Chapter 1
Preliminaries

11. Multilinearity

Let \( (V_i \mid i \in I) \) be a family of linear spaces, we define (see (04.24) of [FDS]), for each \( j \in I \) and each \( v \in \times_{i \in I} V_i \), the mapping \( (v, j) : V_j \to \times_{i \in I} V_i \) by the rule
\[
((v, j))(u)_i := \begin{cases} v_i & \text{if } i \in I \setminus \{j\} \\ u & \text{if } i = j \end{cases}
\quad \text{for all } u \in V_j. \quad (11.1)
\]

**Definition**: Let the family \( (V_i \mid i \in I) \) and \( W \) be linear spaces. We say that the mapping \( M : \times_{i \in I} V_i \to W \) is **multilinear** if, for every \( v \in \times_{i \in I} V_i \) and every \( j \in I \) the mapping \( M \circ (v, j) : V_j \to W \) is linear, so that \( M \circ (v, j) \in \text{Lin}(V_j, W) \).

The set of all multilinear mappings from \( \times_{i \in I} V_i \) to \( W \) is denoted by
\[
\text{Lin}_I(\times_{i \in I} V_i, W). \quad (11.2)
\]

Let linear spaces \( V \) and \( W \) and a set \( I \) be given.

Let \( \text{Perm} I \) be the permutation group, which consists of all invertible mappings from \( I \) to itself. For every permutation \( \sigma \in \text{Perm} I \) we define a mapping \( T_\sigma : V^I \to V^I \) by
\[
T_\sigma(v) = v \circ \sigma \quad \text{for all } v \in V^I, \quad (11.3)
\]
that is \( (T_\sigma(v))_i := v_{\sigma(i)} \) for all \( i \in I \). In view of \( v \circ (\sigma \circ \rho) = (v \circ \sigma) \circ \rho \), we have \( T_{\sigma \rho} = T_\rho \circ T_\sigma \) for all \( \sigma, \rho \in \text{Perm} I \). It is not hard to see that, for every multilinear mapping \( M : V^I \to W \) and every permutation \( \sigma \), the composition \( M \circ T_\sigma \) is again a multilinear mapping from \( V^I \) to \( W \), i.e. \( M \circ T_\sigma \in \text{Lin}_I(V^I, W) \).

**Definition**: A multilinear mapping \( M : V^I \to W \) is said to be (completely) **symmetric** if
\[
M \circ T_\sigma = M \quad \text{for all } \sigma \in \text{Perm} I,
\]
and is said to be (completely) **skew** if
\[
M \circ T_\sigma = \text{sgn}(\sigma) M \quad \text{for all } \sigma \in \text{Perm} I.
\]

The set of all (completely) symmetric multilinear mappings and the set of all (completely) skew multilinear mappings from \( V^I \) to \( W \) will be denoted by \( \text{Sym}_I(V^I, W) \) and by \( \text{Skew}_I(V^I, W) \); respectively.
Both $\text{Sym}_I(V, W)$ and $\text{Skew}_I(V, W)$ are subspaces of the linear space $\text{Lin}_I(V, W)$ with dimensions

$$\dim \text{Sym}_I(V, W) = \left( \frac{\dim V + \#I - 1}{\#I} \right) \dim W \quad (11.4)$$

and

$$\dim \text{Skew}_I(V, W) = \left( \frac{\dim V}{\#I} \right) \dim W. \quad (11.5)$$

For every $k \in \mathbb{N}$, we write $\text{Lin}_k(V^k, W)$, $\text{Sym}_k(V^k, W)$ and $\text{Skew}_k(V^k, W)$ for $\text{Lin}_k(V, W)$, $\text{Sym}_k(V, W)$ and $\text{Skew}_k(V, W)$, respectively.

In applications, we often use the following identifications

$$\text{Lin}_k(V^k, W) \cong \text{Lin}_{k-1}(V^{k-1}, \text{Lin}(V, W)) \cong \text{Lin}(V, \text{Lin}_{k-1}(V^{k-1}, W))$$

and inclusions

$$\text{Sym}_k(V^k, W) \subset \text{Sym}_{k-1}(V^{k-1}, \text{Lin}(V, W)),$$

$$\text{Skew}_k(V^k, W) \subset \text{Skew}_{k-1}(V^{k-1}, \text{Lin}(V, W)).$$

In particular, we shall use $\text{Sym}_2(V^2, W) \cong \text{Sym}(V, V^*) := \text{Sym}(V, \text{Lin}(V, W))$ and $\text{Skew}_2(V^2, W) \cong \text{Skew}(V, V^*) := \text{Skew}(V, \text{Lin}(V, W))$. It can be shown that $\text{Skew}(V, V^*)$ has invertiable mapping if and only if $\dim V$ is even. (See Prop.3 of Sect.87, [FDS].)

Given a number $k \in \mathbb{N}$ and a multilinear mapping $A \in \text{Lin}_k(V^k, W)$, the mapping $\sum_{\sigma \in \text{Perm}_k} (\text{sgn} \, \sigma) \ A \circ T_\sigma : V^k \rightarrow W$ is a completely skew multilinear mapping. Moreover, it can be easily shown that

$$\frac{1}{k!} \sum_{\sigma \in \text{Perm}_k} (\text{sgn} \, \sigma) \ W \circ T_\sigma = W$$

for all skew multilinear mapping $W \in \text{Skew}_k(V^k, W)$.

**Definition:** Given a number $k \in \mathbb{N}$, we define the alternating assignment $\text{Alt} : \text{Lin}_k(V^k, W) \rightarrow \text{Skew}_k(V^k, W)$ by

$$\text{Alt} \, A := \frac{1}{k!} \sum_{\sigma \in \text{Perm}_k} (\text{sgn} \, \sigma) \ A \circ T_\sigma$$

(11.6)

for all linear spaces $V$ and $W$ and all $A \in \text{Lin}_k(V^k, W)$.

Given $p \in \mathbb{N}$. We define, for each $i \in (p+1)^1$, a mapping $\text{del}_i : V^{p+1} \rightarrow V^p$ by

$$(\text{del}_i(v))_j := \begin{cases} v_j & \text{if } 1 \leq i \leq j - 1 \\ v_{i+1} & \text{if } j \leq i \leq p \end{cases}$$

for all $v \in V^{p+1}$. (11.7)
Intuitively, del$_i(v)$ is obtained from $v$ by deleting the $i$-th term.

When the alternating assignment $\text{Alt}$ restricted to the subspace $\text{Lin}(V, \text{Skew}_p(V^p, W))$ of $\text{Lin}(V, \text{Lin}_p(V^p, W)) \cong \text{Lin}_{p+1}(V^{p+1}, W)$, we have

$$(p + 1) (\text{Alt } A)v = \sum_{i \in (p+1)^l} (-1)^{i-1} A(v_i, \text{del}_i v)$$  \hspace{1cm} (11.8)$$

for all $v \in V^{p+1}$ and all $A \in \text{Lin}(V, \text{Skew}_p(V^p, W))$. Similarly, when the alternating assignment $\text{Alt}$ restricted to the subspace $\text{Skew}_p(V^p, \text{Lin}(V, W))$ of $\text{Lin}(V, \text{Lin}_p(V^p, W)) \cong \text{Lin}_{p+1}(V^{p+1}, W)$, we have

$$(p + 1) (\text{Alt } B)v = \sum_{i \in (p+1)^l} (-1)^{p+1-i} B(\text{del}_i v, v_i)$$  \hspace{1cm} (11.9)$$

for all $v \in V^{p+1}$ and all $B \in \text{Skew}_p(V^p, \text{Lin}(V, W))$.

**Definition:** An algebra is a linear space $V$ together with a bilinear mapping $B \in \text{Lin}_2(V^2, V)$. An algebra $V$ is called a Lie Algebra if the bilinear mapping $B$ is skew-symmetric, i.e. $B \in \text{Skew}_2(V^2, V)$, and satisfies the Jacobi identity

$$B(B(v_1, v_2), v_3) + B(B(v_2, v_3), v_1) + B(B(v_3, v_1), v_2) = 0$$  \hspace{1cm} (11.10)$$

for all $v_1, v_2, v_3 \in V$.

