Chapter 5
Geometric Structures.

We assume in this chapter that numbers \( r, s \in \mathbb{R} \), with \( r \geq 3 \) and \( s \in 0..r \), a \( C^r \) manifold \( \mathcal{M} \) and a \( C^s \) linear-space bundle \( \mathcal{B} \) over the manifold \( \mathcal{M} \) are given. We also assume that both \( \mathcal{M} \) and \( \mathcal{B} \) have constant dimensions, and put \( n := \dim \mathcal{M} \) and \( m := \dim \mathcal{B} - \dim \mathcal{M} \). Then we have \( n = \dim T_x \mathcal{M} \) and \( m = \dim B_x \) for all \( x \in \mathcal{M} \).

51. Compatible Connections

Let \( x \in \mathcal{M} \) be fixed. Let \( \Phi \) be an analytic tensor functor and let \( E \in \Phi(\mathcal{B}_x) \) be given.

**Notation:** We define the mapping

\[
\begin{align*}
E^\circ : T\text{lis}_x \mathcal{B} &\to \Phi(\mathcal{B}) \\
E^\circ(T) := \Phi(T)E &\quad \text{for all } T \in T\text{lis}_x \mathcal{B}.
\end{align*}
\]

Since \( \Phi \) is analytic, it is clear that \( E^\circ \) is differentiable at \( 1_{\mathcal{B}_x} \).

**Proposition 1:** We have \( \nabla_{1_{\mathcal{B}_x}} E^\circ \in \text{Lin}(S_x \mathcal{B}, T_x \Phi(\mathcal{B})) \) and, for every bundle chart \( \phi \in \text{Ch}_x(\mathcal{B}, \mathcal{M}) \),

\[
(\nabla_{1_{\mathcal{B}_x}} E^\circ) s = A^{\phi}_x P_s + I_x \Phi_x^\ast(\Lambda(A^{\phi}_x)s)E
\]

for all \( s \in S_x \mathcal{B} \). For the sake of simplicity, we use the following notation

\[
E^\circ := \nabla_{1_{\mathcal{B}_x}} \left( E^\circ|_{\text{Lis}\mathcal{B}_x} \right).
\]

**Proof:** By using (51.2) and the definition (23.21) of gradient, we obtain the desired result.

Taking the gradient of \( E^\circ|_{\text{Lis}\mathcal{B}_x} \) at \( 1_{\mathcal{B}_x} \), we have

\[
(\nabla_{1_{\mathcal{B}_x}} E^\circ|_{\text{Lis}\mathcal{B}_x}) L = (\Phi_x^\ast(L))E
\]

for all \( L \in \text{Lin}\mathcal{B}_x \). For the sake of simplicity, we use the following notation

\[
E^\circ := \nabla_{1_{\mathcal{B}_x}} \left( E^\circ|_{\text{Lis}\mathcal{B}_x} \right).
\]
Given \( r \in \{0\} \), we observe from (51.5) that \((rE)^o = rE^o\) and hence

\[
\text{Null } E^o = \text{Null } (rE)^o.
\]

(51.6)

It is follows from (51.3) and (51.4) that

\[
P_x = P_E(\nabla_{1_{B^x}} E^o) \quad \text{and} \quad (\nabla_{1_{B^x}} E^o) I_x = I_x E^o,
\]

i.e. the diagram

\[
\begin{array}{cccc}
\text{Lin } B_x & \overset{I_x}{\to} & S_x B & \overset{P_x}{\to} & T_x M \\
\downarrow & & \downarrow \nabla_{1_{B^x}} E^o & & \downarrow \\
\Phi(B_x) & \overset{I_E}{\to} & T_E \Phi(B) & \overset{P_E}{\to} & T_x M
\end{array}
\]

commutes. And it also clear from (51.3) that

\[
A_{E^o}^{\Phi(x)} = (\nabla_{1_{B^x}} E^o) A^\phi_x \in R_{\text{con}} E \Phi(B)
\]

(51.8)

for all bundle chart \( \phi \in \text{Ch}(\mathcal{B}, \mathcal{M}) \). More generally, we have

\[
(\nabla_{1_{B^x}} E^o) K \in R_{\text{con}} E \Phi(B) \quad \text{for all } K \in \text{Con}_x \mathcal{B}.
\]

(51.9)

In view of (51.9), the mapping \( \nabla_{1_{B^x}} E^o \) induces the following mapping.

