Theorem 1 (Reals are Uncountable)**.** $|\mathbb{R}| \neq |\mathbb{N}|$

Proof. We will instead show that $(0,1)$ is not countable. This implies the theorem because if there were a bijection from $\mathbb{R}$ to $\mathbb{N}$, one could compose it with a bijection we have from $(0,1)$ to $\mathbb{R}$, and get that $(0,1)$ is countable.

We will go by contradiction. Suppose that $(0,1)$ is countable. Then, fix a bijection $f : \mathbb{N} \to (0,1)$. Then, for each natural number n we have some decimal sequence that n maps to

$$\begin{array}{l}
0 \mapsto 0.\ a_{0,0}\ a_{0,1}\ a_{0,2}\ a_{0,3}\ a_{0,4} \dots \\
1 \mapsto 0.\ a_{1,0}\ a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{1,4} \dots \\
2 \mapsto 0.\ a_{2,0}\ a_{2,1}\ a_{2,2}\ a_{2,3}\ a_{2,4} \dots \\
3 \mapsto 0.\ a_{3,0}\ a_{3,1}\ a_{3,2}\ a_{3,3}\ a_{3,4} \dots \\
4 \mapsto 0.\ a_{4,0}\ a_{4,1}\ a_{4,2}\ a_{4,3}\ a_{4,4} \dots \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad\quad \vdots
\end{array}$$

What we'd like to show this is not actually a bijection. Motivated by the idea that $\mathbb{N}$ is the "smallest" infinity, we must have that $(0,1)$ is "larger" if it's not equal. Therefore, we will try to argue this is not a surjection. To do this, we must find something that the number doesn't hit.

Well, we will construct a decimal which represents something "new" which is not hit. So we build our number at different stages, one for each natural number. At stage i, we will define the ith decimal digit of our number, and we will do it in such a way that $f(i)$ is not going to equal the number we have in question.

We hit our first snag when we realize that decimal numbers do not unique represent real numbers. That is to say, the following reals are equal, but have different decimal representations.

$$.49999\ldots = .50000\ldots$$

Therefore, in representing our numbers about we choose the convention if a number terminates in that way, we will pick the representation which terminates in infinitely many 0's, and not 9's.

Let's begin defining our sequence. First, for $i = 0$, we define the 0th digit of our number, which we will denote b_0. Well, $a_{0,0}$ is some decimal number. We will make it so $b_0 \neq a_{0,0}$. To do this, let's just take b_0 to be 1 if $a_{0,0}$ is not 1, and 2 otherwise. Therefore, b_0 is taken to be different than $a_{0,0}$.

Now, the choices of what I changed it to was arbitrary, except that I wanted to avoid defining things in my sequence 9's since then I may accidently make a sequence terminating in all 9's which will equal a sequence on the list.

We continue in this way; to define b_i, I look at $a_{i,i}$; that is the ith digit of the decimal representation of $f(i)$. And I change it in the same way; if $a_{i,i}$ is not 1, we will make b_i equal to 1, and otherwise we will let it be 2.

$$\begin{array}{l}
0 \mapsto 0.\ a_{0,0}\ a_{0,1}\ a_{0,2}\ a_{0,3}\ a_{0,4} \dots \\
1 \mapsto 0.\ a_{1,0}\ a_{1,1}\ a_{1,2}\ a_{1,3}\ a_{1,4} \dots \\
2 \mapsto 0.\ a_{2,0}\ a_{2,1}\ a_{2,2}\ a_{2,3}\ a_{2,4} \dots \\
3 \mapsto 0.\ a_{3,0}\ a_{3,1}\ a_{3,2}\ a_{3,3}\ a_{3,4} \dots \\
4 \mapsto 0.\ a_{4,0}\ a_{4,1}\ a_{4,2}\ a_{4,3}\ a_{4,4} \dots \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad\quad \vdots
\end{array}$$

As the end we will have defined a sequence $b_0, b_1, b_2, \ldots$. We can read this as a decimal, which we call b:

$$b := 0.\ b_0\ b_1\ b_2\ b_3\ b_4\ \ \ldots$$

We claim this is not on our list. For, if it were $f(i) = b$ for some $i \in \mathbb{N}$. But, the ith decimal digit of $f(i)$ is different then the ith decimal digit of b by the way we defined b, which means they are different numbers.

Thus b is not on our list.

□

Theorem 2 (Cantor's Theorem)**.** *For any set A, we have*

$$|A| \neq |\wp(A)|$$

Proof. Suppose there was such a bijection $f : A \to \wp(A)$. Then for each $a \in A$ we have an associated subset of A, S_a.

We will show, as in the last theorem that this is not surjective by constructing a subset of A which is not hit by f.