By using the inclusion $\text{Skew}_2(V^2, V) \subset \text{Lin}(V, \text{Lin}(V, V))$ and (11.9), we see that (11.10) can be rewritten as

$$\text{Alt } (B \circ B) = 0$$  \hspace{1cm} (11.11)$$

where $(B \circ B)(v_1, v_2, v_3) := B(B(v_1, v_2), v_3)$ for all $v_1, v_2, v_3 \in V$.

**Remark 1:** In the literature the alternating assignment given in (11.6) is often called “skew-symmetric operator” ([B-W]), “complete antisymmetrization” ([F-C]). The symmetric assignment, “symmetric operator” or “complete symmetrization” $\text{Sym} : \text{Lin}_k(V^k, W) \rightarrow \text{Sym}_k(V^k, W)$ is given by

$$\text{Sym } M := \frac{1}{k!} \sum_{\sigma \in \text{Perm} k!} M \circ T_\sigma$$  \hspace{1cm} (11.12)$$

for all linear spaces $V$ and $W$ and all $M \in \text{Lin}_k(V^k, W)$.

**Remark 2:** Both assignments given in (11.6) and (11.12) are “natural linear assignments” from a functor to another functor (see (13.16) of Sect.13). More precisely, the alternating assignment is a natural linear assignment from the functor $\text{Lin}_k$ to the functor $\text{Sk}_k$ and the symmetric assignment is a natural linear assignment from the functor $\text{Lin}_k$ to the functor $\text{Sym}_k$ (see Sect. 13).
12. Isocategories, isofunctors and Natural Assignments

An isocategory* † is given by the specification of a class \( OBJ \) whose members are called objects, a class ISO whose members are called \( \text{ISOmorphisms} \),

(i) a rule that associates with each \( \phi \in \text{ISO} \) a pair \( (\text{Dom} \phi, \text{Cod} \phi) \) of objects, called the domain and codomain of \( \phi \),
(ii) a rule that associates with each \( A \in \text{OBJ} \) a member of ISO denoted by \( 1_A \) and called the identity of \( A \),
(iii) a rule that associates with each pair \( (\phi, \psi) \) in ISO such that \( \text{Cod} \phi = \text{Dom} \psi \) a member of ISO denoted by \( \psi \circ \phi \) and called the composite of \( \phi \) and \( \psi \), with \( \text{Dom}(\psi \circ \phi) = \text{Dom} \phi \) and \( \text{Cod}(\psi \circ \phi) = \text{Cod} \psi \).
(iv) a rule that associates with each \( \phi \in \text{ISO} \) a member of ISO denoted by \( \phi^{-1} \) and called the inverse of \( \phi \).

subject to the following three axioms:

(I1) \( \phi \circ 1_{\text{Dom} \phi} = \phi = 1_{\text{Cod} \phi} \circ \phi \) for all \( \phi \in \text{ISO} \),
(I2) \( \chi \circ (\psi \circ \phi) = (\chi \circ \psi) \circ \phi \) for all \( \phi, \psi, \chi \in \text{ISO} \) such that \( \text{Cod} \phi = \text{Dom} \psi \) and \( \text{Cod} \psi = \text{Dom} \chi \).
(I3) \( \phi^{-1} \circ \phi = 1_{\text{Dom} \phi} \) and \( \phi \circ \phi^{-1} = 1_{\text{Cod} \phi} \) for all \( \phi \in \text{ISO} \).

Given \( \phi \in \text{ISO} \), one writes \( \phi : A \rightarrow B \) or \( A \xrightarrow{\phi} B \) to indicate that \( \text{Dom} \phi = A \) and \( \text{Cod} \phi = B \).

There is one to one correspondence between an object \( A \in \text{OBJ} \) and the corresponding identity \( 1_A \in \text{ISO} \). For this reason, we will usually name an isocategory by giving the name of its class of \( \text{ISOmorphisms} \).

Let isocategories ISO and ISO' with object-classes \( \text{OBJ} \) and \( \text{OBJ}' \) be given. We can then form the product-isocategory ISO \( \times \) ISO' whose object-class \( \text{OBJ} \times \text{OBJ}' \) consists of pairs \( (A,A') \) with \( A \in \text{OBJ} \), \( A' \in \text{OBJ}' \) and \( \text{ISO} \)-morphism-class \( \text{ISO} \times \text{ISO}' \) consists of pairs \( (\phi, \phi') \) with \( \phi \in \text{ISO} \), \( \phi' \in \text{ISO}' \) and the following

(a) For every \( (\phi, \phi') \in \text{ISO} \times \text{ISO}' \), \( \text{Dom}(\phi, \phi') := (\text{Dom} \phi, \text{Dom} \phi') \)
and \( \text{Cod}(\phi, \phi') := (\text{Cod} \phi, \text{Cod} \phi') \).

* A category, introduced by Eilenberg and MacLane, is defined by (i), (ii) and (iii) with the axioms (I1) and (I2). Roughly speaking, an isocategory is a special category whose “morphisms” are called ISO-morphisms.
† Since isocategories are widely used in differential geometry, we introduced them directly instead of making them as a special category.
(b) Composition in $\text{ISO} \times \text{ISO}'$ is defined by termwise composition, i.e. by $(\psi, \psi') \circ (\phi, \phi') := (\psi \circ \phi, \psi' \circ \phi')$ for all $\phi, \psi \in \text{ISO}$ and $\phi', \psi' \in \text{ISO}'$ such that $\text{Dom}(\psi, \psi') = \text{Cod}(\phi, \phi')$.

(c) The identity of a given pair $(\mathcal{A}, \mathcal{A}') \in \text{OBJ} \times \text{OBJ}'$ is defined to be $1_{(\mathcal{A}, \mathcal{A}')} = (1_{\mathcal{A}}, 1_{\mathcal{A}'})$.

The product of an arbitrary family of isocategories can be defined in a similar manner. In particular, if a isocategory $\text{ISO}$ and an index set $I$ are given, one can form the $I$-power-isocategory $\text{ISO}^I$ of $\text{ISO}$; its isomorphism-class consists of all families in $\text{ISO}$ indexed on $I$. In the case when $I$ is of the form $I := n$, we write $\text{ISO}^n := \text{ISO}^n$ for short. For example, we write $\text{ISO}^2 := \text{ISO} \times \text{ISO}$. We identify $\text{ISO}^1$ with $\text{ISO}$ and $\text{ISO}^{m+n}$ with $\text{ISO}^m \times \text{ISO}^n$ for all $m, n \in \text{in the obvious manner.}$ The isocategory $\text{ISO}^0$ is the trivial one whose only object is $\emptyset$ and whose only isomorphism is $1_{\emptyset}$.

A functor $\Phi$ is given by the specification of:

(i) a pair $(\text{Dom} \Phi, \text{Cod} \Phi)$ of categories, called the domain-category and codomain-category of $\Phi$,

(ii) a rule that associates with every $\phi \in \text{Dom} \Phi$ a member of $\text{Cod} \Phi$ denoted by $\Phi(\phi)$,

subject to the following conditions:

(F1) We have $\text{Cod} \Phi(\phi) = \text{Dom} \Phi(\psi)$ and $\Phi(\psi \circ \phi) = \Phi(\psi) \circ \Phi(\phi)$ for all $\phi, \psi \in \text{Dom} \Phi$ such that $\text{Cod} \phi = \text{Dom} \psi$.

(F2) For every identity $1_A$ in $\text{Dom} \Phi$, where $A$ belongs to the object-class of $\text{Dom} \Phi$, $\Phi(1_A)$ is an identity in $\text{Cod} \Phi$.

An isofunctor is a functor whose domain-category and codomain-category are isocategories. In this book we only deal with isofunctors.