\textbf{Definition:} \textit{We define the mapping}

\[
J_E : \text{Con}_x \mathcal{B} \to R_{\text{con}} E \Phi(B)
\]

\textit{by}

\[
J_E(K) := (\nabla_{1_{B^x}} E^o) K \quad \text{for all } K \in \text{Con}_x \mathcal{B}.
\]

(51.10)

\textbf{Proposition 2:} \textit{The mapping } \( J_E \), \textit{defined in (51.10), is flat. Hence, for every } \( D \in \text{Rng } J_E \), \textit{we have } \( J_E \leq (\{D\}) \) \textit{is a flat in } \text{Con}_x \mathcal{B} \textit{with}

\[
\dim J_E \leq (\{D\}) = ????.
\]

Let a cross section \( H : \mathcal{M} \to \Phi(B) \), that is differentiable at \( x \in \mathcal{M} \), be given. The gradient of \( H \) at \( x \) is a tangent connector of \( \Phi(B) \); i.e. \( \nabla_x H \in R_{\text{con}_{\Phi(B)}} \Phi(B) \).
Proposition 3: We have
\[ \nabla_K H = \Lambda (\nabla_{1_{B_x}} H^{\circ}) K \nabla H \] (51.11)
for all \( K \in \text{Con}_x \mathcal{B} \) and hence \( \nabla_K H = 0 \) if and only if \( J_{H(x)}(K) = \nabla_x H \), i.e. \( K \in J^{\less}_H (\{ \nabla_x H \}) \).

Proof: The desired result (51.11) follows from (51.8), (41.11), (42.1) and Remark 1 of Sect. 32.

If \( K \in \text{Con}_x \mathcal{B} \) be such that \( \nabla_K H = 0 \), then it follows from (51.11) that \( \Lambda (\nabla_{1_{B_x}} H^{\circ}) K \nabla H = 0 \). Applying Prop.1 of Sect.14, we see that this can happen if and only if \( (\nabla_{1_{B_x}} H^{\circ}) K = \nabla_x H \). Since \( K \in \text{Con}_x \mathcal{B} \) was arbitrary, the assertion follows.

Now, let a differentiable cross section \( H : \mathcal{M} \to \Phi(\mathcal{B}) \) be given.

Definition: A connection \( C : \mathcal{M} \to \text{Con}\mathcal{B} \) is called a \( H \)-compatible connection if \( \nabla_C H = 0 \) for all \( x \in \mathcal{M} \), i.e.
\[ \nabla_C H = 0. \] (51.12)

It clear from Prop.3 that a connection \( C \) is \( H \)-compatible if and only if
\[ J_{H(x)}(C(x)) = \nabla_x H \quad \text{for all} \quad x \in \mathcal{M}. \] (51.13)

Proposition 4: Let connectors \( K_1, K_2 \in J^{\less}_H (\{ \nabla_x H \}) \) be given and determine \( L \in \text{Lin} (T_x \mathcal{M}, \text{Lin} \mathcal{B}_x) \) such that \( K_1 - K_2 = I_x L \); then we have
\[ H(x)^{\circ} (Lt) = 0 \quad \text{for all} \quad t \in T_x \mathcal{M}. \] (51.14)
52. Riemannian and Symplectic Bundles

We apply Sect. 51 to the case when $\Phi = \text{Smf}_2$ or $\text{Skf}_2$ (see example (4) of Sect. 13).

Let $x \in \mathcal{M}$ be fixed and $E \in \Phi(\mathcal{B}_x)$, $\Phi = \text{Smf}_2$ or $\text{Skf}_2$, be given. We have

$$E^\circ(M) = E \circ (M \times 1_{\mathcal{B}_x}) + E \circ (1_{\mathcal{B}_x} \times M),$$

(52.1)

where $E^\circ$ is given in (51.5), for every $M \in \text{Lin}\mathcal{B}_x$.

