Chapter 2

Manifolds and Bundles

21. Charts, Atlases and Manifolds

Let a set $\mathcal{M}$ and $r \in \mathbb{R}$ be given. A chart $\chi$ for $\mathcal{M}$ is defined to be a bijection whose domain is included in $\mathcal{M}$ and whose codomain is an open subset of a specified flat space, denote by $\text{Pag} \chi$ and called the page of $\chi$. The translation space of $\text{Pag} \chi$ is denoted by

$$V_\chi := \text{Pag} \chi - \text{Pag} \chi.$$  \hfill (21.1)

Let $f$ be a mapping whose domain is a subset of $\mathcal{M}$ and whose codomain is an open subset $\mathcal{D}$ of a specified flat space. We say that $f$ is $C^r$-related to a given chart $\chi$ for $\mathcal{M}$ if

(R1) $\chi_\triangleright (\text{Dom} \chi \cap \text{Dom} f)$ is an open subset of $\text{Pag} \chi$,

(R2) $f \circ \chi^{-1} : (\text{Dom} \chi \cap \text{Dom} f) \to \mathcal{D}$ is of class $C^r$.

We say that two charts $\chi$ and $\gamma$ for $\mathcal{M}$ are $C^r$-compatible if $\gamma$ is $C^r$-related to $\chi$ and $\chi$ is $C^r$-related to $\gamma$.

Pitfall: In general, $C^r$-compatibility is not an equivalence relation.

A class $\mathfrak{A}$ of charts for $\mathcal{M}$ is called a $C^r$-atlas of $\mathcal{M}$ if

(A1) Any two charts in $\mathfrak{A}$ are $C^r$-compatible,

(A2) The domain of the charts in $\mathfrak{A}$ cover $\mathcal{M}$, i.e.

$$\mathcal{M} = \bigcup \{\text{Dom} \chi \mid \chi \in \mathfrak{A}\}.$$  \hfill (21.2)

It is clear that a $C^r$-atlas is also a $C^s$-atlas for every $s \in [0, r]$.

**Proposition 1:** Let $\mathfrak{A}$ be a $C^r$-atlas for $\mathcal{M}$ and let $\chi$ be a chart that is $C^r$-compatible with all charts in $\mathfrak{A}$. If $f$ is a mapping that is $C^r$-related to every chart in $\mathfrak{A}$ then it is also $C^r$-related to $\chi$.

**Proof:** Let $x \in \text{Dom} \chi \cap \text{Dom} f$ be given. By (A2) we may choose $\alpha \in \mathfrak{A}$ such that $x \in \text{Dom} \alpha$. We put

$$\mathcal{G} := \text{Dom} \chi \cap \text{Dom} \alpha \cap \text{Dom} f.$$  \hfill (21.3)

Since $\alpha$ is injective we have

$$\alpha_\triangleright (\mathcal{G}) = \alpha_\triangleright (\text{Dom} \chi \cap \text{Dom} \alpha) \cap \alpha_\triangleright (\text{Dom} f \cap \text{Dom} \alpha).$$
Since \( \chi \) and \( f \) are both \( C^r \)-related to \( \alpha \), it follows from (R1) that both \( \alpha_\succ(\text{Dom } \chi \cap \text{Dom } \alpha) \) and \( \alpha_\succ(\text{Dom } f \cap \text{Dom } \alpha) \) are open subsets of \( \text{Pago} \) and hence that \( \alpha_\succ(\mathcal{G}) \) is also open in \( \text{Pago} \). Since \( \alpha \circ \chi^{-} \) is continuous by (R2), it follows that \( \chi_\succ(\mathcal{G}) = (\alpha \circ \chi^{-})^\prec(\alpha_\succ(\mathcal{G})) \) is an open neighborhood of \( \chi(x) \) in \( \text{Pago} \chi \). Using (0.1) and (0.2) it is easily seen that

\[
(f \circ \chi^{-})\left|_{\chi_\succ(\mathcal{G})}\right. = (f \circ \alpha^{-})\left|_{\alpha_\succ(\mathcal{G})}\right. \circ (\alpha \circ \chi^{-})\left|_{\chi_\succ(\mathcal{G})}\right..
\]

Since both \( f \circ \alpha^{-} \) and \( \alpha \circ \chi^{-} \) are of class \( C^r \) by (R2), it follows from the chain rule that the restriction of \( f \circ \alpha^{-} \) to a neighborhood \( \chi_\succ(\mathcal{G}) \) of \( \chi(x) \) in \( \text{Pago} \chi \) is of class \( C^r \). Since \( x \in \text{Dom } \chi \cap \text{Dom } f \) was arbitrary, it follows that the domain \( \chi_\succ(\text{Dom } \chi \cap \text{Dom } f) \) of \( f \circ \chi^{-} \) is open in \( \text{Pago} \chi \) and that \( f \circ \chi^{-} \) is of class \( C^r \), i.e. that \( f \) is \( C^r \)-related to \( \chi \).

We say that a \( C^r \)-atlas \( \mathfrak{A} \) for \( \mathcal{M} \) is \( C^r \)-saturated if every chart for \( \mathcal{M} \) that is \( C^r \)-compatible with all charts in \( \mathfrak{A} \) already belongs to \( \mathfrak{A} \). The following is an immediate consequence of Prop. 1.

**Proposition 2:** Let \( \mathfrak{A} \) be a \( C^r \)-atlas for \( \mathcal{M} \). Then there is exactly one saturated \( C^r \)-atlas \( \mathfrak{A} \) that includes \( \mathfrak{A} \). In fact, \( \mathfrak{A} \) consists of all charts that are \( C^r \)-compatible with all charts in \( \mathfrak{A} \).

**Definition:** Let \( r \in \mathbb{N} \) be given. A \( C^r \)-manifold is a set \( \mathcal{M} \) endowed with structure by the prescription of a saturated \( C^r \)-atlas for \( \mathcal{M} \), which is called the chart-class of \( \mathcal{M} \) and is denoted by \( \text{Ch}^r \mathcal{M} \), or if no confusion is likely, simply by \( \text{Ch} \mathcal{M} \).

In view of Prop. 2, the structure of a \( C^r \)-manifold on \( \mathcal{M} \) is uniquely determined by specifying a \( C^r \)-atlas included in \( \text{Ch} \mathcal{M} \). Of course, two different such atlases may determine one and the same \( C^r \)-structure.