Let isocategories $\text{ISO}$ and $\text{ISO}'$ with object-classes $\text{OBJ}$ and $\text{OBJ}'$ be given. We say that $\Phi$ is an isofunctor from $\text{ISO}$ to $\text{ISO}'$ and we write $\text{ISO} \xrightarrow{\Phi} \text{ISO}'$ or $\Phi : \text{ISO} \longrightarrow \text{ISO}'$ to indicate that $\text{ISO} = \text{Dom} \Phi$ and $\text{ISO}' = \text{Cod} \Phi$. By (F2), we can associate with each $A \in \text{OBJ}$ exactly one object in $\text{OBJ}'$, denoted by $\Phi(A)$, such that

$$\Phi(1_A) = 1_{\Phi(A)}.$$  \hspace{1cm} (12.1)

It easily follows from (I3), (F1) and (F2) that every isofunctor $\Phi$ satisfies

$$\Phi(\phi^{-1}) = (\Phi(\phi))^{-1} \text{ for all } \phi \in \text{Dom} \Phi.$$  \hspace{1cm} (12.2)

One can construct new isofunctors from given isofunctors in the same way as new mappings are constructed from given mappings. (See, for example, Sect. 03
Thus, if $\Phi$ and $\Psi$ are isofunctors such that $\text{Cod } \Phi = \text{Dom } \Psi$, one can define the **composite isofunctor** $\Psi \circ \Phi : \text{Dom } \Phi \to \text{Cod } \Psi$ by

$$\Psi \circ \Phi(\phi) := \Psi(\Phi(\phi)) \quad \text{for all } \phi \in \text{Dom } \Phi$$

(12.3)

Also, given isofunctors $\Phi$ and $\Psi$, one can define the **product-isofunctor**

$$\Phi \times \Psi : \text{Dom } \Phi \times \text{Dom } \Psi \to \text{Cod } \Phi \times \text{Cod } \Psi$$

of $\Phi$ and $\Psi$ by

$$(\Phi \times \Psi)(\phi, \psi) := (\Phi(\phi), \Psi(\psi))$$

(12.4)

for all $\phi \in \text{Dom } \Phi$ and all $\psi \in \text{Dom } \Psi$.

Product-isofunctors of arbitrary families of isofunctors are defined in a similar way. In particular, if a isofunctor $\Phi$ and an index set $I$ are given, we define the **$I$-power-isofunctor** $\Phi^{\times I} : (\text{Dom } \Phi)^I \to (\text{Cod } \Phi)^I$ of $\Phi$ by

$$\Phi^{\times I}(\phi_i \mid i \in I) = (\Phi(\phi_i) \mid i \in I)$$

(12.5)

for all families $(\phi_i \mid i \in I)$ in $\text{Dom } \Phi$. We write $\Phi^{\times n} := \Phi^{\times n}$ when $n \in \mathbb{N}$.

We now assume that an isocategory $\text{ISO}$ with object-class $\text{OBJ}$ is given. The **identity-isofunctor** $\text{Id} : \text{ISO} \to \text{ISO}$ of $\text{ISO}$ is defined by

$$\text{Id}(\phi) = \phi \quad \text{for all } \phi \in \text{ISO}.$$  

(12.6)

We then have

$$\text{Id}(A) = A \quad \text{for all } A \in \text{OBJ}.$$  

(12.7)

If $I$ is an index set, then the identity-isofunctor of $\text{ISO}^I$ is $\text{Id}^{\times I}$. In particular, the identity-isofunctor of $\text{ISO} \times \text{ISO}$ is $\text{Id} \times \text{Id}$.

Given an object $C \in \text{OBJ}$. The **trivial-isofunctor** $\text{Tr}_C : \text{ISO} \to \text{ISO}$ for $C$ is defined by

$$\text{Tr}_C(\phi) = 1_C \quad \text{for all } \phi \in \text{ISO}.$$  

(12.8)

We then have

$$\text{Tr}_C(A) = C \quad \text{for all } A \in \text{OBJ}.$$  

(12.9)

One often needs to consider a variety of “accounting isofunctors” whose domain and codomain isocategories are obtained from $\text{ISO}$ by product formation. For example, the **switch-isofunctor** $\text{Sw} : \text{ISO}^2 \to \text{ISO}^2$ is defined by

$$\text{Sw}(\phi, \psi) := (\psi, \phi) \quad \text{for all } \phi, \psi \in \text{ISO}.$$  

(12.10)

Given any index set $I$, the **equalization-isofunctor** $\text{Eq}_I : \text{ISO} \to \text{ISO}^I$ is defined by

$$\text{Eq}_I(\phi) := (\phi \mid i \in I) \quad \text{for all } \phi \in \text{ISO}.$$  

(12.11)
We write $E_{q_n} := E_{q_{n-1}}$ when $n \in \mathbb{N}$.

Let a index set $I$ and a family $(\Phi_i \mid i \in I)$ of isofunctors, with $\text{Dom} \Phi_i = \text{ISO}$ for all $i \in I$, be given. We then identify the family $(\Phi_i \mid i \in I)$ with the termwise-formation isofunctor

$$ (\Phi_i \mid i \in I) : \text{ISO} \to \bigotimes_{i \in I} \text{Cod} \Phi_i $$

defined by

$$ (\Phi_i \mid i \in I) := \bigotimes_{i \in I} \Phi_i \circ E_{q_I}, $$

so that

$$ (\Phi_i \mid i \in I)(\phi) = \bigotimes_{i \in I} \Phi_i(\phi), \quad \text{for all } \phi \in \text{ISO}. \quad (12.12) $$

In particular, if $I = 2^I$, we then identify the pair $(\Phi_1, \Phi_2)$ with the pair-formation isofunctor $(\Phi_1, \Phi_2) : \text{ISO} \to \text{Cod} \Phi_1 \times \text{Cod} \Phi_2$.

Let isofunctors $\Phi$ and $\Psi$, both from $\text{ISO}$ to $\text{ISO}'$, be given. A natural assignment $\alpha$ form $\Phi$ to $\Psi$ is a rule that associates with each object $F$ of $\text{ISO}$ a mapping

$$ \alpha_F : \Phi(F) \to \Psi(F), $$

such that

$$ \Psi(\chi) \circ \alpha_{\text{Dom } \chi} = \alpha_{\text{Cod } \chi} \circ \Phi(\chi) \quad \text{for all } \chi \in \text{ISO}; \quad (12.13) $$
i.e. the diagram

$$ \begin{align*}
\Phi(\text{Dom } \chi) & \quad \xrightarrow{\alpha_{\text{Dom } \chi}} \quad \Psi(\text{Dom } \chi) \\
\Phi(\chi) & \downarrow \quad \quad \downarrow \Psi(\chi) \\
\Phi(\text{Cod } \chi) & \xrightarrow{\alpha_{\text{Cod } \chi}} \quad \Psi(\text{Cod } \chi)
\end{align*} $$

is commutative. We write $\alpha : \Phi \to \Psi$ to indicate that $\Phi$ is the domain isofunctor, denoted by $\text{Dmf}_\alpha$, and $\Psi$ is the codomain isofunctor, denoted by $\text{Cdf}_\alpha$.

One can construct new natural assignments from given ones in the same way as new mappings from given ones. Let natural assignments $\alpha : \Phi \to \Psi$ and $\beta : \Psi \to \Theta$ be given. We can define the composite assignment $\beta \circ \alpha : \Phi \to \Theta$, by assigning to each object $F$ of $\text{Dom } \Phi = \text{Dom } \Psi$ the mapping $(\beta \circ \alpha)_F := \beta_F \circ \alpha_F$. If $\alpha, \beta$ are natural assignment, one can define the product-assignment $\alpha \times \beta$ by assigning to each pair $(F, G)$ of objects the mapping $(\alpha \times \beta)_{(F, G)} := \alpha_F \times \beta_G$.