**Proposition 1:** If $E$ is invertiable, then $E^\circ$ is surjective; i.e.

$$\text{Rng } E^\circ = \text{Sym}_2(\mathcal{B}_x^2),$$

(52.2)

when $\Phi = \text{Smf}_2$

i.e., $E \in \text{Sym}_2(\mathcal{B}_x^2)$ and

$$\text{Rng } E^\circ = \text{Skw}_2(\mathcal{B}_x^2),$$

(52.3)

when $\Phi = \text{Skf}_2$

i.e., $E \in \text{Skw}_2(\mathcal{B}_x^2)$.

**Proof:** By using (52.1).

**Proposition 2:** If $E$ is invertiable, then the flat mapping $J_E$ defined in (51.10) is surjective.

**Proof:** The surjectivity follows directly from (51.3), (51.4), (51.5) and the surjectivity of $E^\circ$.

In view of Prop. 2 we see that, for every $D \in \text{Rcon}_E \Phi(\mathcal{B})$, the preimage $J_E(\{D\})$ is a flat in $\text{Con}_x \mathcal{B}$. Let $K_1, K_2 \in J_E(\{D\})$ be given and determine $L \in \text{Lin}(T_x \mathcal{M}, \text{Lin}\mathcal{B}_x)$ such that $K_2 - K_1 = I_x L$. Applying (51.3), we have $0 = J_E(K_2) - J_E(K_1) = E^\circ(L)$, that is $L \in \text{Lin}(T_x \mathcal{M}, \text{Null } E^\circ)$. Since $K_1, K_2 \in J_E(\{D\})$ were arbitrary, we conclude that

$$\dim J_E(\{D\}) = \dim \text{Lin}(T_x \mathcal{M}, \text{Null } E^\circ).$$

(52.4)

**Definition:** A cross section $G : \mathcal{M} \to \text{Smf}_2(\mathcal{B})$ is called a Riemannian field if, for every $x \in \mathcal{M}$, $G(x)$ is invertiable when regard as element of $\text{Sym}(\mathcal{B}_x, \mathcal{B}_x^*)$.

A cross section $S : \mathcal{M} \to \text{Skf}_2(\mathcal{B})$ is called a symplectic field of $\mathcal{B}$ if, for every $x \in \mathcal{M}$, $S(x)$ is invertiable when regard as element of $\text{Skw}(\mathcal{B}_x, \mathcal{B}_x^*)$.

We say that $\mathcal{B}$ is a $C^\ast$ Riemannian linear space bundle if it is endowed with additional structure by the prescription of a $C^\ast$ Riemannian field.

We say that $\mathcal{B}$ is a $C^\ast$ symplectic linear space bundle if it is endowed with additional structure by the prescription of a $C^\ast$ symplectic field.
**Remark 1:** A symplectic field of \( B \) exist if and only if, for every \( x \in M \), \( m := \dim B_x \) is even (see Sect.11). If \( m \) is odd, then

\[
\text{Skw}(B_x, B_x^*) \cap \text{Lis}(B_x, B_x^*) = \emptyset.
\]

**Proposition 3:** If \( G : M \to \text{Smf}_2(B) \) is a Riemannian field, then

\[
\dim J^<_{G(x)}(\{\nabla G\}) = n\left(\frac{m}{2}\right) \quad \text{for all} \quad x \in M.
\]

If \( S : M \to \text{Skf}_2(B) \) is a symplectic field, then

\[
\dim J^<_{S(x)}(\{\nabla S\}) = n\left(\frac{m+1}{2}\right) \quad \text{for all} \quad x \in M.
\]

**Proof:** It following easily from (52.4), (52.2) and (52.3).

**Remark 2:** Let \( G \) be a Riemannian field and \( C : M \to \text{Con}B \) be a \( G \)-compatible connection. Let \( L : M \to \text{Lis}B \) be a cross section with \( \nabla_C L = 0 \) be given. Then, it follows from \( \nabla_C G = 0 \) and \( \nabla_C L = 0 \) that \( \nabla_C (G \circ (L \times L)) = 0 \). Hence, the Riemannian field \( H := G \circ (L \times L) \) satisfies \( \nabla_C H = 0 \).
53. Riemannian and Symplectic Manifolds.