Let \( \mathcal{M} \) be a \( C^r \)-manifold with chart-class \( \text{Ch}^r \mathcal{M} \). Then, for every \( s \in 0..r \), \( \mathcal{M} \) has also the natural structure of a \( C^s \)-manifold, determined by \( \text{Ch}^s \mathcal{M} \) regarded as a \( C^r \)-atlas. Of course, the chart-class \( \text{Ch}^s \mathcal{M} \) of the \( C^s \)-manifold structure includes \( \text{Ch}^r \mathcal{M} \), but we have \( \text{Ch}^s \mathcal{M} \) \( \text{Ch}^s \mathcal{M} \) if \( s < r \).

**Examples of manifold**

**Example 1:** Let \( \mathcal{D} \) be an open subset of a flat space. Then the singleton \( \{1 \mathcal{D}\} \) is a \( C^\omega \)-atlas of \( \mathcal{D} \). It determines on \( \mathcal{D} \) a natural \( C^\omega \)-structure and hence a natural \( C^r \)-structure for every \( r \in \mathbb{N} \).

**Example 2:** (Product manifold) Let \( \mathcal{M} \) and \( \mathcal{N} \) be manifolds of class \( C^r \), then the product \( \mathcal{M} \times \mathcal{N} \) has the natural structure of a \( C^r \) manifold.
We now assume that a $C^r$-manifold $M$ with chart-class $\text{Ch} M$ is given. We use the notation

$$\text{Ch}_x M := \{ \chi \in \text{Ch} M \mid x \in \text{Dom} \chi \}.$$  \hfill (21.4)

It is easily seen that the spaces $\text{Pag} \chi$ and $\mathcal{V}_\chi$, $\chi \in \text{Ch}_x M$, all have the same dimension. This dimension is called the **dimension of $M$ at $x$**, and is denoted by $\text{dim}_x M$.

The $C^r$-manifold $M$ is endowed with a natural topology, namely the coarsest topology that renders all $\chi \in \text{Ch} M$ continuous. A subset $P$ of $M$ is open if and only if, for each $\chi \in \text{Ch} M$, the image $\chi^>(P \cap \text{Dom} \chi)$ is an open subset of $\text{Pag} \chi$. Given $x \in M$, one can construct a neighborhood-basis $\mathfrak{B}_x$ of $x$ in $M$ in the following manner: Choose a chart $\chi \in \text{Ch}_x M$ and a neighborhood-basis $\mathfrak{B}_\chi(x)$ of $\chi(x)$ in $\text{Pag} \chi$. Then put

$$\mathfrak{B}_x := \{ \chi^>(N \cap \text{Cod} \chi) \mid N \in \mathfrak{B}_\chi(x) \}.$$  \hfill (21.5)

**Pitfall:** The natural topology of $M$ need not be separating.

Let $P$ be an open subset of $M$. Then $P$ has the natural structure of a $C^r$-manifold whose chart-class $\text{Ch} P$ is

$$\text{Ch} P := \{ \chi \in \text{Ch} M \mid \text{Dom} \chi \subset P \}.$$  \hfill (21.6)

The natural topology of $P$ as a $C^r$-manifold concides with the topology of $P$ induced by the topology of $M$.

Let $f$ be a mapping whose domain is an open subset of $M$ and whose codomain is an open subset $\mathcal{D}$ of a specified flat space $\mathcal{E}$ with translation space $\mathcal{V} := \mathcal{E} - \mathcal{E}$. We say that $f$ is **of class** $C^s$, with $s \in [0, r]$, if it is $C^s$-related to every chart $\chi \in \text{Ch} M$, i.e. if $f \circ \chi^-$ is of class $C^s$ for all charts $\chi \in \text{Ch} M$. (Since $\text{Dom} f$ is open, $f \circ \chi^- = \chi^>(\text{Dom} \chi \cap \text{Dom} f)$ is automatically open in $\text{Pag} \chi$ when $\chi \in \text{Ch} M$.) It follows from Prop. 1 that $f$ is of class $C^s$ if $f \circ \chi^-$ is of class $C^s$ for every chart $\chi$ in some $C^r$-atlas included in $\text{Ch} M$. If $f$ is of class $C^s$ with $s \geq 1$ and if $\chi \in \text{Ch} M$, we define the **gradient**

$$\nabla_\chi f : \text{Dom} \chi \cap \text{Dom} f \to \text{Lin}(\mathcal{V}_\chi, \mathcal{V})$$

of $f$ in the chart $\chi$ by

$$(\nabla_\chi f)(x) := \nabla_{\chi(x)}(f \circ \chi^-) \quad \text{for all} \quad x \in \text{Dom} \chi \cap \text{Dom} f.$$  \hfill (21.7)

More generally, for every $s \in [1, r]$, the **gradient of order** $s$

$$\nabla_\chi^{(s)} f : \text{Dom} \chi \cap \text{Dom} f \to \text{Sym}_s(\mathcal{V}_\chi, \mathcal{V})$$

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of \( f \) in the chart \( \chi \) defined by

\[
(\nabla^{(s)} f)(x) := \nabla^{(s)}(f \circ \chi^{-1} ) \quad \text{for all} \quad x \in \text{Dom} \chi \cap \text{Dom} f. \tag{21.8}
\]

The following transformation rules are easy consequences of the rules of calculus.

**Proposition 3:** Let \( f \) be a mapping of class \( C^1 \), \( x \in \text{Dom} f \) and \( \chi, \gamma \in \text{Ch}_xM \). Then

\[
(\nabla \gamma f)(x) = (\nabla \chi f)(x)(\nabla \gamma \chi)(x). \tag{21.9}
\]

If \( f \) is also of class \( C^2 \), then

\[
(\nabla^{(2)} \gamma f)(x) = (\nabla^{(2)} \chi f)(x) \circ (\nabla \gamma \chi(x) \times \nabla \gamma \chi(x)) + (\nabla \chi f)(x)(\nabla^{(2)} \gamma \chi)(x). \tag{21.10}
\]

In the case when \( f := \gamma \) the formulas (21.7) and (21.8) reduce to

\[
(\nabla \gamma \gamma)(x) = 1_{V_{\gamma}} \quad \text{and} \quad (\nabla^{(2)} \gamma \gamma)(x) = 0.
\]

Hence Prop. 3 has the following consequence:

**Proposition 4:** Let \( x \in M \) and \( \chi, \gamma \in \text{Ch}_xM \) be given. If \( r \geq 1 \), then

\[
(\nabla \chi \gamma)(x) \in \text{Lin} (V_\chi, V_\gamma) \quad \text{is invertible and}
\]

\[
(\nabla \chi \gamma)(x)^{-1} = (\nabla \gamma \chi)(x). \tag{21.11}
\]

If \( r \geq 2 \), we also have

\[
(\nabla^{(2)} \chi \gamma)(x) = -(\nabla \gamma \chi)(x)((\nabla^{(2)} \gamma \chi)(x) \circ (\nabla \gamma \chi(x) \times \nabla \gamma \chi(x))). \tag{21.12}
\]

If the manifold \( M \) is itself the underlying manifold of an open subset of a flat space (see Example 1 above), then a mapping \( f \) is of class \( C^s \) as described above if and only if it is of class \( C^s \) in the ordinary sense (see Notations).

