

Summary of Day 5

William Gunther

May 23, 2014

1 Objectives

- Solidify understanding of the idea of span and linear combinations.
- Define the notion of linear independence and linear dependence.
- Prove some theorems involving linear dependence/independence.

2 Summary

- Let's begin with an example:

Example Calculate the span of $[0, 1]$ and $[1, 1]$.

Solution. The span is the set of all linear combinations; thus:

$$\text{span}(\{ [0, 1], [1, 1] \}) = \{ \mathbf{v} \in \mathbb{R}^2 \mid \exists c_1, c_2 \in \mathbb{R}. \mathbf{v} = c_1[0, 1] + c_2[1, 1] \}$$

Thus, we need to figure out for which vectors \mathbf{v} we can find a c_1 and c_2 such that the following vector equation holds:

$$c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{v} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We can realize this as a system of linear equations:

$$\begin{aligned} c_2 &= v_1 \\ c_1 + c_2 &= v_2 \end{aligned}$$

which, in term, we can realize as an augmented matrix and use Gauss-Jordan to solve:

$$\left(\begin{array}{cc|c} 0 & 1 & v_1 \\ 1 & 1 & v_2 \end{array} \right) \xrightarrow{R_2 - R_1} \left(\begin{array}{cc|c} 0 & 1 & v_1 \\ 1 & 0 & v_2 - v_1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 0 & v_2 - v_1 \\ 0 & 1 & v_1 \end{array} \right)$$

We can see that this system has a solution no matter what v_1 and v_2 are. Thus, every \mathbf{v} can be written as a linear combination of the vectors $[0, 1]$ and $[1, 1]$. We can even see what the constants are: if you wanted to figure out what linear combination of these vectors hits, $[v_1, v_2]$ it is:

$$(v_2 - v_1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (v_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We can visualize this example as follows:

PICTURE

You can see that the vectors $[0, 1]$ and $[1, 1]$ kind of make their own coordinate system in \mathbb{R}^2 ; saying the span of these vectors is all of \mathbb{R}^2 is saying that *any* vector can be written in this coordinate system. In this case when the span of a set is the whole space we say that the set **spans** the space, and we call the set a **spanning set**.

- The most used spanning set of \mathbb{R}^2 is the vectors $[0, 1]$ and $[1, 0]$ which give us our usual coordinate system of vectors. These vectors are called the **standard unit vectors**, or the **standard basis**. The word basis we will rediscover soon; the 'unit' part of their name just means that they have length 1.

- Let's do another example

Example Calculate the span of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and describe it geometrically.

Solution. As before, we can transform this question into one of a system of linear equations represented by this augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \\ 1 & 1 & c \end{array} \right)$$

Doing row reductions to rref, we get the following matrix:

$$\left(\begin{array}{cc|c} 1 & 0 & 2a - b \\ 0 & 1 & b - c \\ 0 & 0 & c - a \end{array} \right)$$

This system is consistent if and only if $c - a = 0$. Any vector for which that holds, the above has a unique solution. Therefore, the set of vectors in the span is exactly the vectors for which $c = a$, i.e.

$$\text{span}(\{[1, 1, 1], [1, 2, 1]\}) = \{ \mathbf{v} \in \mathbb{R}^3 \mid \mathbf{v} = [v_1, v_2, v_3] \text{ where } v_1 = v_3 \}$$

Another way to view it, is that the span is the set of linear combinations; if \mathbf{x} is a linear combination of $[1, 1, 1]$ and $[1, 2, 1]$ then:

$$\mathbf{x} = s \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

This is a plane in \mathbb{R}^3 ; it's easy to see that the first and last coordinates must be the same. What's not easy to see is that is the only restriction, but our work above shows that is is. So geometrically, this represents the plane in \mathbb{R}^3 given by the above parametric equation, or given by the normal equation $x = z$.

- The above examples allow us to state the following theorem, which is an obvious corollary of the observations we have made.

Theorem A system of linear equations with augmented matrix $(A \mid \mathbf{b})$ is consistent if and only if \mathbf{b} is a linear combination of the columns of A .

- Let's now define two terms which are a generalization of the notion of two vectors being parallel, or three vectors being co-planar. This is the notion of linear independence.

Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$. The set S is said to be **linearly dependent** if there are scalars c_1, \dots, c_m not all zero such that

$$c_1 \mathbf{v}_1 + \dots + c_m \mathbf{v}_m = \mathbf{0}$$

otherwise, S is said to be **linearly independent**.

Remark Why the requirement that the coefficients be nonzero? Well, it would be a trivial restriction if we allowed all 0 scalars, since then any set of vectors would be linearly dependent. The assertion that not all are zero is actually making a nontrivial statement about the vectors.

Example One can check the following two things:

- In \mathbb{R}^2 , two vectors are linearly dependent if and only if they are parallel (i.e. one is a scalar multiple of the other).
- In \mathbb{R}^3 , three vectors are linearly dependent if and only if two are pairwise parallel (i.e. one is a scalar multiple of another) or one is on the plane that the other two make.
- Any set of vectors containing the zero vector is linearly dependent.

- The following theorem is how we usually think about linear dependence:

Theorem A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ is linearly dependent if and only if one of the vectors can be written as linear combination of the rest.

Proof. There are two things to prove since we have asserted an ‘if and only if.’

Let us assume that S is linearly dependent. We want to show that one of the vectors can be written as a linear combination of the rest. Well, since S is linearly dependent we know that there are c_1, c_2, \dots, c_m such that

$$c_1 \mathbf{v}_1 + \dots + c_m \mathbf{v}_m = \mathbf{0}$$

Further, we know that not all the c_j are nonzero; let i be such that $c_i \neq 0$. Then, if we subtract $c_i \mathbf{v}_i$ to the other side, we get:

$$\sum_{\substack{j=1 \\ j \neq i}}^m c_j \mathbf{v}_j = -c_i \mathbf{v}_i$$

as $c_i \neq 0$, we can divide, and we can get:

$$\sum_{\substack{j=1 \\ j \neq i}}^m -\frac{c_j}{c_i} \mathbf{v}_j = \mathbf{v}_i$$

Which is exactly \mathbf{v}_i written as a linear combination of the rest, as we wanted.

Now we need to show the converse. So, let us assume that we can write one of the elements of S , say \mathbf{v}_i as a linear combination of the rest. Then:

$$\sum_{\substack{j=1 \\ j \neq i}}^m c_j \mathbf{v}_j = \mathbf{v}_i$$

Let $c_i := -1$; subtracting \mathbf{v}_i from both sides we get

$$\sum_{j=1}^m c_j \mathbf{v}_j = \mathbf{0}$$

This shows that the set is linearly dependent as not all the constants are 0 (in particular $c_i = -1 \neq 0$). \square

- The question of whether a set of vectors is linearly dependent, since it’s a question of whether a particular vector is a linear combination of others, can be rephrased as a question of systems of equations. We call a system where all the constant terms are 0 a **homogeneous system**.

Theorem A homogeneous system either has $\mathbf{0}$ as a unique solution, or has infinitely many solutions.

Proof. Recall that any system of linear equations either has no, exactly one, or infinitely many solutions. Clearly, $\mathbf{0}$ is one solution. Therefore, either that is a unique solution, or there are infinitely many. \square

Theorem Consider a homogeneous system with m equations and n variables where $n > m$. Then there are infinitely many solutions.

Proof. Clearly, the system is consistent since $\mathbf{0}$ is a solution. The rank theorem tells us the number of free variables is equal to $n - \text{rank}(A)$ where A is the coefficient matrix. Clearly, $\text{rank}(A) \leq m$ since the rank is the number of leading entries, and there can’t be more leading entries than there are rows. Therefore $n - \text{rank}(A) \geq n - m > 0$. Therefore, there is at least one free variable, which means there are infinitely many solutions. \square

Remark The converse is not true; as an exercise come up with a homogeneous system with more equations than unknowns with a unique solution.

- Because every homogeneous system has $\mathbf{0}$ as a solution, we call that the **trivial solution**. Thus the question of whether a set of vectors is linearly dependent can be phrased as such:

Theorem Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^n$. Then S is linearly dependent if and only if the system represented by the augmented matrix $(A \mid \mathbf{0})$ has a nontrivial solution.

Proof. Obvious from what we know about translating systems to and from linear combinations of vectors. \square