A finite goal set in the plane which is not a Winner

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Abstract

J. Beck has shown that if two players alternately select previously unchosen points from $\mathbb{R}^2$, Player 1 can always build a congruent copy of any given (finite) goal set G, in spite of Player 2's efforts to stop him [B]. We give a finite goal set G (it has 5 points) which Player 1 cannot construct *before* Player 2 in this achievement game played in the plane.

1 Introduction

In the G-achievement game played in the plane, two players take turns choosing single points from the plane which have not already been chosen. A player achieves a *weak win* if he constructs a set congruent to $G \subset \mathbb{R}^2$ made up entirely of his own points, and achieves a *strong win* if he constructs such a set before the other player does so. (So a 'win' in usual terms corresponds to a strong win in our terminology.) This is a special case of a positional hypergraph game, where players take turns choosing unchosen points (vertices of the hypergraph) in the hopes of occupying a whole edge of the hypergraph with just their own points. [B96, B] contain results and background in this more general area.

The type of game we are considering here is the game-theoretic cousin of Euclidean Ramsey Theory (see [G] for a survey). Fixing some $r \in \mathbb{N}$ and some finite point set $G \subset \mathbb{R}^2$, the most basic type of question in Euclidean Ramsey Theory is to determine whether it is true that in every r-coloring of the plane, there is some monochromatic congruent copy of G.

Restricting ourselves to 2 colors, the game-theoretic analog asks when Player 1 has a 'win' in the achievement game with G as a goal set. Though one can allow transfinite move numbers indexed by ordinals (see Question 5 in Section 3), it is natural to restrict our attention to games of length ω, in which moves are indexed by the natural numbers. In this case, a weak or strong winning strategy for a player is always a finite strategy (*i.e.*, must always result in weak

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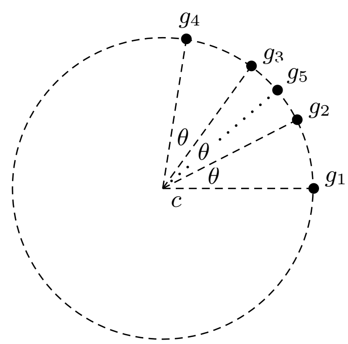

Figure 1: The goal set G. Player 2 can force a draw when the goal is this set, where θ is any irrational multiple of π less than $\frac{\pi}{9}$.

or strong win, respectively, in a finite, though possibly unbounded, number of moves) so long as the goal set G is finite. J. Beck has shown [B] that both players have strategies which guarantee them a weak win in finitely many moves for *any* finite goal set—the proof is a potential function argument related to the classical Erdős-Selfridge theorem [ES]. The question of whether the first player has a strong win—that is, whether he can construct a copy of G first—seems in general to be a *much* harder problem. (A strategy stealing argument shows that the second player cannot have a strategy which ensures him a strong win: see Lemma 3.5.)

For some simple goal sets, it is easy to give a finite strong winning strategy for Player 1. This is the case for any goal set with at most 3 points, for example, or for the 4-point vertex-set of any parallelogram. We give a set G of 5 points for which we prove that the first player cannot have a finite strong win in the G-achievement game (proving, for example, that such finite goal sets do in fact exist).

Fix $\theta = t\pi$, where t is irrational and $t < \frac{1}{9}$. Our set G is a set of 5 points g_i, $1 \leq i \leq 5$, all lying on a unit circle C with center $c \in \mathbb{R}^2$. For $1 \leq i \leq 3$, the angle from g_i to g_{i+1} is θ. The point g_5 (the 'middle point') is the point on C lying on the bisector of the angle $\angle g_2 c g_3$. (See Figure 1.) For this set G, we prove the following:

Theorem 1.1. *There is no finite strong winning strategy for Player 1 in the G-achievement game.*