Let \( f \) be a mapping whose domain is a neighborhood of a given point \( x \in M \) and whose codomain is an open subset of a specified flat space. We say that \( f \) is differentiable at \( x \) if \( f \circ \chi^{-1} \) is differentiable at \( \chi(x) \) for some, and hence all, \( \chi \in \text{Ch}_xM \). If this is the case, (21.7) remains meaningful for the given \( x \in M \) and the transformation formula (21.9) remains valid. The concept of “\( s \) times differentiable at \( x \)” when \( s \in 0..r \) is defined in a similar way.

More generally, let \( C^r \)-manifolds \( M \) and \( M' \) be given. Let \( g \) be a mapping whose domain and codomain are open subsets of \( M \) and \( M' \), respectively. We say that \( g \) is of class \( C^s \) with \( s \in 0..r \) if \( \chi' \circ g \circ \chi^{-1} \) is of class \( C^s \) in the ordinary sense for all \( \chi \in \text{Ch}M \) and all \( \chi' \in \text{Ch}M' \).
**Definition:** Let $\mathcal{M}$ be a $C^r$-manifold and let $\mathcal{P}$ be a subset of $\mathcal{M}$. We say that $\mathcal{P}$ is a submanifold of $\mathcal{M}$ if for each point $x \in \mathcal{P}$ there is a chart $\chi \in \text{Ch}_x \mathcal{M}$ such that $\chi(\mathcal{P} \cap \text{Dom} \chi)$ is an open subset of a flat $F_{\chi}$ of $\text{Pag} \chi$.

Let $\mathcal{P}$ be a $C^r$ submanifold of the manifold $\mathcal{M}$. We left it the readers to show that $\mathcal{P}$ has the natural structure of a $C^r$ manifold. The natural topology of $\mathcal{P}$ as a $C^r$-manifold concides with the topology of $\mathcal{P}$ induced by the topology of $\mathcal{M}$, i.e. $\mathcal{P}$ a topological subspace of $\mathcal{M}$.

Let $f : \mathcal{S} \to \mathcal{M}$ be a $C^s$ mapping from a manifold $\mathcal{S}$ to another manifold $\mathcal{M}$. The mapping $f$ is called a $C^s$ immersion at $x \in \mathcal{S}$ if there exists an open neighborhood $N_x$ of $x$ (in $\mathcal{S}$) such that the restriction $f|_{N_x}$ is injective and $f(N_x)$ is a submanifold of $\mathcal{M}$. We say that $f$ is an immersion if it is an immersion at every $y \in \mathcal{S}$. If $f$ is an immersion, the domain $\mathcal{S}$ called an immersed manifold of $\mathcal{M}$. However, being an immersion is a “local property” and hence the range $\text{Rng } f := f(\mathcal{S})$ of $f$ may not be a submanifold of $\mathcal{M}$. For example (see [L]):

![Figure 11.1](image)

An injective immersion $f$ from manifold $\mathcal{A}$ to manifold $\mathcal{B}$ is an imbedding if the range $\text{Rng } f := f(\mathcal{A})$ is a submanifold of $\mathcal{B}$. The domain of an imbedding is called an imbedded manifold of its codomain manifold. It is clear that for every submanifold $\mathcal{P}$ of a given manifold $\mathcal{M}$ the inclusion $\mathbb{1}_{\mathcal{P} \subset \mathcal{M}}$ is an imbedding.

**Remark:** Let $\mathcal{A}$ and $\mathcal{B}$ be topological spaces and $f : \mathcal{A} \to \mathcal{B}$ be an injection. We say that $f$ is an imbedding if the topology of $\mathcal{A}$ is induced by $f$ from the topology of $\mathcal{B}$.

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**More details on submanifolds**
22. Bundles

We assume that \( r \in \mathbb{N}^\infty \) with \( r \geq 2 \) and a \( C^r \)-manifold \( M \) are given. Let a number \( s \in 0 .. r \) be given and let \( \tau : B \to M \) be a surjective mapping from a given set \( B \) to the manifold \( M \).

Let a concrete isocategory ISO with object class \( OBJ \) be given with the following properties:

(i) Each set in \( OBJ \) has the natural structure of a \( C^s \)-manifold.

(ii) Every isomorphism in ISO is a \( C^s \)-diffeomorphism.

The most important special cases are (1) the isocategory of LIS consisting of all linear isomorphisms, whose object class \( LS \) consist of all (finite dimensional) linear spaces and (2) the isocategory of FIS consisting of all flat isomorphisms, whose object class \( FS \) consist of all flat spaces. The object sets in \( LS \) and \( FS \) have the natural structure of \( C^\infty \)-manifolds and the isomorphisms in LIS and FIS are \( C^\infty \)-diffeomorphisms.

**Definition:** An ISO-bundle chart for \( B \) (for \( \tau \)) is a bijection

\[
\phi : \tau^<(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathcal{V}_\phi,
\]

where \( \mathcal{O}_\phi \) is an open subset of \( M \) and \( \mathcal{V}_\phi \) is a set in \( OBJ \) such that the diagram

\[
\begin{array}{ccc}
\tau^<(\mathcal{O}_\phi) & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \\
\tau^<(\mathcal{O}_\phi) & \downarrow \phi & \downarrow \text{ev}_2 \\
\mathcal{O}_\phi & \text{ev}_1 & \mathcal{V}_\phi
\end{array}
\]

is commutative, i.e. \( \text{ev}_1 \circ \phi = \tau^<(\mathcal{O}_\phi) \).

**Notation:** For every \( y \in M \), we denote \( B_y := \tau^<(\{y\}) \) and for every ISO-bundle chart \( \phi \) we use the following notations

\[
\phi \big|_y := \text{ev}_2 \circ \phi \circ (1_{\mathcal{O}_\phi} \subset \tau^<(\mathcal{O}_\phi)) : B_y \to \mathcal{V}_\phi
\]

for all \( y \in \mathcal{O}_\phi \), i.e. we have the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{V}_\phi & \xrightarrow{\phi} & \mathcal{O}_\phi \times \mathcal{V}_\phi \\
\phi \big|_y & \searrow & \\
B_y & \xrightarrow{\tau^<(\mathcal{O}_\phi)} & \xrightarrow{\phi} \mathcal{O}_\phi \times \mathcal{V}_\phi
\end{array}
\]
Put (22.1) and (22.2) together, we have the following commutative diagram

\[
\begin{array}{ccc}
\phi \downarrow \\
\phi \downarrow \\
\tau \downarrow & \downarrow \\
\tau \downarrow & \downarrow \\
\end{array}
\]

Let \( \phi \) and \( \psi \) be ISO-bundle charts for \( B \). We say that \( \phi \) and \( \psi \) are \( C^* \)-compatible if

\[
\psi \circ \phi^{-1} : (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times \mathcal{V}_\phi \to (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times \mathcal{V}_\psi
\]

is a \( C^* \)-diffeomorphism such that, for every \( y \in \mathcal{O}_\phi \cap \mathcal{O}_\psi \), the mapping

\[
\psi\big| \circ \phi^{-1} : \mathcal{V}_\phi \to \mathcal{V}_\psi
\]

belongs to ISO.

