Summary of Day 21

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1 Objectives

• Look at projections between two vectors, and generalize to projection of a vector on a space.

2 Summary

• In another class you might have explored the idea of a projection of one vector onto another. Let us explore that idea

• We can see the projection of \( v \) onto \( u \) is given by:

\[
\text{proj}_u v = \left( \frac{u \cdot v}{|u|^2} \right) u
\]

• We can extent this idea to the projection of a vector onto a space.

Let \( v \) be a vector of \( \mathbb{R}^n \) and \( W \) a subspace and \( \{ u_1, \ldots, u_k \} \) is an orthogonal basis for \( W \) then we say the orthogonal projection of \( v \) onto \( W \) is defined to be:

\[
\text{proj}_W(v) = \text{proj}_{u_1}(v) + \cdots + \text{proj}_{u_k}(v)
\]

A worry, of course, is that this might depend on the basis. We will see that it does not.

Further we define the component of \( v \) orthogonal to \( W \) as:

\[
\text{perp}_W(v) = v - \text{proj}_W(v)
\]

Example Let \( W \) be a plane in \( \mathbb{R}^3 \) given by the following orthogonal basis:

\[
u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\]
Let
\[ v = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \]
Find the orthogonal projection of \( v \) onto \( W \) and the component of \( v \) orthogonal to \( W \).

• Notice that \( \text{proj}_W(v) + \text{perp}_W(v) = v \). That is, there is a decomposition of \( v \) in terms of a vector on the subspace \( W \) and some other vector.

**Theorem** (Orthogonal Decomposition Theorem) If \( W \) is a subspace of \( \mathbb{R}^n \) and \( v \) is a vector of \( \mathbb{R}^n \) then there are unique vectors \( w \) and \( w \perp \in W^\perp \) such that
\[ v = w + w \perp \]

*Proof.*

\[ \square \]

**Remark** There is a problem with this proof that we’ll have to fix later. Do you see what it is?

This gives us the following as well:

**Theorem** Let \( W \) be a subspace of \( \mathbb{R}^n \). Then:
\[ \dim(W) + \dim(W^\perp) = n \]

*Proof.*

\[ \square \]
• The above actually gives a quite short proof of the rank nullity theorem since 
(row((A)))^\perp = (A).

• We will now try to fix the problem presented in the last section: we don’t know how
to find orthogonal bases for spaces. We actually don’t even know if it is in principle
always to find them.

• Idea: We want to be able to take a basis for some subspace $W$ of $\mathbb{R}^n$ and transform
it to an orthogonal set of vectors. We will do this using an algorithm called The
Gram-Schmidt process.

The Gram-Schmidt process is iterative. We will construct our vectors one at a time.
The input to our algorithm is a basis \{x_1, \ldots, x_k\}. The output of our algorithm
will be a list of $k$ (why $k$?) many vectors \{v_1, \ldots, v_k\}. Here’s how it goes:

1. First choose $v_1 := x_1$. Note: span($v_1$) = span($x_1$). Set $W_1 := \{x_1\}$.
2. For choose $v_2$ we choose it as follows:
   \[ v_2 = x_2 - \text{proj}_{v_1}(x_2) = \text{perp}_{v_1}(x_2) \]
   Note: span($v_1, v_2$) = span($x_1, x_2$) Set $W_2 := \{x_1, x_2\}$
3. Iterating... Choose $v_i$ as follows:
   \[ v_i = x_i - \sum_{j=1}^{i-1} \text{proj}_{v_j}(x_i) \]
   \[ = x_i - \text{proj}_{W_{i-1}}(x_i) \]
   \[ = \text{perp}_{W_{i-1}}(x_i) \]

**Theorem** Gram-Schmidt is correct; meaning, the set \{v_1, \ldots, v_k\} is a orthogonal
basis.

**Proof.** (this will be a more informal argument)
Example Use Gram-Schmidt to construct an orthonormal basis for the span of:

\[ \mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \mathbf{x}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix} \]