

Assignment 7

Friday June 23, 2012

1 Counting

This section is intended to be completed Tuesday June 20th. Here we will practice all the basic counting techniques we have learned. Fully explanations of how one counted is necessary for **any** points. One can leave your final answer in terms of basic operations (factorials, combinations, multiplication, addition, subtraction, exponentiation are all fine)

Problem 1. Let A be a finite set of size n

- (a) Count the number of subsets of A of size k
- (b) Use the above and rule of sums to count the number of subsets of A .
- (c) Recall we already proved $|\wp(A)| = 2^n$. Equate the formula you got from the last part and 2^n .

Problem 2. At a party there are 15 men and 20 women.

- (a) How many ways can they form 15 couples consisting of 1 man and 1 woman?
- (b) How many ways can they form 10 couples consisting of 1 man and 1 woman?

Problem 3. Suppose there are n people at a dinner party.

- (a) Count the number of ways they can sit at a circular table. Note, the only thing that distinguishes an arrangement is the relative position of the people (eg. an arrangement where Bob is between Alice and Charlotte is different than if he's between Alice and Chuck).
- (b) Alice hates Bob. Count the number of ways the n people can sit at the circular table so Alice doesn't have to sit next to Bob.
- (c) Alice changed her mind. She likes Bob now, but doesn't want him to be too close to the right side of her face. Count the number of ways n people can sit at the circular table so that Bob doesn't sit next to Alice's right.
- (d) Alice and Bob decided they're best friends now. Count the number of ways to arrange them so that Alice and Bob do sit next to each other (he can sit to her right or left).

Problem 4. A woman is on the southwest corner of a city at 1st Street and 1st Avenue in a city with a nice city grid (not Pittsburgh, for example). She wants to get to 20th Street and 5th Avenue only moving one block at a time east and north. How many ways are there to do this?

Problem 5. Consider binary strings of length $2n$.

- (a) Count the number of strings.
- (b) Count the number of strings, where there are exactly n ones.
- (c) Count the number of strings, where there are exactly n ones and none are consecutive.

2 More Counting

This section is intended to be completed Wednesday June 21st.

Problem 6. Count:

- (a) The number of ways to arrange the letters in the word ADDRESSES.
- (b) The number of ways to put 10 distinguishable balls in 3 distinguishable bins, such that the first bin has 3 balls, the second has 2, and the third has 5.
- (c) Do the same as part (b), but with indistinguishable balls.

Problem 7. You're going to a donut shop, and you gotta buy a dozen donuts. There are 5 different varieties.

- (a) How many ways are there to do this?
- (b) How many ways are there to do as in part (a), given that you want at least one chocolate?
- (c) How many ways are there to do as in part (a), given that there is only 2 chocolate donuts left.

Problem 8. Consider the equation:

$$x_1 + x_2 + x_3 + x_4 = 20$$

- (a) How many solutions are there to this, under the restrictions each x_i is a natural number.
- (b) How many solutions, under the restriction each x_i is a positive integer?
- (c) How many solutions, under the restrictions that each x_i is natural, except for x_1 which is an integer ≥ -3

3 Counting Proofs

This section is intended to be completed Thursday June 22nd.

Problem 9. Prove by counting in both ways

- (a) $n^2 = n + n(n - 1)$
- (b) $n^3 = n + \binom{n}{2}n(n - 1) + n(n - 1)(n - 2)$

Problem 10. Prove that

$$n(n - 1)\binom{n - 2}{k} = \binom{n}{k + 2}(k + 2)(k + 1)$$

Problem 11. Prove that

$$\binom{n}{3} = \sum_{i=2}^{n-1} (i - 1)(n - i)$$