A class \( \mathfrak{A} \) of ISO-bundle charts for \( B \) is called a \( C^* \) ISO-bundle atlas for \( B \) if

- (BA1) every two ISO-bundle charts in \( \mathfrak{A} \) are \( C^* \)-compatible,
- (BA2) for every \( x \in M \) there is a bundle chart \( \phi \in \mathfrak{A} \) with \( x \in \mathcal{O}_\phi \); i.e. we have

\[
M = \bigcup_{\phi \in \mathfrak{A}} \mathcal{O}_\phi .
\]

**Proposition 1:** Let \( \mathfrak{A} \) be a ISO-bundle atlas for \( B \) and let \( \phi \) be a ISO-bundle chart that is \( C^* \)-compatible with all ISO-bundle charts in \( \mathfrak{A} \). If \( \psi \) is a ISO-bundle chart that is \( C^* \)-compatible with every ISO-bundle chart in \( \mathfrak{A} \) then it is also \( C^* \)-compatible with \( \phi \).

**Proof:** Let \( x \in \mathcal{O}_\phi \cap \mathcal{O}_\psi \) be given. By (BA2), we may choose a ISO-bundle chart \( \theta \in \mathfrak{A} \) such that \( x \in \mathcal{O}_\theta \). Put \( \mathcal{O} := \mathcal{O}_\phi \cap \mathcal{O}_\psi \cap \mathcal{O}_\theta \). Since both \( \phi \) and \( \psi \) are \( C^* \)-compatible with \( \theta \), we see that the restriction

\[
\psi \circ \phi^{-1} \big|_{\phi(\tau < \{\mathcal{O}\})} = (\psi \circ \theta^{-1}) \big|_{\phi(\tau < \{\mathcal{O}\})} \circ (\theta \circ \phi^{-1}) \big|_{\phi(\tau < \{\mathcal{O}\})}
\]

on \( \phi(\tau < \{\mathcal{O}\}) \) is a \( C^* \)-diffeomorphism and the induced mapping

\[
\psi\big|_x \circ \phi^{-1}_x = (\psi\big|_x \circ \theta^{-1}_x) \circ (\theta\big|_x \circ \phi^{-1}_x)
\]
is a ISO-isomorphism. Since $x \in O_\phi \cap O_\psi$ was arbitrary, we conclude that $\psi$ and $\phi$ are $C^\ast$-compatible.

We say that a ISO-bundle atlas $\mathcal{A}$ of $B$ is $C^\ast$-saturated if every ISO-bundle chart for $B$ that is $C^\ast$-compatible with all ISO-bundle charts in $\mathcal{A}$ already belongs to $\mathcal{A}$. The following is an immediate consequence of Prop. 1.

**Proposition 2:** Let $\mathcal{A}$ be a $C^\ast$ ISO-bundle atlas for $B$. Then there is exactly one $C^\ast$-saturated ISO-bundle atlas $\mathcal{A}$ that includes $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all ISO-bundle charts that are $C^\ast$-compatible with all ISO-bundle charts in $B$.

Let $\mathcal{A}$ be a saturated ISO-atlas for $B$ and let $\phi$ be a ISO-bundle chart in $\mathcal{A}$. On each fibre $B_x$, $x \in O_\phi$, we can transport the ISO-structure of $V_\phi$ by means of $\phi|^x_B : B_x \to V_\phi$. The result is independent of the choice of $\phi$, since every pair of bundle charts $\phi$ and $\psi$ in $\mathcal{A}$ are compatible and hence $\psi|^x_B \circ \phi|^x_B^{-1} : V_\phi \to V_\psi$ is a ISO-isomorphism.

**Definition:** A $C^\ast$ ISO-bundle over $M$ is a set $B$ and a mapping $\tau : B \to M$ endowed with structure by the prescription of a saturated $C^\ast$ ISO-bundle atlas for $B$, which is called the bundle structure for $B$ and is denoted by $\text{Ch}(B, M)$. We denote the ISO-bundle by $(B, \tau, M)$ or simply by $B$.

The mapping $\tau$ is called the bundle-projection. For every $x \in M$, $B_x := \tau^{-1}(\{x\})$ is called the fiber over $x$ and the inclusion mapping of $B_x$ in $B$ is called the bundle inclusion at $x$. Right inverses of $\tau$ are called cross sections of $B$. We also use the following notation

$$\text{Ch}_x(B, M) := \{ \phi \in \text{Ch}(B, M) \mid x \in O_\phi \}.$$ (22.5)

As explained above, for every $x \in M$, the fiber $B_x$ is naturally endowed with the structure of a ISO-set in such a way that $\phi|^x_B : B_x \to V_\phi$ is in ISO (is an isomorphism) for all $\phi \in \text{Ch}_x(B, M)$. Thus the dimension of $B_x$ can be obtained from all $\phi \in \text{Ch}_x(B, M)$.

Locally (relative to $M$), the manifold structure of the bundle manifold $B$ is completely determined by the manifold structure of the base manifold $M$ and the manifold structures of $V_\phi$ for a single $\phi \in \text{Ch}(B, M)$. Every bundle chart $\phi$ in $\text{Ch}(B, M)$ transports the manifold structure from $O_\phi \times V_\phi$ to $\tau^{-1}(O_\phi)$, and hence a manifold chart can be easily obtained from $\phi$.

Let $b \in B$ be given and put $x := \tau(b)$. The dimension of $B$ at $b$ can be obtained from the codomain of each bundle chart $\phi \in \text{Ch}_x(B, M)$. We have

$$\dim_b B = m + n,$$
where \( \dim_x \mathcal{M} = m \) and \( \dim_y \mathcal{B}_x = n \).