Given a natural assignment $\alpha : \Phi \to \Psi$ and a isofunctor $\Theta$ such that $\text{Cod } \Theta = \text{Dom } \Phi = \text{Dom } \Psi$, one can define the composite assignment
\( \alpha \circ \Theta : \Phi \circ \Theta \rightarrow \Psi \circ \Theta \) by assigning to each object \( \mathcal{F} \) of \( \text{Dom} \Phi = \text{Dom} \Psi \) the mapping \( (\alpha \circ \Theta)_\mathcal{F} := \alpha_{\psi(\mathcal{F})} \).

### 13. Tensor Functors

We say that an isocategory ISO is **concrete** if ISO consists of mappings, the object-class \( \text{OBJ} \) consists of sets, and if domain and codomain, composition, identity and inverse have the meaning they are usually given for sets and mappings. (See, e.g. Sect. 01 – 04 of [FDS]).

**Examples of concrete isocategory**

The following are some concrete isocategories to be used in this book:

(A) The category FIS whose object-class \( FS \) consists of all finite dimensional flat spaces over and whose ISOmorphism-class FIS consists of all flat isomorphism from one such space onto another or itself.

(B) Fix a field and we consider the concrete isocategory whose object-class \( LS \) consists of all finite dimensional linear spaces over and whose ISOmorphism-class LIS consists of all linear isomorphism from one such space onto another or itself.

(C) Given \( s \in \), the category DIF\( s \) whose object-class \( DF \) consists of all \( C^s \) manifolds and whose ISOmorphism-class DIF\( s \) consists of all diffeomorphism from one such manifold onto another or itself.

From now on, *in this section*, we will deal only with LIS and the categories obtained from it by product formation, such as \( LIS^m \times LIS^n \) when \( m, n \in \). We use the term **tensor functor of degree** \( n \in \) for functor from \( LIS^n \) to LIS. (Under this definition, composition of tensor functors is somewhat strange: the second one of those functors must be of degree 1!!!!!!!!!!!!)

**Examples of tensor functor**

Here is a list of important tensor functors used in linear algebra and differential geometry:

(1) The **product-space functor** \( \text{Pr} : LIS^2 \rightarrow LIS \). It is defined by

\[
\text{Pr}(A, B) := A \times B \quad \text{for all} \quad (A, B) \in LIS^2.
\]

We have \( \text{Pr}(\mathcal{V}, \mathcal{W}) := \mathcal{V} \times \mathcal{W} \) (the *product-space* of \( \mathcal{V} \) and \( \mathcal{W} \)) for all \( \mathcal{V}, \mathcal{W} \in LS \).
(2) Given $k \in \mathbb{N}$, the **$k$-lin-map-functor** $\text{Lin}_k : \text{LIS}^k \times \text{LIS} \to \text{LIS}$. It assigns to each list $(\mathcal{V}_i | i \in k^1)$ in $\text{LS}$ and each $\mathcal{W} \in \text{LS}$ the linear space

$$\text{Lin}_k((\mathcal{V}_i | i \in k^1), \mathcal{W}) := \text{Lin}_k\left(\times_{i \in k^1} \mathcal{V}_i, \mathcal{W}\right)$$

(13.2)

of all $k$-multilinear mappings from $\times_{i \in k^1} \mathcal{V}_i$ to $\mathcal{W}$, and it assigns to every list $(\mathcal{A}_i | i \in k^1)$ in $\text{LIS}$ and each $\mathcal{B} \in \text{LIS}$ the linear mapping

$$\text{Lin}_k((\mathcal{A}_i | i \in k^1), \mathcal{B})$$

(13.3)

from $\text{Lin}_k\left(\times_{i \in k^1} \text{Dom} \mathcal{A}_i, \text{Dom} \mathcal{B}\right)$ to $\text{Lin}_k\left(\times_{i \in k^1} \text{Cod} \mathcal{A}_i, \text{Cod} \mathcal{B}\right)$ defined by

$$\text{Lin}_k((\mathcal{A}_i | i \in k^1), \mathcal{B})T := \text{BT} \circ \times_{i \in k^1} \mathcal{A}_i^{-1}$$

(13.4)

for all $T \in \text{Lin}\left(\times_{i \in k^1} \text{Dom} \mathcal{A}_i, \text{Dom} \mathcal{B}\right)$. When $k = 1$, $\text{Lin}_1 : \text{LIS} \times \text{LIS} \to \text{LIS}$ is called the **lin-map-functor** and abbreviated by $\text{Lin} := \text{Lin}_1$.

(3) Given $k \in \mathbb{N}$, the **$k$-multilin-functor** $\text{Ln}_k : \text{LIS}^2 \to \text{LIS}$. It is defined by

$$\text{Ln}_k := \text{Lin}_k \circ (\text{Eq}_k \times \text{Id}).$$

(13.5)

We have

$$\text{Ln}_k(\mathcal{A}, \mathcal{B})T := \text{BT} \circ (\mathcal{A}^{-1})^{\times k}$$

(13.6)

for all $\mathcal{A}, \mathcal{B} \in \text{LIS}$ and all $T \in \text{Lin}_k((\text{Dom} \mathcal{A})^k, \text{Dom} \mathcal{B})$. and

$$\text{Ln}_k(\mathcal{V}, \mathcal{W}) := \text{Lin}_k(\mathcal{V}^k, \mathcal{W})$$

(13.7)

for all $\mathcal{V}, \mathcal{W} \in \text{LS}$

There are two very important “**subfunctors**” (see [E-M]), $\text{Sm}_k$ and $\text{Sk}_k$, given in following. The **symmetric-$k$-multilin-functor** $\text{Sm}_k : \text{LIS}^2 \to \text{LIS}$ assigns to every pair of linear spaces $(\mathcal{V}, \mathcal{W}) \in \text{LS}^2$ the linear space

$$\text{Sm}_k(\mathcal{V}, \mathcal{W}) := \text{Sym}_k(\mathcal{V}^k, \mathcal{W})$$

(13.8)

of all symmetric $k$-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$. It is clear that

$$\text{Sm}_k(\mathcal{A}, \mathcal{B})T := \text{BT} \circ (\mathcal{A}^{-1})^{\times k}$$

(13.9)

for all $\mathcal{A}, \mathcal{B} \in \text{LIS}$ and all $T \in \text{Sym}_k((\text{Dom} \mathcal{A})^k, \text{Dom} \mathcal{B})$. The **skew-$k$-multilin-functor** $\text{Sk}_k : \text{LIS}^2 \to \text{LIS}$ is defined in the same manner as $\text{Sm}_k$, except that $\text{Sym}_k(\mathcal{V}^k, \mathcal{W})$ in (13.8) is replaced by the linear space $\text{Skew}_k(\mathcal{V}^k, \mathcal{W})$ of all skew $k$-multilinear mappings from $\mathcal{V}^k$ to $\mathcal{W}$.
Given \( n \in \mathbb{N} \), the \textbf{\( k \)-linform-functor} \( \text{Ln}_k \), the \textbf{\( k \)-symform-functor} \( \text{Smf}_k \), the \textbf{\( k \)-skewform-functor} \( \text{Skf}_k \), all from LIS to LIS. They are defined by

\[
\text{Ln}_k := \text{Ln} \circ (\text{Id}, \text{Tr}) , \quad \text{Smf}_k := \text{Sm} \circ (\text{Id}, \text{Tr}) , \quad \text{Skf}_k := \text{Sk} \circ (\text{Id}, \text{Tr}).
\]

Given \( V \in \mathcal{L}S \), we have

\[
\text{Ln}_k(V) := \text{Lin}_k(V, \cdot),
\]

the space of all \( k \)-multilinear forms on \( V^k \). We have

\[
\text{Ln}_k(A) \omega := \omega \circ (A^{-1})^k \quad \text{for all} \quad \omega \in \text{Lin}_k((\text{Dom} A)^k, \cdot)
\]

and all \( A \in \text{LIS} \). The formulas (13.11) and (13.12) remain valid if Lin is replaced by Sym or Skew and Ln by Smf or Skf correspondingly.

