P}, \mathcal{P}^b) = F(\mathcal{P}) \quad \text{for all } \mathcal{P} \in \Omega.
\]  

(14)

The theorem is an immediate consequence of the following:

**Lemma:** For every interaction \( I \) and every pair \((\mathcal{P}, \mathcal{Q}) \in (\Omega \times \Omega)_{\text{sep}}, \) we have
\[ I(P, Q) + I(Q, P) = I(P, P^b) + I(Q, Q^b) - I(P \lor Q, (P \lor Q)^b). \quad (15) \]

**Proof:** Put \( R := P \lor Q \). By Prop. 3 of Section 11 we then have \((R^b, P), (R^b, Q) \in (\Omega \times \Omega)_{sep}\) and \( Q^b = R^b \lor P, P^b = R^b \lor Q \). The additivity of \( I(P, \cdot) \) and \( I(Q, \cdot) \) gives

\[ I(P, P^b) + I(Q, Q^b) = I(P, Q) + I(P, R^b) + I(Q, P) + I(Q, R^b). \]

Using the additivity of \( I(\cdot, R^b) \) we get the desired result \((14)\). \( \square \)

**Corollary:** Every balanced interaction is skew.

### 1.13 Fit Regions and Transplants

Before giving a precise mathematical definition of a “continuous body”, we will first specify two classes: (i) a class \( Fr \) consisting of subsets of Euclidean spaces, subsets which are candidates for regions occupied by a continuous body when placed in a frame of reference, (ii) a class \( Tp \) of mappings which are candidates for the changes of placement of a body in a given frame of reference or from one frame to another.

We use the term “Euclidean space” to mean what is called a “Genuine Euclidean space” in Sect. 46 of [FDSI].

We say that a subset \( D \) of a given Euclidean space \( E \) is a **fit region** in \( E \) if

\( F_1 \) \( D \) is a bounded subset of \( E \),

\( F_2 \) \( D \) is regularly open in the sense that \( D = \text{Int} \text{ Clo} \ D \),

\( F_3 \) \( D \) has a negligible boundary, i.e., \( Bry D \) is a negligible set as defined in Sect. 51 of [FDSII],

\( F_4 \) \( D \) has finite perimeter as defined, for example, in Sect. 4 of [NV].

Let a Euclidean space \( E \) be given. We denote the set of all fit regions in \( E \) by \( \text{Fr}(E) \). If a subset \( \mathcal{J} \) of \( E \) is given, we use the notation

\[ \text{Fr}(\mathcal{J}) := \{ D \in \text{Fr}(E) \mid D \subset \mathcal{J} \} \quad (16) \]

for the set of all fit regions included in \( \mathcal{J} \). The class of all fit regions, in all conceivable Euclidean spaces, is denoted by \( \text{Fr} \).

**Proposition 1:** Let a Euclidean space \( E \) and \( C, D \in \text{Fr}(E) \) be given. Then

\[ C \cap D \in \text{Fr}(E) \quad (17) \]
\[ C \lor D := \text{Int Clo } (C \cap D) \in \text{Fr}(E) \tag{18} \]

and

\[ C \Delta D := \text{Int}(C \setminus D) \in \text{Fr}(E). \tag{19} \]

We call \( C \lor D \) the \textbf{join} of \( C \) and \( D \) (see also 11.7) and \( C \Delta D \) the \textbf{difference-region} of \( C \) and \( D \). In fact, \( C \cap D \) is the largest fit region that is included in both \( C \) and \( D \), \( C \lor D \) is the smallest fit region in \( E \) that includes both \( C \) and \( D \), and \( C \) and \( C \Delta D \) is the largest fit region included in \( C \) and disjoint from \( D \).

The proof of Proposition 1, which is quite difficult, is given in [NV] (Theorem 4 of Section 5).

We say that a mapping \( \lambda \) is a \textbf{transplacement} if it satisfies the following requirements:

1. \( \lambda \) is an invertible mapping whose domain \( \text{Dom} \lambda \) and range \( \text{Rng} \lambda \) are subsets of Euclidean spaces \( \text{Dsp} \lambda \) and \( \text{Rsp} \lambda \), respectively. We call \( \text{Dsp} \lambda \) the \textbf{domain-space} and \( \text{Rsp} \lambda \) the \textbf{range-space} of \( \lambda \).

2. We have \( \text{Dom} \lambda \in \text{Fr}(\text{Dsp} \lambda) \).

3. There is a \( C^2 \)-diffeomorphism \( \varphi : \text{Dsp} \lambda \to \text{Rsp} \lambda \) such that \( \lambda = \gamma \big|_{\text{Rng} \lambda}^{\text{Dom} \lambda} \).