Let ISO-bundles \((B', \tau', \mathcal{M}')\) and \((B, \tau, \mathcal{M})\) be given. We say that 
\((B', \tau', \mathcal{M}')\) is a ISO-subbundle of \((B, \tau, \mathcal{M})\) provided \(B'\) is a submanifold of \(B\), \(\mathcal{M}'\) is a submanifold of \(\mathcal{M}\) and \(\tau' = \tau|_{\mathcal{M}'}\) such that, for each bundle chart \(\varphi \in \text{Ch}(B', \mathcal{M}')\), we have \(\varphi = \varphi|_{\text{Dom}\varphi} \circ \varphi|_{\text{Cod}\varphi} \) for some bundle chart \(\phi \in \text{Ch}(B, \mathcal{M})\).

It is easily seen that for every open subset \(\mathcal{P}\) of \(\mathcal{M}\), \((\tau^{\mathcal{P}}, \tau|_{\mathcal{P}})\) is an open subbundle of \((B, \tau, \mathcal{M})\).

**Definition:** A cross section on \(\mathcal{O}\) of \(B\), where \(\mathcal{O}\) is an open submanifold of \(\mathcal{M}\), is a mapping \(s: \mathcal{O} \to B\) such that \(\tau \circ s = 1_{\mathcal{O} \subset \mathcal{M}}\). For every \(p \in 0..s\), we denote the collection of all \(C^p\) cross sections of \(B\) by \(\text{Sec}^p B\).

If ISO is the category DIF, that consists of all \(C^s\)-diffeomorphisms between \(C^s\) manifolds, we call \(B\) a \(C^s\)-bundle. If ISO = FIS, we call \(B\) a flat-space bundle. If ISO = LIS, we call \(B\) a linear-space bundle.

**Proposition 3:** Let \(D\) be an open subset of a flat space \(E\) and let \(V, W\) be linear spaces. Let \(F: D \to \text{Lin}(V, W)\) be given. If \(f: D \times V \to W\) is defined by
\[
f(x, v) := F(x)v \quad \text{for all } (x, v) \in D \times V
\]
then \(f\) is of class \(C^p\), \(p \in \), if and only if \(F\) is of class \(C^p\).

**Proof:** The assertion follows from the Partial Gradient Theorem [FDS].

If \(B\) is a linear-space bundle, then it follows from (22.3), (22.4) and Prop. 3 that for every pair of bundle charts \(\phi, \psi \in \text{Ch}(B, \mathcal{M})\), the mapping
\[
\psi \circ \phi: \mathcal{O}_{\phi} \cap \mathcal{O}_{\psi} \to \text{Lin}(V_{\psi}, V_{\phi})
\]
defined by
\[
(\psi \circ \phi)(x) := \psi \circ \phi^{-1} \quad \text{for all } x \in \mathcal{O}_{\phi} \cap \mathcal{O}_{\psi}
\]
is of class \(C^s\).

Before closing this section, we give two examples of constructing a new bundle from given ones. We omit the details.

**Examples :**

1. **Trivial bundles:** \(\mathcal{M} \times \mathcal{G}\), where \(\mathcal{G} \in \text{OBJ}\). The fiber \(\mathcal{B}_x = \{x\} \times \mathcal{G}\) at \(x \in \mathcal{M}\) is \(\mathcal{G}\) tagged with \(x\).
(2) **Fiber-product bundles**: Let two bundles \((A, \alpha, M)\) and \((B, \beta, M)\) over the same base manifold \(M\) be given. Put

\[
A \times_M B := \bigcup_{x \in M} A_x \times B_x
\]

\[
\alpha \times_M \beta := \alpha \circ \text{ev}_1 = \beta \circ \text{ev}_2
\]

\[
\begin{array}{ccc}
A \times_M B & \xrightarrow{ev_2} & B \\
\downarrow & & \downarrow \beta \\
\alpha & \xrightarrow{\alpha} & M
\end{array}
\]

The bundle \((A \times_M B, \alpha \times_M \beta, M)\) is called the **fiber-product bundle** of \((A, \alpha, M)\) and \((B, \beta, M)\). The bundle projection \(\alpha \times_M \beta : A \times_M B \to M\) is given by

\[
\alpha \times_M \beta(v) := \{ y \mid v \in A_y \times B_y \}.
\]

Let bundle charts \(\phi \in \text{Ch}(A, M)\) and \(\psi \in \text{Ch}(B, M)\) be given. The mapping

\[
\phi \times_M \psi : (\tau_1 \times_M \tau_2)^\circ (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \to (\mathcal{O}_\phi \cap \mathcal{O}_\psi) \times (\mathcal{V}_\phi \times \mathcal{V}_\psi)
\]

given by

\[
\phi \times_M \psi(v) = \left( y, (\phi|_y \times \psi|_y)v \right) \quad \text{for all} \quad v \in A \times_M B
\]

is a bundle chart for \((A \times_M B, \phi \times_M \psi, M)\).
23. The tangent bundle

Let \( r \in ^\infty \), a \( C^r \)-manifold \( M \), and a point \( x \in M \) be given.

**Definition:** The tangent space of \( M \) at \( x \) is defined to be

\[
T_xM := \left\{ t \in \times \right\} \text{ where the condition (23.2) holds },
\]

where the condition (23.2) is given by

\[
t_\chi = \nabla_\chi \gamma(x) t_\chi \quad \text{for all } \chi, \gamma \in \text{Ch}_xM.
\]

\( T_xM \) is endowed with the natural structure of a linear space as shown below and \( \dim T_xM = \dim_xM \).

For every \( \chi \in \text{Ch}_xM \), define the evaluation mapping \( \text{ev}_\chi : T_xM \to V_\chi \) by

\[
\text{ev}_\chi(t) := t_\chi \quad \text{for all } t \in T_xM.
\]

It follows from (21.10) that the evaluation mapping \( \text{ev}_\chi \) is invertible and that its inverse \( \text{ev}_\chi^{-1} : V_\chi \to T_xM \) is given by

\[
(\text{ev}_\chi^{-1})(u) = \left( \nabla_\chi \alpha(x) u \right| \alpha \in \text{Ch}_xM \) \quad \text{for all } u \in V_\chi.
\]

Hence we have

\[
\text{ev}_\chi \circ \text{ev}_\chi^{-1} = \nabla_\chi \chi(x) \in \text{Lis} (V_\gamma, V_\chi)
\]

for all \( \gamma, \chi \in \text{Ch}_xM \). It follows from that the linear-space structure on \( T_xM \) obtained from that of \( V_\chi \) by \( \text{ev}_\chi \) does not depend on the choice of \( \chi \in \text{Ch}_xM \) and hence is intrinsic to \( T_xM \). We consider \( T_xM \) to be endowed with this structure.

