1 Objectives

- Do an example of diagonalization.
- Rediscover inner products and talk about orthogonality

2 Summary

- We begin where we left off yesterday: exploring diagonalization.

**Theorem** If $A$ is a $n \times n$ matrix with $n$ eigenvalues with multiplicity, then the following are equivalent:

1. $A$ is diagonalizable
2. $\mathbb{R}^n$ has a basis of eigenvectors of $A$.
3. Each eigenvalue of $A$ has algebraic multiplicity equal to its geometric multiplicity.

**Example** Determine if these matrices are diagonalizable. If they are, they diagonalize them.

(a) \[
\begin{pmatrix}
0 & 1 \\
2 & 1 \\
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 0 \\
0 & 1 & 2 \\
0 & 0 & 2 \\
\end{pmatrix}
\]
• We will now talk about orthogonality (this is 5.1). We begin with revisiting the notion of the dot product.

• Recall: For vectors \( v = [v_1, \ldots, v_n] \) and \( u = [u_1, \ldots, u_n] \) of \( \mathbb{R}^n \) we define the **dot product** of \( u \) and \( v \) by:

\[
\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^{n} v_i u_i
\]

We say they are **orthogonal** if \( \mathbf{v} \cdot \mathbf{u} = 0 \). We now extend this definition to a set.

• A set of vectors \( S = \{v_1, \ldots, v_m\} \) of \( \mathbb{R}^n \) is an **orthogonal set** if the vectors in the set are pairwise orthogonal. That is:

\[
v_i \cdot v_j = 0 \text{ if } i \neq j
\]

**Example** The following three vectors form an orthogonal set:

\[
\begin{align*}
v_1 &= \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \\
v_2 &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\
v_3 &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}
\end{align*}
\]

• Geometrically, the next theorem is fairly intuitive:

**Theorem** If \( S = \{v_1, \ldots, v_k\} \) is an orthogonal set of vectors then \( S \) is linearly independent.

**Proof.**

• Recall that a basis is a linearly independent set that spans the space. The most used bases is the standard basis which is:

\[
\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}
\]

These vectors form an orthogonal set and a basis. Such bases are very useful, which motivates the next definition:
• A basis \( B \) is an **orthogonal basis** of a subspace \( W \) of \( \mathbb{R}^n \) if it is also orthogonal.

**Example**

• The standard basis is pretty useful because we can easily write vectors as a linear combination of it. For example, \([2, 3] = 2[1, 0] + 3[0, 1] \). All bases enjoy the property of being able to write every member uniquely, but most of the time it requires solving a system to find the coefficients. For the standard basis, this is not the case.

This is a property of all orthogonal bases.

**Theorem** Let \( W \) be subspace of \( \mathbb{R}^n \) with orthogonal basis \( \{v_1, \ldots, v_k\} \). Then for each \( w \in W \) there is a unique \( c_1, \ldots, c_k \) such that

\[
  c_1 v_1 + \cdots + c_k v_k = w
\]

Moreover:

\[
  c_i = \frac{w \cdot v_i}{v_i \cdot v_i}
\]

**Proof.**

**Remark** The formula above may look familiar if you took any classes that talked about vector geometry. It is the projection of \( w \) onto \( v_i \). We will talk about this soon, no worries.

• Something else from the above formula looks familiar. Recall that we can define a **norm** of \( \mathbb{R}^n \) in the following way:

\[
  \|x\| = x \cdot x
\]

We say that a vector is a **unit vector** if it has norm 1.

**Remark** The standard basis consists of orthogonal unit vectors. This motivates the next definition:

• A basis is called an **orthonormal basis** if it is an orthogonal basis consisting of unit vectors.

**Remark** Let \( \{v_1, \ldots, v_k\} \) be such a basis. Complete the following formula:

\[
  v_i \cdot v_j = \left\{ \begin{array}{ll}
  1 & \text{if } i = j \\
  0 & \text{if } i \neq j
\end{array} \right.
\]

In the event we have an orthonormal basis, the above theorem gets simpler:

**Theorem** Let \( W \) be subspace of \( \mathbb{R}^n \) with orthonormal basis \( \{v_1, \ldots, v_k\} \). Then for each \( w \in W \) there is a unique \( c_1, \ldots, c_k \) such that

\[
  c_1 v_1 + \cdots + c_k v_k = w
\]

Moreover:

\[
  c_i = w \cdot v_i
\]