To detertemine the subest we will just decide for every $a \in A$ whether a will be in the subset we are creating or not. Call the subet we are creating S. Take $a \in A$. We want to ensure that the S we are creating is not a surjection, so we need to make sure it is different than everything. When considering whether a is in S or not, we will make sure that a does not hit S.

So, we look at where a is sent to. a is sent to some subset of A, S_a. S_a either has a as a member, or not. If $a \in S_a$, then we will not put a in S to make sure S_a is different than S. Similarly, if $a \notin S_a$ we will put a in S.

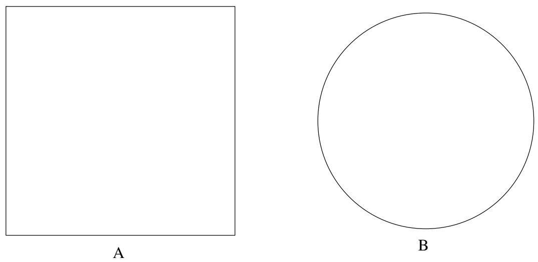
Therefore, we have decided $a \in S$ if and only if $a \notin f(a)$ for every $a \in A$. We claim A is not hit by the function f. For if it were there would be some $a \in A$ such that $f(a) = S$. But, then ask: is $a \in S$? We have put a is S only when $a \notin f(a)$. Thus if $a \in S$ then this must be because $a \notin f(a) = S$, which is a contradiction. Similarly, we excluded a from S only when $a \in f(a) = S$. Thus we must have that S was not hit by f. □

Corollary 1. *There is no "largest" set.*

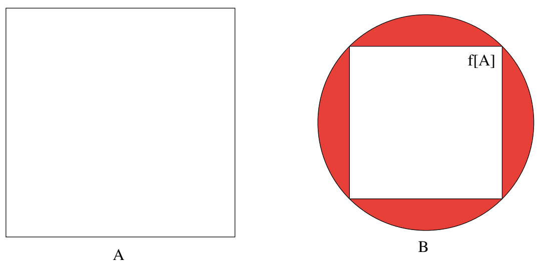
Theorem 3 (Cantor-Schroeder-Bernstein)**.** *Suppose that $f : A \to B$ and $g : B \to A$ are injections. Then there is a bijection from A to B.*

Proof Sketch. Here is morally the idea:

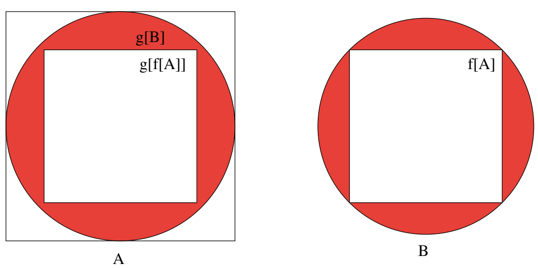
Our philosophy will be to do as little as we need to in order for it to work. f is already an injection, so we don't need to do much other than make sure it is surjective.



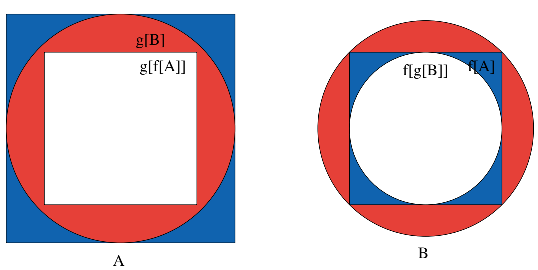
$f[A]$ is already good; everything is hit, and everything is hit exactly once. But, unfortunately, there may be things outside of $f[A]$. That is, there may be things in $B \setminus f[A]$. We need to hit these things, and we have to find a way to do it other than through f, since f doesn't even touch these objects!



This is where g comes in; look at the image of this set under g, that is $g[B \setminus f[A]]$. This is a subset of A, and it is exactly the stuff in A that is hit by g in this section of B that is not hit by f. So, for the stuff in this set, which as g is injective is the same as $g[B] \setminus g[f[A]]$, we will follow the lines for g instead of f.



Then, for the things in $A \setminus g[B]$ we can just map across f, which will hit things if $f[A] \setminus f[g[B]]$



This makes a big problem though: Now, the things that were hit by the things we "re-routed" are no longer hit. That is, everything in $f[g[B]] \setminus f[g[f[A]]]$ certainly will no longer by hit by f.