Let \( f \) be a mapping whose domain \( D \) is a neighborhood of \( x \) in \( M \) and whose codomain is an open subset of a flat space with translation space \( V \). It follows from (23.3) and (21.7) that

\[
\nabla_\chi f(x) \circ \text{ev}_\chi \in \text{Lin}(T_xM, V)
\]

is the same for all \( \chi \in \text{Ch}_xM \). Hence we may define the **gradient of \( f \) at \( x \)** by

\[
\nabla f := \nabla_\chi f(x) \circ \text{ev}_\chi \in \text{Lin}(T_xM, V)
\]

for all \( \chi \in \text{Ch}_xM \). In particular, if we put \( f := \chi \) we get \( \nabla_\chi \chi = \text{ev}_\chi \) and hence

\[
(\nabla_\chi \chi)t = t_\chi \quad \text{for all } \chi \in \text{Ch}_xM.
\]

Also, if \( f \) is given as above, we have

\[
\nabla f = \nabla_\chi f(x) \nabla_\chi \chi \quad \text{for all } \chi \in \text{Ch}_xM.
\]
Let $\mathcal{P}$ be an open neighborhood of $x$ in $\mathcal{M}$. By (21.6) we have $\text{Ch}_x \mathcal{P} \subset \text{Ch}_x \mathcal{M}$ and the mapping
\[
(t \mapsto t|_{\text{Ch}_x \mathcal{P}}) : T_x \mathcal{M} \to T_x \mathcal{P}
\]
is a natural bijection; we use it to identify
\[
T_x \mathcal{P} \cong T_x \mathcal{M}. \tag{23.7}
\]

**Definition:** The **tangent bundle** $T\mathcal{M}$ of $\mathcal{M}$ is defined to be the union of all the tangent spaces of $\mathcal{M}$:
\[
T\mathcal{M} := \bigcup_{x \in \mathcal{M}} T_x \mathcal{M}. \tag{23.8}
\]

It is endowed with the natural structure of a $C^{r-1}$-linear-space bundle as shown below.

In view of the identifications (23.7) we may regard $TP$ as a subset of $T\mathcal{M}$ when $\mathcal{P}$ is an open subset of $\mathcal{M}$.

Let $\mathcal{D}$ be an open subset of a flat space $\mathcal{E}$ with translation space $\mathcal{V} := \mathcal{E} - \mathcal{E}$. Then the singleton $\{1_{\mathcal{D}}\}$ is a $C^\infty$-atlas of $\mathcal{D}$. It determines on $\mathcal{D}$ a natural $C^\infty$-manifold structure and hence a natural $C^r$-manifold structure for every $r \in \mathbb{R}$. Given $x \in \mathcal{D}$, the linear isomorphism $\text{ev}_{1_{\mathcal{D}}} : T_x \mathcal{D} \to \mathcal{V}$ will be used for the identification
\[
T_x \mathcal{D} \cong \{x\} \times \mathcal{V}. \tag{23.9}
\]
Let $f$ be a mapping whose domain is an open neighborhood of $x$ and whose codomain is an open subset of a flat space $\mathcal{E}'$ with translation space $\mathcal{V}'$. If $f$ is differentiable at $x \in \mathcal{D}$ then the gradient $\nabla_x f$ in the ordinary sense of (23.4) belongs to $\text{Lin}(\{x\} \times \mathcal{V}, \mathcal{V}')$ when the identification (23.9) is used. No confusion is likely because we have
\[
\nabla_x f(x, v) = \nabla_x f(v) \quad \text{for all } v \in \mathcal{V} \tag{23.10}
\]
when $\nabla_x f$ is used with both meanings.

If $\mathcal{D}$ is the underlying manifold of an open subset of a flat space, then (23.9) gives rise to the identification
\[
T\mathcal{D} \cong \mathcal{D} \times \mathcal{V}. \tag{23.11}
\]

Note that the family $(T_x \mathcal{M} \mid x \in \mathcal{M})$ is disjoint. The **bundle projection** $pt : T\mathcal{M} \to \mathcal{M}$ of the tangent bundle is given by
\[
pt(t) : = \{ x \in \mathcal{M} \mid t \in T_x \mathcal{M} \}. \tag{23.12}
\]
Every manifold chart \( \chi \in \text{Ch}_M \) induces a bundle chart for \( T_M \) as shown in the following. We define the **tangent-bundle chart**

\[
tgt_\chi : \text{pt}^\ast(\text{Dom } \chi) \rightarrow \text{Dom } \chi \times \mathcal{V}_\chi
de fined by
\]

\[
tgt_\chi(t) = (z, (\nabla_z \chi)(t)) \quad \text{where} \quad z := \text{pt}(t).
\]

It is easily seen that \( tgt_\chi \) is invertible and that

\[
tgt_\chi^{-1}(z, u) = (\nabla_z \chi)^{-1}u
\]

for all \( z \in \text{Dom}_\chi \) and all \( u \in \mathcal{V}_\chi \). Let \( \chi, \gamma \in \text{Ch}_M \) be given. It follows from (21.7) and (23.6) that

\[
\nabla_\chi(z)(\gamma \circ \chi^{-1}) = (\nabla_\chi \gamma)(z) = (\nabla_\chi \nabla_z \chi)(\gamma^{-1}u)
\]

for all \( z \in \text{Dom } \gamma \cap \text{Dom } \chi \). Hence, by (23.14) and (23.15) with \( \chi \) replaced by \( \gamma \), we have

\[
(tgt_\gamma \circ tgt_\chi^{-1})(z, u) = (z, \nabla_\chi(z)(\gamma \circ \chi^{-1})u)
\]

for all \( z \in \text{Dom } \gamma \cap \text{Dom } \chi \) and all \( u \in \mathcal{V}_\chi \). It is clear that \( tgt_\gamma \circ tgt_\chi^{-1} \) is of class \( C^{r-1} \). Since \( \chi, \gamma \in \text{Ch}_M \) were arbitrary, it follows from (23.17) that

\[
\{ \text{tgt}_\alpha \mid \alpha \in \text{Ch}_M \}
\]

is a \( C^{r-1} \) bundle-atlas of \( T_M \). We consider \( T_M \) has being endowed with the \( C^{r-1} \) linear space bundle structure determined by this atlas.

It is also easily seen that \( \{ (\alpha \times 1_{\mathcal{V}_\alpha}) \circ \text{tgt}_\alpha \mid \alpha \in \text{Ch}_M \} \) is a \( C^{r-1} \) manifold-atlas of \( T_M \). If \( \chi \in \text{Ch}_M \) then the page of the manifold chart \( (\chi \times 1_{\mathcal{V}_\chi}) \circ \text{tgt}_\chi \) is

\[
\text{Pag} ((\chi \times 1_{\mathcal{V}_\chi}) \circ \text{tgt}_\chi) = \text{Pag} \chi \times \mathcal{V}_\chi
\]

and we have

\[
\mathcal{V}_{(\chi \times 1_{\mathcal{V}_\chi}) \circ \text{tgt}_\chi} = (\mathcal{V}_\chi)^2
\]

and hence

\[
\dim T_M = 2 \dim_{\text{pt}(t)} M \quad \text{for all} \quad t \in T_M.
\]

It is easily seen that the bundle projection \( \text{pt} : T_M \rightarrow M \) defined by (23.12) is of class \( C^{r-1} \).