We denote the class of all transplacements by \( \mathcal{T}_p \). Given two Euclidean spaces \( E \) and \( E' \), we use the notation \( \mathcal{T}_p(E, E') \) for the set of all transplacements with domain-space \( E \) and range-space \( E' \), so that

\[ \mathcal{T}_p(E, E') = \{ \lambda \in \mathcal{T}_p \mid \text{Dom} \lambda \subset E, \text{Rng} \lambda \subset E' \}. \tag{20} \]

**Proposition 2:** The range of every transplacement is a fit region; i.e., \( \text{Rng} \lambda \in \text{Fr} \) for every \( \lambda \in \mathcal{T}_p \).

The proof of Prop. 2, which is very difficult, is given in [NV] (Theorem 5 of Sect. 5). The proofs of the following three results are now very easy.

**Proposition 3:** The composite of two transplacements is again a transplacement, i.e., for all \( \lambda, \mu \in \mathcal{T}_p \) with \( \text{Rng} \lambda = \text{Dom} \mu \) we have \( \mu \circ \lambda \in \mathcal{T}_p \).

**Proposition 4:** The inverse of every transplacement is again a transplacement, i.e., for all \( \lambda \in \mathcal{T}_p \) we have \( \lambda^{-1} \in \mathcal{T}_p \).

**Proposition 5:** For every \( \lambda \in \mathcal{T}_p \) and every \( D \in \text{Fr}(\text{Dom} \lambda) \) we have \( \lambda \big|_{\lambda^{-1}(D)} \in \mathcal{T}_p \).
Remark 1: Strictly speaking, $T_p$ is the class of morphisms of a category whose objects are pairs $(D, E)$ where $E$ is a Euclidean space and $D \in Fr(E)$.

The proof of the following result is not very hard.

Proposition 6: Let Euclidean spaces $E$ and $E'$, with distance functions $d$ and $d'$, respectively, be given. For every transplacement $\lambda \in T_p(E, E')$ there exist $k, k' \in \mathbb{P}$ such that

$$k d(x, y) \leq d(\lambda(x), \lambda(y)) \leq k'd(x, y)$$

for all $x, y \in \text{Dom}\lambda$.

Remark 2: For certain purposes, for example when dealing with bodies subject to constraints, one might wish to modify the definitions of $T_p$ or of $Fr$, or both.

1.14 Continuous Bodies

Definition 1: continuous body $B$ is a non-empty set endowed with structure by the specification of a non-empty class $Pl(B)$ satisfying the following requirements.

- $B_1$ Each $\kappa \in Pl(B)$ is an invertible mapping with $\text{Dom}\kappa = B$ and $\text{Rng}\kappa \in Fr$.
- $B_2$ For all $\kappa, \gamma \in Pl(B)$ we have $\kappa \circ \gamma^{-1} \in T_p$.
- $B_3$ For every $\kappa \in Pl(B)$ and $\lambda \in T_p$ such that $\text{Rng}\kappa = \text{Dom}\lambda$, we have $\lambda \circ \kappa \in Pl(B)$.

The elements of $B$ will be called material points. The members of the class $Pl(B)$ will be called the placements of $(B)$. Given a placement $\kappa \in Pl(B)$ we denote the Euclidean space in which $\text{Rng}\kappa$ is a fit region by $\text{Frm}\kappa$ and call it the frame-space of $\kappa$, so that $\text{Rng}\kappa \in Fr(\text{Frm}\kappa)$. The translation-space of $\text{Frm}\kappa$ will be denoted by $V_{\text{rm}}\kappa$ and the members of $V_{\text{rm}}\kappa$ will be called frame-vectors for $\kappa$. Given $\kappa \in Pl(B)$ and $X \in B$, we call $\kappa(X) \in \text{Frm}\kappa$ the place of the material point $X$ in the placement $\kappa$; the set of all these places, i.e., $\text{Rng}\kappa$, is also called the region occupied by the body in the placement $\kappa$.

The axiom ($B_1$) states, roughly, that continuous bodies can only occupy fit regions. Axiom ($B_2$) states that the transition from one placement to another must be a transplacement in the sense of Sect. 13. Axiom ($B_3$) states that a transplacement from a given placement always gives a new placement.

We now assume that a continuous body with placement-class $Pl(B)$ is given. Given $\kappa \in Pl(B)$, it follows from ($B_2$) and ($B_3$) that

$$Pl(B) = \{\lambda \circ \kappa \mid \lambda \in T_p, \text{Dom}\lambda = \text{Rng}\kappa\}.$$
The subsets of $\mathcal{B}$ belonging to
\[ \Omega_B := \{ P \in \text{Sub}_B \mid \kappa > (P) \in \text{Fr} \text{ for some } \kappa \in \text{Pl}(\mathcal{B}) \} \tag{23} \]
are called the parts of $\mathcal{B}$.

