Exam 2 Review

June 13, 2014

1 Technique Question

The following is a list of question. Contained in this list are some that will be chosen to put on the test. Note that none of these proofs needs to be particularly long. Most are about a paragraph in length, some only a few sentences.

1. Let V, W be subspaces of $\mathbb{R}^n$.
   - Prove that it is not necessarily true that $V \cup W = \{\, \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \in V \text{ or } \mathbf{x} \in W \,\}$ is a subspace of $\mathbb{R}^n$ by constructing a V, W subspaces where this set (the union) is not a subspace.
   - Prove that $\{\, \mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} = c\mathbf{v} + d\mathbf{w} \text{ for some scalars } c, d \text{ and } \mathbf{v} \in V \text{ and } \mathbf{w} \in W \,\}$ is a subspace of $\mathbb{R}^n$.
2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation, and suppose that $T\mathbf{v} = [1, 0]$ and $T\mathbf{u} = [0, 1]$. Prove that:
   (a) T is invertible.
   (b) $\{\mathbf{u}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$ (i.e. show that it spans $\mathbb{R}^2$ and that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. This definition will not be provided on the test).
3. Let B be a basis for V. Prove that every element in V can be uniquely written as linear combination of elements from the basis B.
4. Let A be an $n \times n$ matrix. Show the following are equivalent:
   - There is a $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{v} \neq \mathbf{w}$ such that $A\mathbf{v} = A\mathbf{w}$
   - A is not invertible.
5. Suppose A is is an $n \times n$ matrix with rank k, and let B be an invertible $n \times n$ matrix. Prove that AB and BA have rank k. (Hint: You may want to prove that if B is invertible and if $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent then $B\mathbf{v_1}, \ldots, B\mathbf{v_m}$ are linearly independent). (Hint^2: You should use that the rank is the dimension of the column/row space rather than anything about matrix representation)
6. Suppose A is a $n \times n$ matrix with n eigenvalues (not necessarily distinct). Prove that $\det(A)$ is the product of the eigenvalues. (Hint: consider a factored version of the characteristic polynomial). (Note: we stated/proved this is class only for diagonalizable matrices. Here A is not necessarily diagonalizable).
7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$ corresponding to eigenvectors $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ respectively. Prove that $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ are linearly independent.
8. Prove (using the definition of eigenvalue) that λ is an eigenvalue for A if and only if $\det(A - \lambda I) = 0$.
9. Prove that if $A \sim B$ then $A^T \sim B^T$.
10. Prove that if A is diagonalizable then A^T is diagonalizable.

2 Other Review

2.1 Definitions

Be sure that you can define all of the important words that we defined in class. If you look through my notes, they are in bold. Here is a sampling (I'm not precisely sure it is complete).

> algebraic multiplicity, basis, characteristic polynomial, closure under addition, closure under scalar multiplication, codomain, cofactor, column space, coordinate vector with respect to B, determinant, diagonalizable, dimension, domain, eigenspace, eigenvalue, function composition, geometric multiplicity, inverse matrix, linear transformation, minor, null space, nullity, range, rank, row space, similar matrix, spectrum, subspace.

Since a lot of this math is built on older math, you should make sure you know the last exam material too (especially the definitions and algorithms).

2.2 Major Theorems

A component of the test will be able to state/use easy corollaries to theorems in a multiple choice or true/false manner. For example, a question on the test might be:

> True/False: If A is an $n \times n$ matrix with trivial null space then it's range is $\mathbb{R}^n$

This is true, and is a straightforward applications of theorems we did involving dimension and rank/nullity.

The fundamental theorem of invertible matrices (as presented in 4.3) is extremely important, but you should go through the book and notes and make sure you understand all of the theorems. The technique questions should help since their proofs are usually quite pithy if you can use the correct theorems.

2.3 Major Computational Problems

We've done a lot of computational problems, and of course this will be a major component of the test. Here's a few things you should be able to do (this list is probably not exhaustive)

- Verify a subset of a space is a subspace.
- Verify a function on spaces is a linear transformation.
- Find a matrix representing a linear transformation.
- Find a basis for a space/subspace (like the column space, row space, null space, eigenspace).
- Find the dimension of a space/subspace.
- Find a coordinate vector with respect to a different basis.
- Calculate the determinant of a matrix (using row reductions for large matrices).
- Find the spectrum of a matrix.
- For each eigenvalue of a matrix, finding a basis for its eigenspace.
- Determining if a matrix is diagonalizable, and if so diagonalizing it.