Let \( r \in \) and \( C^r \)-manifolds \( M \) and \( M' \) be given. Let \( g \) be a mapping whose domain and codomain are open subsets of \( M \) and \( M' \), respectively. We say that \( g \) is of class \( C^s \) with \( s \in [0, r] \) if \( \chi' \circ g \circ \chi^{-1} \) is of class \( C^s \) in the ordinary sense for all \( \chi \in \text{Ch}_M \) and all \( \chi' \in \text{Ch}_M' \). This is the case if and only if \( \chi' \circ g \circ \chi^{-1} \) is of class \( C^s \) in the sense of Sect.21 for all \( \chi' \in \text{Ch}_M' \). Also, \( g \) is of class \( C^s \) if \( \chi' \circ g \circ \chi^{-1} \) is of class \( C^s \) for all \( \chi \) in some atlas included in \( \text{Ch}_M \) and for all
χ′ in some atlas included in ChM′. The notion of differentiability of g is defined in a similar way.

Assume that g is differentiable at x ∈ M. It follows from (23.16) that

$$\nabla_x g := (\nabla_{g(x)} \chi')^{-1} \nabla_{\chi(x)} (\chi' \circ g \circ \chi^{-1}) \nabla_x \chi$$  (23.21)

does not depend on the choice of χ ∈ Ch_xM and χ′ ∈ Ch_{g(x)}M′. We call

$$\nabla_x g \in \text{Lin (T}_x\text{M}, T_{g(x)}\text{M′)}$$  (23.22)

the gradient of g at x. Appropriate versions of the chain rule apply to gradients in this sense. If M′ is an open subset of a flat space E′ with translation space V′, then the gradient ∇x g in the sense of (23.22) is related to the gradient ∇x g in the sense of (23.4) by

$$(\nabla_x g)t = (g(x), (\nabla_x g)t) \quad \text{for all} \quad t \in T_xM$$  (23.23)

when the identification T_{g(x)}M′ ≅ \{g(x)\} × V′ is used.

**Definition:** A mapping h : M → TM is called a vector-field on M if it is a right-inverse of pt, i.e. if

$$h(x) \in T_xM \quad \text{for all} \quad x \in M.$$  (23.24)

If h and k are vector-fields, then h + k is the vector-field defined by value-wise addition, i.e. by (h + k)(x) := h(x) + k(x) for all x ∈ M. If h is a vector-field and f a real-valued function on M (often called a “scalar-field”), then f h is defined by value-wise scalar multiplication, i.e. by (f h)(x) := f(x)h(x) for all x ∈ M.

The set of all real-valued functions of class C^s, s ∈ 0..(r − 1), on M will be denoted by C^s(M). The set of all vector-fields of class C^s, s ∈ 0..(r − 1), on M will be denoted by X^s(TM). Using value-wise addition and multiplication, C^s(M) acquires the natural structure of a commutative algebra over . The constants form a subalgebra of C^s(M) that is isomorphic to . Using value-wise addition and multiplication, X^s(TM) acquires the natural structure of a C^s(M)-module.

Let h : M → TM be a vector-field and χ ∈ ChM. Define h^χ : Domχ → Vχ by

$$h^\chi(y) := (\nabla_\chi \chi)h(y) \quad \text{for all} \quad y \in \text{Dom} \chi.$$  (23.25)

Given x ∈ Dom χ, we define

$$\nabla_x h^\chi := (\nabla_x \chi)^{-1} \nabla_x h^\chi \in \text{Lin T}_x\text{M}.$$  (23.26)
It is easily seen from
\[
(\nabla_x \chi)^{-1} \nabla_x h^\chi = (\nabla_x \chi)^{-1} (\nabla_x h^\chi(x)) \nabla_x \chi
\]
that \(\nabla_x h^\chi\) is simply the ordinary gradient of \(h^\chi\) in the chart \(\chi\), transported from \(\text{Lin} V^\chi\) to \(\text{Lin} T_x M\) by \(\nabla_x \chi\).

A continuous mapping \(p : J \rightarrow M\) from some genuine interval \(J \in \mathbb{R}\) into the manifold \(M\) will be called a **process**. If \(p\) is differentiable at a given \(t \in J\), then
\[
\partial_t p := (\nabla_{p(t)} \chi)^{-1} \partial_t (\chi \circ p) \quad (23.27)
\]
does not depend on the choice of \(\chi \in \text{Ch}_{p(t)} M\). We call \(\partial_t p \in T_{p(t)} M\) the **derivative of \(p\) at \(t\)**. If \(p\) is differentiable, we define the **derivative** (-process) \(p' : J \rightarrow TM\) by
\[
p'(t) := \partial_t p \quad \text{for all } \quad t \in J. \quad (23.28)
\]

### 24. Tensor Bundles

We now assume that a number \(s \in \mathbb{R}\) and a \(C^s\) linear-space bundle \((B, \tau, M)\) are given.

With each analytic tensor functor \(\Phi\) one can construct what is called the associated **\(\Phi\)-bundle** of \(B\)
\[
\Phi(B) := \bigcup_{y \in M} \Phi(B_y). \quad (24.1)
\]
It has the natural structure of a \(C^s\) linear-space bundle over \(M\). For every open subset \(P\) of \(M\), we also use the following notation
\[
\Phi(\tau^<(P)) := \bigcup_{y \in P} \Phi(B_y). \quad (24.2)
\]

We define the **bundle projection** \(\tau^\Phi : \Phi(B) \rightarrow M\) of the bundle \(\Phi(B)\) by
\[
\tau^\Phi(v) \in \{ \ y \in M \mid v \in \Phi(B_y) \}. \quad (24.3)
\]
For every bundle chart \(\phi : \tau^<(\mathcal{O}_\phi) \rightarrow \mathcal{O}_\phi \times \mathcal{V}_\phi\), we have
\[
\phi(v) = (\ y, \phi|_y(t) \ ) \quad \text{where} \quad y := \tau(t)
\]
We define the mapping
\[
\Phi(\phi) : \Phi(\tau^<(\mathcal{O}_\phi)) \rightarrow \mathcal{O}_\phi \times \Phi(\mathcal{V}_\phi) \quad (24.4)
\]
by
\[(\Phi(\phi))(v) := (y, \Phi(\phi|_y)v) \quad \text{when} \quad y := \tau^\Phi(v). \quad (24.5)\]
It follows from the analyticity of the mapping \((L \mapsto \Phi(L))\) that
\[
\{ \Phi(\phi) \mid \phi \in \text{Ch}(\mathcal{B}, \mathcal{M}) \}
\]
is a \(C^s\)-bundle-atlas of \(\Phi(\mathcal{B})\). It determines the \(C^s\) linear-space bundle structure of \((\Phi(\mathcal{B}), \tau^\Phi, \mathcal{M})\).