**Definition:** We say that $\mathcal{M}$ is a **Riemannian manifold** if the tangent bundle $T\mathcal{M}$ is endowed with additional structure by the prescription of a $C^{r-1}$ Riemannian field.

We say that $\mathcal{M}$ is a **symplectic manifold** if the tangent bundle $T\mathcal{M}$ is endowed with additional structure by the prescription of a $C^{r-1}$ symplectic field.

Let a Riemannian field $G : \mathcal{M} \to \text{Sym}^\text{inv}(T\mathcal{M}, T\mathcal{M}^*)$ of class $C^{r-1}$ be given.

**Proposition 1:** For every $x \in \mathcal{M}$, the restriction

$$ T_x|_{J^C_{\mathcal{G}(x)}((\nabla_x G))} : J^C_{\mathcal{G}(x)}((\nabla_x G)) \to \text{Skw}_2(T_x \mathcal{M}^2, T_x \mathcal{M}) \quad (53.1) $$

of the torsion mapping $T_x$ is bijective.

**Proof:** Given $x \in \mathcal{M}$. If $K_1, K_2 \in \text{Con}_{T_x}(T\mathcal{M}, \mathcal{M})$, then we have $T_x(K_1) = T_x(K_2)$ if and only if $K_1 - K_2 = I_x L$ for some $L \in \text{Sym}_2((T_x \mathcal{M})^2, T_x \mathcal{M})$ and hence

$$ (G(x)L)(t, b, d) = (G(x)L)(b, t, d) \quad (53.2) $$

for all $t, b, d \in T_x \mathcal{M}$.

Let $K_1, K_2 \in J^C_{\mathcal{G}(x)}((\nabla_x G))$ with $T_x(K_1) = T_x(K_2)$ be given and determining $L \in \text{Lin}_2((T_x \mathcal{M})^2, T_x \mathcal{M})$ such that $K_1 - K_2 = I_x L$. Applying (52.1), (51.14) and (53.2), we have

$$ (G(x)L)(t, b, d) = -(G(x)L)(t, d, b) = -(G(x)L)(d, t, b) = (G(x)L)(d, b, t) = -(G(x)L)(b, t, d) = -(G(x)L)(b, d, t) $$

for all $t, b, d \in T_x \mathcal{M}$. This shown that $G(x)L = 0$. Since $G(x)$ is invertible, we observe that $L = 0$ and hence $K_1 = K_2$. In other words, the restriction

$$ T_x|_{J^C_{\mathcal{G}(x)}((\nabla_x G))} : J^C_{\mathcal{G}(x)}((\nabla_x G)) \to \text{Skw}_2(T_x \mathcal{M}^2, T_x \mathcal{M}) \quad (53.3) $$

of the flat mapping $T_x$ is injective and hence bijective. Since $x \in \mathcal{M}$ was arbitrary, the assertion follows. ☐
**Proposition 2:** For every \( x \in \mathcal{M} \), we have
\[
J_{G(x)}^\leq(\{\nabla x G\}) = \{ K - \frac{1}{2} I_x G(x)^{-1} (S(\nabla K G)) \mid K \in \text{Con}_x(TM, \mathcal{M}) \} \quad (53.4)
\]
where
\[
(S(\nabla K G)) = \nabla K G + \nabla K G^{-1(1,2)} - \nabla K G^{-1(3,3)}.
\]
Moreover, if \( K_1, K_2 \in \text{Con}_x(TM, \mathcal{M}) \) with \( T_x(K_1) = T_x(K_2) \), i.e.
\[
K_1 - K_2 \in \{ I_x \} \text{Sym}_2(T_x M^2, T_x M),
\]
then we have
\[
K_1 - \frac{1}{2} I_x G(x)^{-1}(\nabla K_1 G + \nabla K_1 G^{-1(1,2)} - \nabla K_1 G^{-1(3,3)}) = K_2 - \frac{1}{2} I_x G(x)^{-1}(\nabla K_2 G + \nabla K_2 G^{-1(1,2)} - \nabla K_2 G^{-1(3,3)}). \quad (53.5)
\]