Thus we must, sort of, iterate this procedure forever. It may be the case that we have stuff that is at the bottom of this long chain of images back and forth. If that's the case, we can safely use f to carry them across. $\square$

Proof With Explanation. We define a sequence of sets which are supposed to mimic our exploration of our sketch. We first look at $A_0 = A$ and $B_0 = B$. Then, we are repeatedly taking imaged under g of the last B to get new A's, and vice-a-versa. That is, $A_{n+1} = g[B_n]$ and $B_{n+1} = f[A_n]$. Notice, it's fairly clear that $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$.

Now, we need to decide where x maps to, for every $x \in A$. So, we look at which i has the property that $x \in A_i \setminus A_{i-1}$. That is look for the last i which x appears in. There is no reason to believe that x is not in all the A_i's so we must make an allowence for that as well.

But, if x does end up in $A_i \setminus A_{i-1}$, we do cases on whether i is even or odd. If i is even, then our goal is to be ordinary and send x across f. If i is odd however, then we instead look at g, and map x "backwards" through g.

So, we define $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_i \setminus A_{i+1} \text{ for } i \text{ even, or } x \in \bigcup_{i\in\mathbb{N}} A_i \\ c & \text{if } x \in A_i \setminus A_{i+1} \text{ for } i \text{ odd, where } c \in g^{-1}[\{x\}] \end{cases}$$

We may be worried about this rule being well-defined. It is however, since g is injective and for any odd i, if $x \in A_i$ then x is in the image of g, there is one and only one c which that clause could find.

It's left only to verify that this is injective and surjective. Morally, this should be easy, as the at every stage we had equality with sets in our sketch; for example, as f and g are injective, $f[g[B] \setminus g[f[A]]] = f[g[B]] \setminus f[g[f[A]]]$. In practice however, this can be difficult to state.

First we will do surjective. Take some $y \in B$. Then as the B's are decreasing, $B = B_0 \supseteq B_1 \supseteq B_2 \supseteq \cdots$ there is either a last i in which y appears, or y is in every B_i. Note, if the last i that y is even, then this is hit by whatever x is hit by y through g. Otherwise, i is odd, but then it is certainly in the image of f (everything with in B_i for $i \geq 1$ is in the image of f), so there is some x that hits it, and this is what our function does.

If y is in all of the B_i, then there is something which hits it which is in all the A_i.

For injective, this is rather tedious to check very formally. If two things $x, y \in A$ hit the same thing $h(x) = h(y) = b$, then one does as above and checks the last i such that b is in B_i (if there is such a thing). If it is in all the B_i, then b is hit through f, so $h(x) = f(x)$ and $h(y) = f(y)$, and as f is injective, $x = y$. Similarly if the last i is odd.

If the last i is even, then the path is taken through g, and as g is a function, x and y must be identical. □

Proof As One Would Write It. Let $f : A \to B$ and $g : B \to A$ be injections. Let $A_0 = A$ and $B_0 = B$ and by induction we define $A_{i+1} = g[B_i]$ and $B_{i+1} = f[A_i]$. Let $A' = \bigcap_{i\in\mathbb{N}} A_i$ and $B' = \bigcap_{i\in\mathbb{N}} B_i$. Then, define:

$$h(x) = \begin{cases} f(x) & \text{if } x \in A' \cup \bigcup_{i\in\mathbb{N}} (A_{2i} \setminus A_{2i+1}) \\ g^{-1}(x) & \text{otherwise} \end{cases}$$

We claim this is a bijection from A to B.

To show surjective, take $b \in B$. If $b \in B'$ then $b \in B_1 = \mathrm{im}(f)$ so there is $a \in A$ such that $b = f(a) = h(a)$. Otherwise, $b \in B_i \setminus B_{i+1}$ for some $i \in \mathbb{N}$. If i is odd then there is $a \in A_{i-1} \setminus A_i$ such that $b = f(a)$; as $a \in A_{i-1} \setminus A_i$ and $i - 1$ is even, $h(a) = f(a) = b$. If i is even then $g(b) = a \in A_{i+1} \setminus A_{i+2}$ for $i + 1$ odd, so $h(a) = g^{-1}(a) = b$. Thus, h is surjective.

To show injective, suppose that $h(x) = h(y) = b$. If $b \in B'$ we're done, as $f(x) = h(x) = h(y) = f(y)$, and as f is injective, $x = y$. Otherwise, $b \in B_i \setminus B_{i+1}$. If i is odd, then $x, y \in A_{i-1} \setminus A_i$, so $f(x) = h(x) = h(y) = f(y) = b$, and as f is injective, $x = y$. If i is even, then $x, y \in A_{i+1} \setminus A_{i+2}$ (in particular, $x, y \notin A' \cup \bigcup_{i\in\mathbb{N}} (A_{2i} \setminus A_{2i+1})$). Thus $g^{-1}(x) = h(x) = h(y) = g^{-1}(y) = b$, so $x = y$. □