The bundle projection \(\tau^\Phi : \Phi(\mathcal{B}) \to \mathcal{M}\) defined by (24.3) is easily seen to be of class \(C^s\).

**Notation:** For every \(p \in 0..s\), we denote the collection of all \(C^p\) cross sections of \(\Phi(\mathcal{B})\) by \(X^p(\Phi(\mathcal{B}))\). The collection of all differentiable cross sections of \(\Phi(\mathcal{B})\) is denoted by \(X(\Phi(\mathcal{B}))\).

In the special case \(\mathcal{B} = T\mathcal{M}\), we call \(\Phi(T\mathcal{M})\) the **tensor bundle** of \(\mathcal{M}\) of type \(\Phi\). A cross section of the tensor bundle \(\Phi(T\mathcal{M})\) is called a **tensor-field** of type \(\Phi\). When \(\Phi := Dl\) is the duality functor (see Sect.13), we call \(Dl(T\mathcal{M})\) the **cotangent bundle** of \(\mathcal{M}\) which will be denoted by \(T^*\mathcal{M}\).

**Remark:** Let \(\mathcal{M}\) be a \(C^\infty\)-manifold. With every \(h \in X^\infty(T\mathcal{M})\) we can then associate a mapping \(h^\nabla : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})\) defined by
\[
h^\nabla(f) := (\nabla f)h \quad \text{for all} \quad f \in C^\infty(\mathcal{M}) \quad (24.6)
\]
where the gradient \(\nabla f\) of \(f\) is the covector field of class \(C^\infty\) given by \(\nabla f(x) := \nabla_x f\) for all \(x \in \text{Dom } f\). It is clear that \(h^\nabla\) is \(-\)linear. By using the product rule \(\nabla(fg) = f\nabla g + g\nabla f\), we have
\[
h^\nabla(fg) = fh^\nabla(g) + gh^\nabla(f) \quad \text{for all} \quad f, g \in C^\infty(\mathcal{M}). \quad (24.7)
\]
This shows that \(h^\nabla\) is a derivation of the module \(C^\infty(\mathcal{M})\). One can prove that every derivation of \(C^\infty(\mathcal{M})\) can be obtained in this manner. (The proof is fairly difficult.)

Let a cross section section \(H : \mathcal{M} \to \Phi(\mathcal{B})\) be given. For every bundle chart \(\phi \in \text{Ch}(\mathcal{B}, \mathcal{M})\) we define the mapping
\[
H^\phi : O_\phi \to \Phi(V_\phi)
\]
by
\[
H^\phi(y) := \Phi(\phi|_y)H(y), \quad \text{for all} \quad y \in O_\phi. \quad (24.8)
\]
Given \(x \in O_\phi\), we define
\[
\nabla^\phi x H := \Phi(\phi|_x^{-1})\nabla_x H^\phi \in \text{Lin}(T_x \mathcal{M}, \Phi(\mathcal{B}_x)). \quad (24.9)
\]

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When $\Phi = \text{Id}$ and $\mathcal{B} = T\mathcal{M}$, we have $\nabla^\mathcal{B}_x h = \nabla^\chi_x h$ for all $\chi \in \text{Ch}\mathcal{M}$ and all $x \in \text{Dom} \chi$.

One defines value-wise addition of cross sections of $\Phi(\mathcal{B})$ and value-wise scalar multiplication of a real function on $\mathcal{M}$ and a cross section of $\Phi(\mathcal{B})$ in the obvious manner. $\mathcal{X}^p\Phi(\mathcal{B})$ has the natural structure of a $C^p(\mathcal{M})$-module, where $C^p(\mathcal{M})$ is the ring of all real-valued functions of class $C^p$ on $\mathcal{M}$.

Let $(\mathcal{L}_1, \tau_1, \mathcal{M})$ and $(\mathcal{L}_2, \tau_2, \mathcal{M})$ be linear-space bundles over $\mathcal{M}$ and let $\mathcal{L}_1 \times_{\mathcal{M}} \mathcal{L}_2$ be the fiber product bundle of $\mathcal{L}_1$ and $\mathcal{L}_2$. For every tensor bifunctor $\Upsilon$, it follows form (24.5) that for each bundle chart $\phi_1 \in \text{Ch}(\mathcal{L}_1, \mathcal{M})$ and each bundle chart $\phi_2 \in \text{Ch}(\mathcal{L}_2, \mathcal{M})$

$$\Upsilon(\phi_1 \times_{\mathcal{M}} \phi_2)(v) = (y, \Upsilon(\phi_1|_y \times \phi_2|_y)v)$$  \hspace{1em} (24.10)

where $y := (\tau_1 \times_{\mathcal{M}} \tau_2)\Upsilon(v)$ (see 24.3).

Let a cross section $H : \mathcal{M} \to \Upsilon(\mathcal{L}_1 \times_{\mathcal{M}} \mathcal{L}_2)$ be given. For each bundle chart $\phi_1 \in \text{Ch}(\mathcal{L}_1, \mathcal{M})$ and each bundle chart $\phi_2 \in \text{Ch}(\mathcal{L}_2, \mathcal{M})$, we define the mapping

$$H^{\phi_1, \phi_2} : \mathcal{O}_\phi \to \Upsilon(\mathcal{V}_{\phi_1} \times \mathcal{V}_{\phi_2})$$

by

$$H^{\phi_1, \phi_2}(y) := H(\phi_1|_y)H(y), \text{ for all } y \in \mathcal{O}_{\phi_1} \cap \mathcal{O}_{\phi_2}. \hspace{1em} (24.11)$$

Given $x \in \mathcal{O}_{\phi_1} \cap \mathcal{O}_{\phi_2}$, we define

$$\nabla^{\phi_1, \phi_2}_x H := \Upsilon(\phi_1|_x^{-1} \times \phi_2|_x^{-1})\nabla_x H^{\phi_1, \phi_2}$$  \hspace{1em} (24.12)

which is in $\text{Lin}(T_x\mathcal{M}, \Upsilon(\mathcal{L}_{1x} \times \mathcal{L}_{2x}))$. 

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