**Proof:** By (41.8), we have
\[
(\Box_x G) I_x G(x)^{-1}(\nabla K G)(s, t, u) = \nabla K G(s, t, u) + \nabla K G(s, u, t),
\]
\[
(\Box_x G) I_x G(x)^{-1}(\nabla K G^{-1(1,2)})(s, t, u) = \nabla K G(t, s, u) + \nabla K G(u, s, t), \quad (53.6)
\]
\[
(\Box_x G) I_x G(x)^{-1}(\nabla K G^{-1(3,3)})(s, t, u) = \nabla K G(t, u, s) + \nabla K G(u, t, s);
\]
for all \( x, t, u \in T_x M \). Observing \( \nabla K G \in \text{Lin} (T_x M, \text{Sym}_2(T_x M^2, \mathcal{M})) \), we see that (53.4) follows easily from (53.6).

The more general version of “the fundamental theorem of Riemannian geometry” follows immediately from Prop. 1:

**Fundamental Theorem of Riemannian Geometry (with torsion):**

For every prescribed torsion field \( L : \mathcal{M} \to \text{Skw}_2(TM^2, TM) \) of class \( C^s \), \( s \in 0..r - 2 \), there is exactly one \( G \)-compatible connection \( C \), i.e. one satisfying \( \nabla C G = 0 \), such that \( T(C) = L \). \( C \) is of class \( C^s \).

**Remark 1:** When \( L = 0 \), the corresponding connection is called the Levi-Civitā connection.

**Remark 2:** It follows from Theorem 3 that for every connection \( C' : \mathcal{M} \to \text{Con} TM \) of class \( C^s \), \( s \in 0..r - 2 \), there is exactly one connection \( C : \mathcal{M} \to \text{Con} TM \) such that \( T(C) = T(C') \) and \( \nabla C G = 0 \). Moreover, in view of Prop. 2, we have
\[
C = C' - \frac{1}{2} IG^{-1}(\nabla C' G - \nabla C' G^{-1(1,2)} + \nabla C' G^{-1(3,3)}). \quad (53.7)
\]
Now let a connection \( C : \rightarrow \text{Con}\,T\mathcal{M} \) be given. We may define, for each \( x \in \mathcal{M} \), a mapping

\[
\mathcal{A}_x^C : \text{Con}_xT\mathcal{M} \rightarrow \text{Sym}_2(T_x\mathcal{M}^2, T_x\mathcal{M})
\]

by

\[
\mathcal{A}_x^C(K) := \Lambda(C(x))K + (\Lambda(C(x))K) \sim \text{ for all } K \in \text{Con}_xT\mathcal{M}.
\]

Let a symplectic field \( S : \mathcal{M} \rightarrow \text{Skw}_{\text{inv}}^\text{inv}(T\mathcal{M}, T^*\mathcal{M}) \) of class \( C^{r-1} \) be given.

**Proposition 3:** For every \( x \in \mathcal{M} \), the restriction

\[
\mathcal{A}_x^C|_{J_{S(x)}(\{\nabla_x S\})} : J_{S(x)}(\{\nabla_x S\}) \rightarrow \text{Sym}_2(T_x\mathcal{M}^2, T_x\mathcal{M})
\]

of the mapping \( \mathcal{A}_x^C \) is bijective.

**Proof:** Similar to the proof of Prop. 1.

**Proposition 4:** For every connection \( C \) and each prescribed symmetric field \( L : \mathcal{M} \rightarrow \text{Sym}_2(T\mathcal{M}^2, T\mathcal{M}) \) of class \( C^s \), \( s \in 0..r-2 \), there is exactly one \( S \)-compatible connection \( K \), i.e. one satisfying \( \nabla_K S = 0 \), such that \( \mathcal{A}_x^C(K) = L \).

**Proof:** It follows immediately from Prop.3.

**Notes 53**

1. The proof of the Fundamental Theorem of Riemannian Geometry given here is modelled on the proof given by Noll in [N1].

2. In [Sp], Spivak, M. stated: “Perhaps its only defect [of the fundamental theorem of Riemannian geometry] is the restriction to symmetric connections.” We show that this restriction is not needed.
54. Identities

Let a $C^r$, $r \geq 2$, Riemannian manifold $\mathcal{M}$ with the Riemannian-field $G$ be given. Assume that $\dim \mathcal{M} \geq 2$.