When \( k = 1 \), we have \( \text{Lnf}_1 = \text{Smf}_1 = \text{Skf}_1 \) which is called the \textbf{duality-functor} and denoted by \( Dl : \text{LIS} \to \text{LIS} \).

(5) The \textbf{lineon-functor} \( \text{Ln} : \text{LIS} \to \text{LIS} \). It is defined by

\[
\text{Ln} := \text{Lin} \circ \text{Eq}_2.
\]

We have

\[
\text{Ln}(V) := \text{Lin}(V, V) \quad \text{for all} \quad V \in \mathcal{L}S
\]

and

\[
\text{Ln}(A)T := AT A^{-1} \quad \text{for all} \quad A \in \text{LIS} \quad \text{and} \quad T \in \text{Ln}(\text{Dom} A).
\]

Remark : In much of the literature (see [K-N], Sect. 2 of Ch.I or [M-T-W], §3.2) the use of the term “tensor” is limited to tensor functors of the form \( T^r_s := \text{Lin} \circ (\text{Lnf}_r, \text{Lnf}_s) : \text{LIS} \to \text{LIS} \) with \( r, s \in \mathbb{N} \), or to tensor functors that are naturally equivalent to one of this form. Given \( V \in \mathcal{L}S \) a member of the linear space \( T^r_s(V) \) is called a “tensor of contravariant order \( r \) and covariant order \( s \).”

Let a family of tensor functors \( \{ \Phi_i \mid i \in k^l \} \) and a tensor functor \( \Psi \) with \( \text{Dom} \times_{i \in k^l} \Phi_i = \text{LIS}^k = \text{Dom} \Psi \) be given. We say that a natural assignment \( \beta : \times_{i \in k^l} \Phi_i \to \Psi \) is a \textbf{\( k \)-linear assignment} if, for every \( F \in \mathcal{L}S^k \), the mapping

\[
\beta_F : \times_{i \in k^l} \Phi_i(F_i) \to \Psi(F)
\]

is \( k \)-linear.

The following are examples for bilinear natural assignments.
(6) Given \( k \in \), the **alternating assignment** \( \text{Alt}: \text{Lin}_k \to \text{Sk}_k \) it assigns each pair \((V, W) \in \text{LS}^2\) the mapping

\[
\text{Alt}_{(V, W)} A := \sum_{\sigma \in \text{Perm } k} (\text{sgn } \sigma) A \circ T_\sigma
\]

(13.17)

where \(\text{Perm } k\) is the permutation group of \( k\) and \( T_\sigma \) is defined as in (11.3), for all \( A \in \text{Lin}_k(V^k, W)\).

(7) The **tensor product** \( tpr: \text{Id} \times \text{Id} \to \text{Lin} \circ (\text{Dl} \times \text{Id}) \circ \text{Sw} \) assigns each pair \((V, W) \in \text{LS}^2\) the mapping

\[
tpr_{(V, W)} : V \times W \to \text{Lin}(W^*, V)
\]

(13.18)

defined by

\[
tpr_{(V, W)}(v, w) := v \otimes w \quad \text{for all} \quad v \in V \text{ and } w \in W,
\]

(13.19)

where \( v \otimes w \) is the tensor product defined according to Def. 1 of Sect. 25, [FDS], with the identification \( W \cong W^{**} \).

We use \( v \otimes w \in \text{Lin}(W^*, V) \) but others use \( v \otimes w \in \text{Lin}(V^*, W) \) (see e.g. [B-W]). Our definition of \( \otimes \) bring up the switch functor Sw here!!!!!!!!!!!!!!!

The **wedge product** \( wpr: \text{Id} \times \text{Id} \to \text{Lin} \circ (\text{Dl} \times \text{Id}) \circ \text{Sw} \) is defined by

\[
wpr_{(V, W)}(v, w) := v \wedge w \quad \text{for all} \quad v \in V \text{ and } w \in W,
\]

(13.20)

where \( v \wedge w \) is the wedge product defined according to (12.9) of Sect. 12, [FDS], Vol.2, with the identification \( W \cong W^{**} \).

We have \( wpr = \frac{1}{2} \text{Alt} \circ tpr \). Need more development!!!!!!!!!!!!!!!!!

We now assume that the field relative to which \( \text{LS} \) and \( \text{LIS} \) are defined in above is the field of real number. Given \( V, W \in \text{LS} \), the set

\[
\text{Lis}(V, W) := \{ A \in \text{LIS} \mid \text{Dom } A = V, \text{Cod } A = W \}
\]

(13.21)

is then an open subset of the linear space \( \text{Lin}(V, W) \). (See, for example, the Differentiation Theorem for Inversion Mappings in Sect.68 of [FDS].).

Let a tensor functor \( \Phi \) be given. For every pair of objects \((V, W) \in \text{Dom } \Phi\), we define the mapping

\[
\Phi_{(V, W)} : \text{Lis}(V, W) \to \text{Lis}(\Phi(V), \Phi(W))
\]

(13.22)

by

\[
\Phi_{(V, W)}(A) := \Phi(A) \quad \text{for all} \quad A \in \text{Lis}(V, W).
\]

(13.23)
Indeed, we can view (13.22) as a bilinear assignment from Lin = Ln_1 to Lin ◦ (Φ × Φ). The one to be used in (13.27)

$$\Phi_{(V,V)} : \text{Lis}(V) \to \text{Lis}(\Phi(V))$$

is a linear assignment from Ln to Ln ◦ Φ and hence whose gradient is also a linear assignment from Ln to Ln ◦ Φ.

We say that the tensor functor Φ is analytic if Φ(V, W) is an analytic mapping for every pair of objects (V, W) of Dom Φ. We say that a natural assignment α : Φ → Ψ is an analytic assignment if the mapping α_Φ : Φ(F) → Ψ(F) is an analytic mapping for every object F of Dom Φ. All the tensor functors listed in above are in fact analytic. (The fact that they are of class C^∞ can easily be inferred from the results of Ch.6 of [FDS]. Proofs that they are analytic can be inferred, for example, from the results that will be presented in Ch.2 of Vol.2 of [FDS].)

**Theorem:** Let an analytic tensor functor Φ be given and associate with each V ∈ Dom Φ the mapping

$$\Phi^\bullet_V : \text{Ln}(V) \to \text{Ln}(\Phi(V))$$

(13.24)

defined by

$$\Phi^\bullet_V := \nabla_1 \Phi_{(V,V)}.$$  (13.25)

(The gradient-notation used here is explained in [FDS], Sect.63.) Then $\Phi^\bullet$ is a linear assignment from Ln to Ln ◦ Φ. We call $\Phi^\bullet$ the derivative of Φ.

**Proof:** Let a pair of objects (V, W) of Dom Φ and A ∈ Lis(V, W) be given. It follows from (13.23), from axiom (F1), and from (12.2) that

$$\Phi_{(W,W)}(A L A^{-1}) = \Phi(A) \Phi_{(V,V)}(L) \Phi(A)^{-1}$$  (13.26)

for all L ∈ Lis(V, V). By (13.15) we may write (13.26) as

$$((\Phi_{(W,W)} \circ \text{Ln}(A))(L) = (\text{Ln}(\Phi(A)) \circ \Phi_{(V,V)})(L)$$  (13.27)

for all L ∈ Lis(V, V). Taking the gradient of (13.27) with respect to L at L := 1_V yields

$$\Phi^\bullet_W \circ \text{Ln}(A) = (\text{Ln} \circ \Phi)(A) \circ \Phi^\bullet_V.$$  (13.28)

In view of (12.13) it follows that $\Phi^\bullet$ is a natural assignment from Ln to Ln ◦ Φ. The linearity of $\Phi^\bullet$ follows from the definition of gradient.