Idea: Let $\theta_c^n(x)$ denote the image of $x \in \mathbb{R}^2$ under the rotation $n\theta$ about the point c. An important property of our choice of G is that once a player has threatened to build a copy of G by selecting all the points g_1, g_2, g_3, g_4, he can give a new threat by choosing the point $\theta_c(g_4)$ or $\theta_c^{-1}(g_1)$. Furthermore, since θ is an irrational multiple of π, the player can continue to do this indefinitely, tying up his opponent (who must continuously block the new threats by selecting the corresponding middle points) while failing himself to construct a copy of G. If Player 1 is playing for a finite strong win, he cannot let Player 2 indefinitely

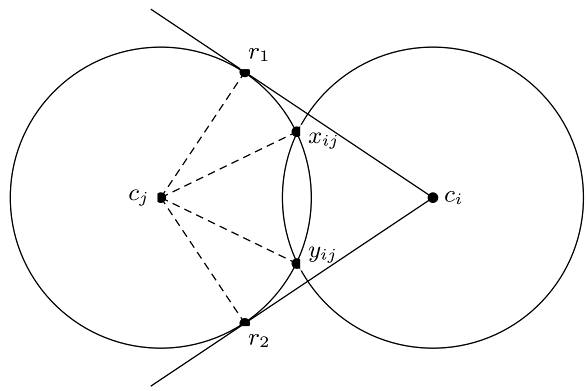

Figure 2: Proving Lemma 2.1

force in this manner. However, to deny Player 2 that possibility, we will see that Player 1's only option is the same indefinite forcing, which leaves him no better. The rest of the rigorous proof is a case study.

2 The Proof

For the proof of Theorem 1.1, we will need the following lemma.

Lemma 2.1. *There are no three unit circles C_2, C_3, C_4 so that the pairs C_i,C_j each intersect at 2 distinct points x_{ij} and y_{ij}, so that the angles $\angle x_{ij} c_i y_{ik}$ are less than $\frac{\pi}{3}$ for all $j \neq i \neq k$.*

Proof. Let B_i denote the unit ball whose boundary is C_i for each i, and choose C_i and C_j from $\{C_2, C_3, C_4\}$ so that the area $A(B_i \cap B_j)$ is maximal. In Figure 2, for any C_k intersecting the circle C_j at points x_{jk}, y_{jk} lying on C_j between r_1 and r_2, we would have $A(B_i \cap B_k) > A(B_i \cap B_j)$, a contradiction. The maximum angle between the points r_1 and r_2 on C_j is $\frac{2\pi}{3}$. □

We are now ready to prove Theorem 1.1.

It is clear that Player 2 can either play indefinitely or reach a point where it is his move, he has a point h_1 at least 10 units away from any of Player 1's points, and Player 1 has no more than 2 points in any given (closed) ball of radius 10. (For example: on each turn until he has reached this point, Player 2 moves 30 units away from any of Player 1's points.) Reaching this point, Player 2 begins to build a copy of G; that is, he arbitrarily designates some 'center point' c at unit distance from the point h_1, and chooses as his move a point h_2 which is an angle θ away from h_1 on the unit circle C centered at c. In fact, h_1 and h_2 lie on two unit circles which are disjoint except at h_1, h_2, and so Player 1's response can lie on only one of them.

($\star$) Thus without loss of generality, we see that Player 2 can reach a situation where it is his turn and he has points h_1, h_2 separated by an angle θ on a unit circle C centered at c. Moreover, Player 2 ensures that Player 1 controls at most 3 points of any unblocked copy of G and that any such copy must have its center point at least 9 units from h_1, and that Player 1 has at most 1 point within 8 units of h_1—and, if he does have such a point which is close to h_1, then we have as well that Player 1 has in fact just 2 points of any unblocked copy of G.

From this point, Player 2 naturally continues his construction choosing the point h_3 which lies on the circle C and is separated from the points h_1, h_2 by angles $2\theta, \theta$, respectively. We classify the rest of the proof into three different cases depending on Player 1's response.