For every $A, B \in \mathfrak{X}(T\mathcal{M})$ and a connection $\mathbf{C} : \mathcal{M} \to \text{Con}(T\mathcal{M})$, we use the following notations

$$(A, B) := G(A, B) \quad \text{and} \quad \nabla_A B := (\nabla_C B)A.$$ 

**Proposition 1:** A connection $\mathbf{C}$ on a Riemannian manifold $\mathcal{M}$ is compatible with the Riemannian-field $G$ if and only if

$$A(B, D) = \langle \nabla_A B, D \rangle + \langle B, \nabla_A D \rangle$$

for all $A, B, D \in \mathfrak{X}(T\mathcal{M})$.

**Proof:** Taking the covariant gradient of $G \circ (B, D)$ with respect to $\mathbf{C}$, we obtain

$$(\nabla_{G \circ (B, D)} \mathbf{C})A = G((\nabla_C B)A, D) + G(B, (\nabla_C D)A).$$

The equation (I.1) holds if and only if $\nabla_C G = 0$.

For the sake of simplification, we adapt the following notation

$$\langle \langle X, Y, Z, T \rangle \rangle := \langle R(X, Y)Z, T \rangle$$

for all $X, Y, Z, T \in \mathfrak{X}(T\mathcal{M})$,

where $R := R(C)$ is the curvature field for a given connection $\mathbf{C}$. Also recall that

$$R(X, Y, Z) = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(T\mathcal{M})$.

**Proposition 2:** Let $\mathbf{C}$ be a connection on a Riemannian manifold $\mathcal{M}$ which is compatible with the Riemannian-field $G$, then we have

$$\langle \langle X, Y, Z, T \rangle \rangle = -\langle \langle X, Y, T, Z \rangle \rangle$$

for all $X, Y, Z, T \in \mathfrak{X}(T\mathcal{M})$.

**Proof:** To prove (I.2) is equivalent to show

$$0 = \langle \langle X, Y, Z, Z \rangle \rangle = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z \rangle.$$

Applying (I.1), we have

$$\langle \nabla_Y \nabla_X Z, Z \rangle = Y\langle \nabla_X Z, Z \rangle - \langle \nabla_X Z, \nabla_Y Z \rangle$$
and
\[
\langle \nabla_X \nabla_Y Z, Z \rangle = X \langle \nabla_Y Z, Z \rangle - \langle \nabla_Y Z, \nabla_X Z \rangle.
\]
Hence
\[
\langle \langle X, Y, Z, Z \rangle \rangle = Y \langle \nabla_X Z, Z \rangle - X \langle \nabla_Y Z, Z \rangle + \langle [X, Y] Z, Z \rangle.
\]
It follows from (I.1) and the symmetry of the Riemannian-field \(G\) that
\[
\frac{1}{2} A(D, D) = \langle \nabla_A D, D \rangle \quad \text{for all} \quad A, D \in \mathfrak{X}(T \mathcal{M}). \quad (54.3)
\]
And hence
\[
\langle \langle X, Y, Z, Z \rangle \rangle = \frac{1}{2} Y \langle X(Z, Z) \rangle - \frac{1}{2} X \langle Y(Z, Z) \rangle + \frac{1}{2} [X, Y] \langle Z, Z \rangle
\]
\[
= -\frac{1}{2} [X, Y] \langle Z, Z \rangle + \frac{1}{2} [X, Y] \langle Z, Z \rangle = 0.
\]
Since \(X, Y, Z \in \mathfrak{X}(T \mathcal{M})\) were arbitrary, the equation (I.2) follows.

Let \(C\) be a compatible connection with the Riemannian-field \(G\).

Given \(x \in \mathcal{M}\). Since \(R_x(C) \in \text{Skw}_2(T_x \mathcal{M}^2, \text{Lin } T_x \mathcal{M})\), we observe from Prop. 2 that
\[
\langle \langle \cdot, \cdot, \cdot, \cdot \rangle \rangle \in \text{Skw}_2(T_x \mathcal{M}^2, \text{Skw}_2(T_x \mathcal{M}^2, \cdot)).
\]

**Lemma:** Let an inner-product space \(T\), with \(\dim T \geq 2\), and a two-dimensional subspace \(S\) of \(T\) be given. If both \(\{u, v\}\) and \(\{s, t\}\) are bases for \(S\), then we have
\[
\frac{W(u, v, u, v)}{(u \wedge v)(u, v)} = \frac{W(s, t, s, t)}{(s \wedge t)(s, t)} \quad (54.4)
\]
for all \(W \in \text{Skw}_2(T^2, \text{Skw}_2(T^2, \cdot))\).