We now list the derivatives of a few analytic tensor functors. The formulas given are valid for every V ∈ LS.
(6) $\text{Ln}_V^\bullet : \text{Ln}(\mathcal{V}) \to \text{Ln}(	ext{Ln}(\mathcal{V}))$ is given by
\[
(\text{Ln}_V^\bullet L)M = LM - ML \quad \text{for all} \quad L, M \in \text{Ln}(\mathcal{V}) \tag{13.29}
\]
(This formula is an easy consequence of (13.15) and, [FDS] (68.9).)

(7) Let $k \in \mathbb{N}$ be given. In order to describe
\[
(\text{Ln}_k^\bullet)^\bullet : \text{Ln}(\mathcal{V}) \to \text{Ln}(\text{Lin}_k(\mathcal{V}^k)),
\]
we define, for every $L \in \text{Ln}(\mathcal{V})$ and every $j \in k^\prime$, $D_j(L) \in (\text{Lin}_k(\mathcal{V}^k))^k$ by
\[
(D_j(L))_i := \begin{cases} 
L & \text{if } i = j \\
1_V & \text{if } i \neq j
\end{cases} \quad \text{for all} \quad i \in k^\prime. \tag{13.31}
\]
We then have
\[
((\text{Ln}_k^\bullet)^\bullet L)\omega = -\sum_{j \in k^\prime} \omega \circ D_j(L) \quad \text{for all} \quad \omega \in \text{Lin}_k(\mathcal{V}^k), \tag{13.32}
\]
and all $L \in \text{Ln}(\mathcal{V})$. The formula (13.32) remains valid if $\text{Ln}_f$ is replaced by $\text{Sm}_f$ or $\text{Sk}_f$ and $\text{Lin}$ by $\text{Sym}$ or $\text{Skew}$, correspondingly.

The General Chain Rule for gradients (see [FDS], Sect.63) and the definition (13.25) immediately lead to the following

**Chain Rule for Analytic Tensor Functors**

Let $\Phi$ and $\Psi$ be analytic tensor functors. Then the composite functor $\Psi \circ \Phi$ is also an analytic tensor functor and we have
\[
(\Psi \circ \Phi)^\bullet = (\Psi^\bullet \circ \Phi) \circ \Phi^\bullet, \tag{13.33}
\]
where the composite assignments on the right are explained in the end of Sect.12.

For example, (13.33) shows that, for each $\mathcal{V} \in \mathcal{LS}$,
\[
(\text{Ln} \circ \text{Ln})^\bullet \mathcal{V} : \text{Ln}(\mathcal{V}) \to \text{Ln}(	ext{Ln}(\text{Ln}(\mathcal{V})))
\]
is given by
\[
(\text{Ln} \circ \text{Ln})^\bullet \mathcal{V} = \text{Ln}_{\text{Ln}(\mathcal{V})}^\bullet \text{Ln}_\mathcal{V}^\bullet. \tag{13.34}
\]
In view of (13.29.) above, (13.34) gives
\[
(((\text{Ln} \circ \text{Ln})^\bullet \mathcal{V} L)K)M = ((\text{Ln}_\mathcal{V}^\bullet L)K - K(\text{Ln}_\mathcal{V}^\bullet L))M \\
= L(KM) - (KM)L - K(LM - ML) \tag{13.35}
\]
for all \( V \in L\mathcal{S} \), all \( K \in \text{Ln}(\text{Ln}(V)) \), and all \( L, M \in \text{Ln}(V) \).

If \( \Phi \) and \( \Psi \) are analytic tensor functors so is \( \text{Pr} \circ (\Phi, \Psi) \) and we have

\[
(\text{Pr} \circ (\Phi, \Psi))^*_V = (\Phi^*_V L) \times 1_{\Psi(V)} + 1_{\Psi(V)} \times (\Phi^*_V L)
\]

for all \( V \in L\mathcal{S} \) and all \( L \in \text{Ln}(V) \).

Let \( \alpha \) be an analytic assignment of degree \( n \in \mathbb{N} \). If we associate with each \( V \in L\mathcal{S} \) the mapping \((\nabla \alpha)_V := \nabla(\alpha_V)\), the gradient of the mapping \( \alpha_V \), then \( \nabla \alpha \) is again an analytic assignment of degree \( n \) and we have \( \text{Dmf}_{\nabla \alpha} = \text{Dmf}_\alpha \) and \( \text{Cdf}_{\nabla \alpha} = \text{Lin} \circ (\text{Dmf}_\alpha, \text{Cdf}_\alpha) \). We call \( \nabla \alpha \) the gradient of \( \alpha \).

Let tensor functors \( \Phi_1, \Phi_2, \Psi \), all of degree \( n \in \mathbb{N} \) but not necessarily analytic, be given. Each bilinear assignment \( \beta : \text{Pr} \circ (\Phi_1, \Phi_2) \rightarrow \Psi \) is then analytic and its gradient \( \nabla \beta : \text{Pr} \circ (\Phi_1, \Phi_2) \rightarrow \text{Lin} \circ (\text{Pr} \circ (\Phi_1, \Phi_2), \Psi) \) is given by

\[
((\nabla \beta)_V(v_1, v_2))(u_1, u_2) = \beta_v(v_1, u_2) + \beta_v(u_1, v_2)
\]

for all \( V \in L\mathcal{S} \), all \( v_1, u_1 \in \Phi_1(V) \), and all \( v_2, u_2 \in \Phi_2(V) \).

If \( \alpha \) is an analytic assignment of degree \( n \in \mathbb{N} \) and if \( \Phi \) is any isofunctor from \( \text{LIS}^k \) to \( \text{LIS}^n \) with \( k \in \mathbb{N} \), then \( \alpha \circ \Phi \) is an analytic assignment of degree \( k \) and we have \( \nabla(\alpha \circ \Phi) = (\nabla \alpha) \circ \Phi \).
14. Short Exact Sequences

Let a pair \((I, P)\) of mappings be given such that \(\text{Cod} \, I = \text{Dom} \, P\). We often write
\[
U \xrightarrow{I} V \xrightarrow{P} W \quad \text{or} \quad W \xleftarrow{P} V \xleftarrow{I} U \tag{14.1}
\]
to indicate that \(U = \text{Dom} \, I, V = \text{Cod} \, I = \text{Dom} \, P\) and \(\text{Cod} \, P = W\). If \(U, V\) and \(W\) are linear spaces and if \(I\) is injective linear mapping, \(P\) is surjective linear mapping with
\[
\text{Rng} \, I = \text{Null} \, P,
\]
we say that \((I, P)\), or \((14.1)\), is a short exact sequence *. In the literature, a short exact sequence is often expressed as
\[
0 \rightarrow U \xrightarrow{I} V \xrightarrow{P} W \rightarrow 0.
\]

Let a short exact sequence \(U \xrightarrow{I} V \xrightarrow{P} W\) be given.

**Notation:** The set of all linear right-inverses of \(P\) is denoted by
\[
\text{Riv}(P) := \{ K \in \text{Lin} (W, V) \mid PK = 1_W \}, \tag{14.2}
\]
and the set of all linear left-inverses of \(I\) is denoted by
\[
\text{Liv}(I) := \{ D \in \text{Lin} (V, U) \mid DI = 1_U \}. \tag{14.3}
\]

**Proposition 1:** There is a bijection \(\Lambda : \text{Riv}(P) \rightarrow \text{Liv}(I)\) such that, for every \(K \in \text{Riv}(P)\)
\[
U \xleftarrow{K} V \xleftarrow{\Lambda(K)} W \tag{14.4}
\]
is again a short exact sequence. We have
\[
KP + I \Lambda(K) = 1_V \quad \text{for all} \quad K \in \text{Riv}(P). \tag{14.5}
\]

**Proof:** It is easily seen that \((K \mapsto \text{Rng} \, K)\) is a bijection from \(\text{Riv}(P)\) to the set of all supplements of \(\text{Null} \, P = \text{Rng} \, I\) in \(V\). Also, \((D \mapsto \text{Null} \, D)\) is a bijection from \(\text{Liv}(I)\) to the set of all supplements of \(\text{Rng} \, I = \text{Null} \, P\) in \(V\). The mapping \(\Lambda\) is the composite of the first of these bijections with the inverse of the second one.