It is an immediate consequence of (B$_2$) and Prop. 2 of Sect. 13 that
\[ \Omega_B = \{ P \in \text{Sub}_B \mid \kappa > (P) \in \text{Fr}(Rng\kappa) \text{ for all } \kappa \in \text{Pl}(\mathcal{B}) \} . \tag{24} \]

The non-empty parts of $\mathcal{B}$ are also called subbodies, which is justified by the following fact.

**Theorem 1:** Every part $P \in \Omega_B$ acquires the natural structure of a continuous body by the specification
\[ \text{Pl}(P) := \left\{ \kappa \mid \kappa \in \text{Pl}(\mathcal{B}) \right\} \tag{25} \]
for the placement-class of $P$.

**Proof:** The fact that Pl($P$) satisfies (B$_1$) follows directly from (25). Given $\kappa, \gamma \in \text{Pl}(\mathcal{B})$ we have
\[ \gamma\mid_P \circ \left( \kappa \mid_P^{(P)} \right) \stackrel{\text{tr}}{=} \left( \gamma \circ \kappa^{-} \right) \mid_{\kappa>(P)} \].

Hence, since $\gamma \circ \kappa^{-} \in \text{Tp}$ because Pl($\mathcal{B}$) satisfies (B$_2$), it follows from Prop. 5 of Sect. 13, with $D := \kappa>(P)$ and $\lambda := \gamma \circ \kappa^{-}$, that Pl($P$) also satisfies (B$_2$). Now let $\kappa \in \text{Pl}(P)$ and $\lambda \in \text{Tp}$ be given such that $Rng\left( \kappa \mid_P^{(P)} \right) = \kappa>(P) = \text{Dom}\lambda$. By (T$_3$) we may choose $\varphi : \text{Dsp}\lambda \rightarrow \text{Rsp}\lambda$ such that $\lambda = \varphi\mid_{Rng\kappa}$. Putting $\lambda := \varphi\mid_{Rng\kappa}$, we have $\lambda \in \text{Tp}$ and hence, since Pl($\mathcal{B}$) satisfies (B$_3$), $\lambda \in \text{Pl}(\mathcal{B})$. Therefore, $\left( \lambda \circ \kappa \right) \mid_{\kappa>(P)} = \lambda \circ \left( \kappa \mid_{\kappa>(P)} \right) \in \text{Pl}(P)$ satisfies (B$_3$). \hfill \Box

The following theorem shows that $\Omega_B$ satisfies the axioms for a material system as given in Sect. 11.

**Theorem 2:** The collection $\Omega_B$, when ordered by inclusion, has the following properties:

(i) The intersection of any two parts of $\mathcal{B}$ is a part of $\mathcal{B}$, i.e., for all $P, Q \in \Omega_B$ we have $P \cap Q \in \Omega_B$.

(ii) For any two parts $P, Q \in \Omega_B$ there is a smallest part of $\mathcal{B}$ that includes both. This smallest member of $\Omega_B$ that includes both $P$ and $Q$ is called the join of $P$ and $Q$ and is denoted by $P \lor Q$. 

8
(iii) For every part of $P \in \Omega_B$, there is a largest part of $B$ that is disjoint from $P$. This largest member of $\Omega_B$ that is disjoint from $P$ is called the \textit{exterior} of $P$ in $B$ and is denoted by $P^b$, so that

$$P \cap P^b = \emptyset$$

(26)

and

$$(P \cup Q = \emptyset \Rightarrow Q \supset P^b) \text{ for all } Q \in \Omega_B.$$  

(27)

For every $\kappa \in \text{Pl}(B)$ and all $P, Q \in \Omega_B$, we have

$$\kappa > (P \cap Q) = \kappa > (P) \cap \kappa > (Q),$$

(28)

$$\kappa > (P \lor Q) = \kappa > (P) \lor \kappa > (Q) = \text{Int Clo}(\kappa > (P) \cup \kappa > (Q)),$$

(29)

$$\kappa > (P^b) = \text{Rng} \kappa \upharpoonright \kappa > (P) = \text{Int}(\text{Rng} \kappa \setminus \kappa > (P)).$$

(30)

\textbf{Proof:} Let $\kappa \in \text{Pl}(B)$ be given. It follows from (24) that $(\kappa >)_{\text{FrRng} \kappa} = \{\kappa > (P) \mid P \in \Omega_B\}$ is the set Fr(Rng $\kappa$) of all fit regions in Rng $\kappa$ defined according to (16). Hence, since $\kappa$ is invertible, $\kappa >_{\text{FrRng} \kappa}$ is an order-isomorphism from $\Omega_B$ to Fr(Rng $\kappa$), the order being inclusion. Now, it follows from Prop. 1 of Sect 13 that Fr(Rng $\kappa$) has properties analogous to (i), (ii), and (iii). Hence $\Omega_B$ has these properties and (28)–(30) hold. $\square$