**Proof:** By calculations.

Applying the above Lemma, we arrive the following definition.

**Definition:** Let \(V \subset T_x \mathcal{M}\) be a two-dimensional subspace of \(T_x \mathcal{M}\). Let \(\{u, v\}\) be a basis for \(S\). The sectional curvature of \(S\) at \(x\) is defined by
\[
K_x(S) := \frac{\langle \langle u, v, u, v \rangle \rangle}{(u \wedge v)(u, v)} \quad (54.5)
\]
which does not depend on the choice of \( \{u,v\} \).

**Remark:** The definition of sectional curvature “does not” require the assumption of the compatible connection \( C \) to be torsion-free.

**Proposition 4:** Let \( C \) be a connection on a Riemannian manifold \( M \) which is compatible with the Riemannian-field \( G \), then we have

\[
\langle [X,Y,Z,W]\rangle - \langle [Z,W,X,Y]\rangle = V(X,Y,Z,W) \quad (54.6)
\]

for all \( X,Y,Z,W \in \mathfrak{X}(T_M) \).

**Proof:**

\[
\begin{align*}
R(X,Y)Z \cdot W + R(Y,Z)X \cdot W + R(Z,X)Y \cdot W \\
+ R(Y,Z)W \cdot X + R(Z,W)Y \cdot X + R(W,Y)Z \cdot X \\
+ R(Z,W)X \cdot Y + R(W,X)Z \cdot Y + R(X,Z)W \cdot Y \\
+ R(W,X)Y \cdot Z + R(X,Y)W \cdot Z + R(Y,W)X \cdot Z \\
= \nabla T(X,Y,Z) \cdot W + \nabla T(Y,Z,X) \cdot W + \nabla T(Z,X,Y) \cdot W \\
+ \nabla T(Y,Z,W) \cdot X + \nabla T(Z,W,Y) \cdot X + \nabla T(W,Y,Z) \cdot X \\
+ \nabla T(Z,W,X) \cdot Y + \nabla T(W,X,Z) \cdot Y + \nabla T(X,W,Z) \cdot Y \\
+ \nabla T(W,X,Y) \cdot Z + \nabla T(Y,W,X) \cdot Z \\
+ \nabla T(Y,Z,W) \cdot X + \nabla T(T(Y,Z),X) \cdot W + \nabla T(T(Z,X),Y) \cdot W \\
+ \nabla T(T(Y,Z),W) \cdot X + \nabla T(T(Z,W),Y) \cdot X + \nabla T(T(W,Y),Z) \cdot X \\
+ \nabla T(T(Z,W),X) \cdot Y + \nabla T(T(W,X),Z) \cdot Y + \nabla T(T(X,Z),W) \cdot Y \\
+ \nabla T(T(W,X),Y) \cdot Z + \nabla T(T(X,Y),W) \cdot Z + \nabla T(T(Y,W),X) \cdot Z
\end{align*}
\]

**Proposition 5:** Let \( C \) be a connection on a Riemannian manifold \( M \) which is compatible with the Riemannian-field \( G \), then we have

\[
\text{tr} \left( R(x)(s,\cdot) t - R(x)(t,\cdot) s + R(x)(t,s) \right) = ?? \quad (54.7)
\]

for all \( s,t \in T_x M \).