* The term short exact sequence comes from the more general concept of an “exact sequence” which is not needed here.
Let $K \in \text{Riv}(P)$ be given. Both $KP$ and $I\Lambda(K)$ are idempotents with $\text{Rng} KP = \text{Rng} K$ and $\text{Rng} I\Lambda(K) = \text{Rng} I$. Since $\text{Rng} K$ and $\text{Rng} I$ are supplementary in $V$, it follows that

$$KP + I\Lambda(K) = 1_V. \quad (14.6)$$

Since $K \in \text{Riv}(P)$ was arbitrary, the assertion follows.$\blacksquare$

**Proposition 2:** $\text{Riv}(P)$ is a flat in $\text{Lin}(W, V)$ whose direction space is

$$\{ IL \mid L \in \text{Lin}(W, U) \}.$$

$\text{Liv}(I)$ is a flat in $\text{Lin}(V, U)$ whose direction space is

$$\{- LP \mid L \in \text{Lin}(W, U) \}.$$

**Proof:** Given $K, K' \in \text{Riv}(P)$, we have $1_V = PK = PK'$ and hence $P(K - K') = 0$. It follows that $\text{Rng} (K - K') \subset \text{Null} P = \text{Rng} I$ and hence $K - K' = IL$ for some $L \in \text{Lin}(W, U)$. On the other hand, given $K \in \text{Riv}(P)$ and $L \in \text{Lin}(W, U)$, we have $P(IL) = 0$ and hence $1_W = PK = P(K + IL)$, which implies $K + IL \in \text{Riv}(P)$. These facts show that $\text{Riv}(P)$ is a flat in $\text{Lin}(W, V)$ with direction space $\{ IL \mid L \in \text{Lin}(W, U) \}$.

Similar arguments show that $\text{Liv}(I)$ is a flat in $\text{Lin}(V, U)$ with direction space $\{- LP \mid L \in \text{Lin}(W, U) \}$.$\blacksquare$

**Proposition 3:** Let $K$ and $K'$ in $\text{Riv}(P)$ be given and determine $L \in \text{Lin}(W, U)$ such that $K - K' = IL$. Then

$$\Lambda(K) - \Lambda(K') = -LP. \quad (14.7)$$

**Proof:** It follows from (14.5) that $KP + I\Lambda(K) = 1_V = K'P + I\Lambda(K')$ and hence

$$I(\Lambda(K) - \Lambda(K')) = -(K - K')P.$$ 

Since $K - K' = IL$ and $I$ is injective, we obtain $\Lambda(K) - \Lambda(K') = -LP$.$\blacksquare$

It follows from the injectivity of $I$ and from the surjectivity of $P$ that both the direction space $\{ I \} \text{Lin}(W, U)$ of $\text{Riv}(P)$ and the direction space $\text{Lin}(W, U) \{ P \}$ of $\text{Liv}(I)$ are naturally isomorphic to $\text{Lin}(W, U)$. Hence we may and will consider $\text{Lin}(W, U)$ to be the external translation space (see Conventions and Notations) of both $\text{Riv}(P)$ and $\text{Liv}(I)$. We have

$$\dim \text{Riv}(P) = (\dim W)(\dim U) = \dim \text{Liv}(I). \quad (14.8)$$
Proposition 4: The mapping $\Lambda : \text{Riv}(P) \to \text{Liv}(I)$, as described in Prop. 1, is a flat isomorphism whose gradient $\nabla \Lambda \in \text{Lin}(\text{Lin}(W,U))$ is $-\mathbf{1}_{\text{Lin}(W,U)}$, so that
\[ \nabla \Lambda(L) = -L \quad \text{for all} \quad L \in \text{Lin}(W,U). \quad (14.9) \]

Proof: It follows from Prop. 2 and the identification $\text{Lin}(W,U)\{P\} \cong \text{Lin}(W,U)$ that $\Lambda : \text{Riv}(P) \to \text{Liv}(I)$ is a flat isomorphism with $\nabla \Lambda = -\mathbf{1}_{\text{Lin}(W,U)}$. \hfill \qed

Notation: Let $K \in \text{Riv}(P)$ be given. We define the mapping $\Gamma^K : \text{Riv}(P) \to \text{Lin}(W,U)$ by
\[ \Gamma^K(K') := -\Lambda(K)K' \quad \text{for all} \quad K' \in \text{Riv}(P). \quad (14.10) \]

Proposition 5: For every $K \in \text{Riv}(P)$, the mapping $\Gamma^K : \text{Riv}(P) \to \text{Lin}(W,U)$ is a flat isomorphism whose gradient $\nabla \Gamma^K \in \text{Lin}(\text{Lin}(W,U))$ is $-\mathbf{1}_{\text{Lin}(W,U)}$; i.e.
\[ \nabla \Gamma^K(L) = -L \quad \text{for all} \quad L \in \text{Lin}(W,U). \]

Proof: Let $K_1, K_2 \in \text{Riv}(P)$ be given; then we determine $L \in \text{Lin}(W,U)$ such that $K_1 - K_2 = IL$. It follows from (14.10) and $\Lambda(K)I = \mathbf{1}_U$ that
\[ \Gamma^K(K_1) - \Gamma^K(K_2) = -\Lambda(K)(K_1 - K_2) = -\Lambda(K)(IL) = -L. \]

Since $K_1, K_2 \in \text{Riv}(P)$ were arbitrary, the assertion follows. \hfill \qed

Proposition 6: We have
\[ K - K' = I \Gamma^K(K') \]
\[ \Lambda(K) - \Lambda(K') = -\Gamma^K(K')P \quad (14.11) \]
and hence $\Gamma^K'(K) = -\Gamma^K(K')$ for all $K, K' \in \text{Riv}(P)$. Moreover,
\[ \Gamma^{K_1}(K_3) - \Gamma^{K_2}(K_3) = \Gamma^{K_1}(K_2) \quad (14.12) \]
for all $K_1, K_2, K_3 \in \text{Riv}(P)$.

Proof: In view of (14.5) and (14.10), we have
\[ K - K' = (KP - 1_U)K' = -I\Lambda(K)K' = I\Gamma^K(K') \]
17.
for all $K', K \in \operatorname{Riv}(P)$. The second equation (14.11) follows from (14.11) and Prop. 2 with $L$ replaced by $\Gamma^K(K')$.

We observe from (14.11) that

$$I\Gamma^K_1(K_2) = K_1 - K_2 = (K_1 - K_3) - (K_2 - K_3)$$

for all $K_1, K_2, K_3 \in \operatorname{Riv}(P)$. Since $I$ is injective, (14.12) follows.

Remark: We consider $\operatorname{Lin}(W, U)$ to be the external translation space of $\operatorname{Riv}(P)$. Given $K \in \operatorname{Riv}(P)$, in view of (14.11), we have

$$\Gamma^K(K') = K - K' \quad \text{for all} \quad K' \in \operatorname{Riv}(P).$$

Roughly speaking, the flat isomorphism $\Gamma^K: \operatorname{Riv}(P) \to \operatorname{Lin}(W, U)$ identify $\operatorname{Riv}(P)$ with $\operatorname{Lin}(W, U)$ by choosing $K$ as the “zero” (or “origin”).

15. Brackets and Twists

We assume now that linear spaces $V, W$ and $Z$ and a short exact sequence

$$\operatorname{Lin}(W, Z) \xrightarrow{I} V \xrightarrow{P} W$$

are given. Recall from Prop. 1 of Sec. 14 that to every linear right-inverse $K$ of $P$ there corresponds exactly one linear left-inverse $\Lambda(K)$ of $I$ such that

$$\operatorname{Lin}(W, Z) \xrightarrow{\Lambda(K)} V \xrightarrow{K} W$$

is again a short exact sequence. In view of the identification

$$\operatorname{Lin}(W, \operatorname{Lin}(W, Z)) \cong \operatorname{Lin}_2(W^2, Z)$$

we may identify the external translation space $\operatorname{Lin}(W, \operatorname{Lin}(W, Z))$ of $\operatorname{Riv}(P)$ with $\operatorname{Lin}_2(W^2, Z)$.

Assumption: From now on, we assume that in this section, a flat $F$ in $\operatorname{Riv}(P)$ with direction space $\{I\} \operatorname{Sym}_2(W^2, Z)$ is given. Here $\operatorname{Sym}_2(W^2, Z)$ is regarded as a subspace of $\operatorname{Lin}_2(W^2, Z) \cong \operatorname{Lin}(W, \operatorname{Lin}(W, Z))$.

**Proposition 1:** For every $K_1, K_2 \in F$,

$$(\Lambda(K_1)v)(Pv') - (\Lambda(K_1)v')(Pv) = (\Lambda(K_2)v)(Pv') - (\Lambda(K_2)v')(Pv)$$

holds for all $v, v' \in V$. 18
Proof: Let \( K_1, K_2 \in \mathcal{F} \) be given. Then we determine \( L \in \text{Sym}_2(W^2, Z) \) such that \( K_1 - K_2 = IL \). It follows from Prop.3 of Sect.14 that

\[
(\Lambda(K_1)v)(Pv') - (\Lambda(K_2)v)(Pv') = -L(Pv, Pv')
\]

holds for all \( v, v' \in V \). By interchanging \( v \) and \( v' \) and observing that \( L \) is symmetric, we conclude that (15.4) follows. \( \blacksquare \)

Definition: In view of Prop. 1, the \( \mathcal{F} \)-bracket \( B_\mathcal{F} \in \text{Skw}_2(V^2, Z) \) can be defined such that

\[
B_\mathcal{F}(v, v') := (\Lambda(K)v)(Pv') - (\Lambda(K)v')(Pv) \quad \text{for all } v, v' \in V \quad (15.5)
\]

is valid for all \( K \in \mathcal{F} \). Using the identification (15.3) we also have

\[
B_\mathcal{F} \in \text{Lin}(V, \text{Lin}(V, Z)).
\]

Proposition 2: The \( \mathcal{F} \)-bracket \( B_\mathcal{F} \in \text{Lin}(V, \text{Lin}(V, Z)) \) satisfies

\[
\begin{align*}
B_\mathcal{F}(IM) &= MP \quad \text{for all } M \in \text{Lin}(W, Z), \\
(B_\mathcal{F}v)K &= \Lambda(K)v \quad \text{for all } K \in \mathcal{F} \text{ and all } v \in V.
\end{align*}
\]

(15.6)

If \( \dim Z \neq 0 \), then \( B_\mathcal{F} \) is injective; i.e. \( \text{Null } B_\mathcal{F} = \{0\} \).

Proof: The equations (15.6)_1 and (15.6)_2 follow from Definition (15.5) together with \( \Lambda(K)I = 1_{\text{Lin}(W, Z)} \) and \( PK = 1_V \), respectively.

Let \( v \in \text{Null } B_\mathcal{F} \) be given, so that \( B_\mathcal{F}v = 0 \) and hence

\[
0 = (B_\mathcal{F}v)IM = B_\mathcal{F}(v, IM) = -(B_\mathcal{F}(IM)v)
\]

for all \( M \in \text{Lin}(W, Z) \). Using (15.6)_1, it follows that \( -MPv = 0 \) for all \( M \in \text{Lin}(W, Z) \), which can happen, when \( \dim Z \neq 0 \), only if \( Pv = 0 \) and hence \( v \in \text{Null } P = \text{Rng } I \). Thus we may choose \( M' \in \text{Lin}(W, Z) \) such that \( v = IM' \) and hence \( B_\mathcal{F}(IM') = 0 \). Using (15.6)_1 again, it follows that \( M'P = 0 \). Since \( P \) is surjective , we conclude that \( M' = 0 \) and hence \( v = 0 \). Since \( v \in \text{Null } B_\mathcal{F} \) was arbitrary, it follows that \( \text{Null } B_\mathcal{F} = \{0\} \). \( \blacksquare \)

Definition: The \( \mathcal{F} \)-twist

\[
T_\mathcal{F} : \text{Riv}(P) \to \text{Skw}_2(W^2, Z)
\]

is defined by

\[
T_\mathcal{F}(K) := -B_\mathcal{F} \circ (K \times K) \quad \text{for all } K \in \text{Riv}(P),
\]

(15.7)
where $B_F$ is the $F$-bracket defined by (15.5).

**Proposition 3:** For every $H \in F$, we have

$$T_F = \Gamma^H - \Gamma^H^-$$

(15.9)

where $\sim$ denotes the value-wise switch, so that $\Gamma^H^-(K)(s, t) = \Gamma^H(K)(t, s)$ for all $K \in \text{Riv}(P)$ and all $s, t \in W$.

**Proof:** Let $K \in \text{Riv}(P)$ and $s, t \in W$ be given. By (15.8) and (15.5), we see that for every $H \in F$ we have

$$T_F(K)(s, t) = -B_F(Ks, Kt) = -\Lambda(H)(Ks)P(Kt) + \Lambda(H)(Kt)P(Ks).$$

(15.10)

We conclude from $PK = 1_W$, (15.10) and (14.10) that

$$T_F(K)(s, t) = \Gamma^H(K)(s, t) - \Gamma^H(K)^-(s, t).$$

Since $s, t \in W$ and $K \in \text{Riv}(P)$ were arbitrary, (15.9) follows.

**Remark:** It is clear from (15.9) and (11.6) that

$$T_F = 2 \text{ Alt} \circ \Gamma^H \quad \text{for all } H \in F.$$

The numerical factor 2 is conventional which reduces numerical factors in calculations.

**Proposition 4:** The $F$-torsion $T_F$ is a surjective flat mapping whose gradient

$$\nabla T_F \in \text{Lin}\left(\text{Lin}_2(W^2, Z), \text{Skw}_2(W^2, Z)\right)$$

is given by

$$(\nabla T_F)L = L^- - L$$

(15.11)

for all $L \in \text{Lin}_2(W^2, Z)$.

**Proof:** Let $H \in F$ be given. It follows from (15.8) and (15.5)

$$T_F(H - \frac{1}{2}IL) = L \quad \text{for all } L \in \text{Skw}_2(W^2, Z)$$

and hence $T_F$ is surjective.

Prop. 3 together with Prop. 4 in Sec. 14 shows that the $F$-torsion $T_F$ is a flat mapping whose gradient is given by (15.11).

In view of definitions (15.8), (15.5) and (15.11), we have $T_F^<(\{0\}) = F$.  

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**Definition:** We say that $K \in \text{Riv}(P)$ is $\mathcal{F}$-twist-free (or $\mathcal{F}$-symmetric) if $T_{\mathcal{F}}(K) = 0$, i.e. if $K \in \mathcal{F}$.

$\mathcal{F}$ is a flat in Riv($P$) with the (external) direction space Sym$_2 (W^2, Z)$ and hence

$$\dim T_{\mathcal{F}} \subseteq \{0\} = \dim \text{Sym}_2 (W^2, Z) = \frac{n(n + 1)}{2} m, \quad (15.12)$$

where $n := \dim W$ and $m := \dim Z$. The mapping

$$S_{\mathcal{F}} := \left(1_{\text{Riv}(P)} + \frac{1}{2} IT_{\mathcal{F}}\right) \mid T_{\mathcal{F}} \subseteq \{0\} \quad (15.13)$$

is the projection of Riv($P$) onto $T_{\mathcal{F}} \subseteq \{0\}$ with Null $\nabla S_{\mathcal{F}} = \text{Skw}_2 (W^2, Z)$. If $K \in \text{Riv}(P)$, we call

$$S_{\mathcal{F}}(K) = K + \frac{1}{2} I(T_{\mathcal{F}}(K))$$

the $\mathcal{F}$-symmetric part of $K$. 

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