**Second Proof of Pro. 2:**

In view of (I.1) we have, for all \( X,Y,Z,T \in \mathfrak{X}(T_M) \),

\[
\langle \nabla_Y \nabla_X Z, T \rangle = Y \langle \nabla_X Z, T \rangle - \langle \nabla_X Z, \nabla_Y T \rangle,
\]

\[
\langle \nabla_X \nabla_Y Z, T \rangle = X \langle \nabla_Y Z, T \rangle - \langle \nabla_Y Z, \nabla_X T \rangle
\]
and
\[ \langle \nabla_{[X,Y]}Z, T \rangle = [X,Y]\langle Z, T \rangle - \langle Z, \nabla_{[X,Y]}T \rangle. \]

Hence
\[ \langle \langle X, Y, Z, T \rangle \rangle = \langle \nabla_{\nabla X}Z, T \rangle - \langle \nabla_{\nabla Y}Z, T \rangle + \langle \nabla_{[X,Y]}Z, T \rangle \]
\[ = Y\langle \nabla_{X}Z, T \rangle - X\langle \nabla_{Y}Z, T \rangle + [X,Y] \langle Z, T \rangle - \langle Z, \nabla_{[X,Y]}T \rangle \]
\[ = -Y\langle \nabla_{X}Z, T \rangle + X\langle \nabla_{Y}Z, T \rangle + \langle \nabla_{Y}Z, \nabla_{X}T \rangle - \langle \nabla_{[X,Y]}T, Z \rangle \]
\[ = -\langle \nabla_{Y}\nabla_{X}Z, T \rangle + \langle \nabla_{X}\nabla_{Y}Z, T \rangle - \langle \nabla_{[X,Y]}T, Z \rangle \]
\[ = -\langle X, Y, T, Z \rangle. \]

Since \( X, Y, Z, T \in \mathfrak{X}(T\mathcal{M}) \) was arbitrary, the assertion of Prop. 2 follows.

55. Einstein-tensor field

Let a \( C^r \) manifold \( \mathcal{M} \), with \( r \geq 2 \) and \( \dim \mathcal{M} \geq 2 \), and a \( C^r \) connection \( \nabla : \mathcal{M} \to \text{Con}(T\mathcal{M}) \) be given. Assume that \( G : \mathcal{M} \to \text{Sym}_2(T\mathcal{M}^2) \) be a Riemannian-field compatible with the connection \( \nabla \).

Let \( x \in \mathcal{M} \) be given and assume that the following condition hold
\[ \text{tr} \left( R(x)(s, \cdot) t - R(x)(t, \cdot) s + R(x)(t, s) \right) = 0, \quad (55.1) \]
i.e. we have
\[ \text{tr} (R(x)(s, \cdot) t) - \text{tr} (R(x)(t, \cdot) s) + \text{tr} (R(x)(t, s)) = 0. \]

Since \( R(x)(t, s) \) is skew-symmetric with respect to \( G \), we obtain that
\[ \text{tr} (R(x)(s, \cdot) t) = \text{tr} (R(x)(t, \cdot) s) \quad \text{for all} \quad s, t \in T_x \mathcal{M}. \]

**Definition**: The Ricci-tensor field \( \text{Ric} : \mathcal{M} \to \text{Sym}_2(T\mathcal{M}^2) \) is defined by
\[ \text{Ric}(x)(s, t) := \text{tr} (R(x)(s, \cdot) t) \quad (55.2) \]
for all \( x \in \mathcal{M} \) and all \( s, t \in T_x \mathcal{M} \).
**Definition**: The Einstein-tensor field $\text{Ein} : \mathcal{M} \rightarrow \text{Sym}_2(T\mathcal{M}^2,)$ is defined by

$$\text{Ein}(x) := \text{Ric}(x) - \frac{1}{2} \text{tr} \left( \text{G}^{-1}(x) \text{Ric}(x) \right) \text{G}(x) \quad (55.3)$$

for all $x \in \mathcal{M}$. (The factor $1/2$ is determined by the assumption $\dim T_x \mathcal{M} = 4$)

It follows from the 2nd Bianchi Identity (this condition should be weakened) that

$$\text{div}_\nabla \text{Ein} = 0. \quad (55.4)$$

**Remark**: The matter tensor field $\text{Mat} : \mathcal{M} \rightarrow \text{Sym}_2(T\mathcal{M}^2,)$ satisfying

$$\text{Ein}(x) = \kappa \text{Mat}(x) \quad (55.5)$$

where $\kappa \in \mathbb{R}$ is the universal gravitational constant. It follows from (Ein.4) and (Ein 5) that

$$\text{div}_\nabla \text{Mat} = 0 \quad (55.6)$$

(balance of world-momentum).