Case 1: A natural response for Player 1 might be to play on the circle C, thus attempting to prevent Player 2 from building a significant threat. Since no point is a rotation about the point c by both positive and negative integer multiples of $\frac{\theta}{2}$, we may assume WLOG that Player 1 does not choose any rotations of h about c by positive integer multiples of $\frac{\theta}{2}$. Thus Player 2 responds by choosing the point h_4 on C which is at an angle $\theta, 2\theta, 3\theta$ from the points h_3, h_2, h_1, respectively. Since Player 2 is now threatening to build a copy of G on his next move and Player 1 is not (he has at most 3 points in any ball of radius 10 at this point), Player 1 must take the point on C which together with h_1, h_2, h_3, h_4 complete a copy of G. Player 2's response is naturally to choose the point h_5 on C at angle $\theta, 2\theta, 3\theta$ from h_4, h_3, h_2, and we are in essentially the same situtation: Player 1 has always at most 3 points in any unblocked congruent copy of G (since he has only one point 'near' C which is not on C, and any set congruent to G and not on C interesects C in at most 2 points), and Player 2 can force indefinitely.

Case 2: Another response for Player 1 which may be possible is to play within the vicinity of his previously chosen points such that he controls 4 points of an unblocked copy G. In this case all of his points are necessarily at least 8 units away from the point h_1 controlled by Player 2. In this case, Player 2 is forced to choose the corresponding 5th point of the copy of G which Player 1 is threatening to build. But now we are in essentially the same situtaion as one full turn earlier: since Player 1 has at most 3 points in any unblocked copy of G (and the center point still lies at least 9 units away from h_1) and no points within 8 units of the point h, we are in the situation described by the starred ($\star$) paragraph above. Thus Player 1 makes no progress in this case.

Case 3: Finally, we consider the case where Player 1 does 'none of the above'; that is, he chooses a point not on the circle C, but which nevertheless does not increase to 4 the number of points he controls in some congruent copy of G. Perhaps surprisingly, this is a bit trickier than the previous two cases, and it is here that we make use of Lemma 2.1.

Player 1 now has as many as two points within 8 units distance of the point h_1. By choosing successively points h_4, h_5, h_6, etc., as in Case 1, Player 2 hopes to successively force Player 1 to take the corresponding fifth point of each congruent copy of G that Player 2 threatens to build at each step. The only snag is this: it is conceivable that Player 1, in taking these corresponding 'fifth' points, builds his own threat. He already has two points in the vicinity,

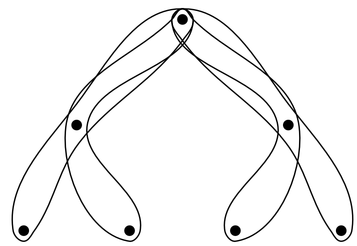

Figure 3: The hypergraph $\mathcal{H}_T$, in the case where T is the balanced binary directed tree of depth 2.

and it is possible that they lie on a congruent copy of G which intersects the circle C in two points which Player 1 will eventually be forced to take by Player 2's moves. In this case, Player 2 would have to respond and could conceivably end up losing the game if Player 1 is able to break is forcing sequence.

Of course, this is only truly a problem if Player 1 is threatening this in 'both directions'—that is, regardless of whether h_4 is at angles $\theta, 2\theta, 3\theta$ to the points h_3, h_2, h_1, respectively, or to the points h_1, h_2, h_3, respectively. However, such a double threat is immediately ruled out by Lemma 2.1, since this would require two sets $S_1, S_2 \cong G$ (each a subset of a 3θ-arc of a unit circle) intersecting each other in two points (previously chosen by Player 1) and each also each interescting C in two places. This completes the proof. □

3 Further Questions

1. Our (rather crude) methods here are not suited to much larger goal sets, and certainly not to arbitrarily large goal sets. So we ask: are there arbitrarily large goal sets G for which Player 1 cannot force a finite strong win in the G-achievement game played in $\mathbb{R}^2$?

2. We have examples of 4-point sets for which Player 1 has strong winning strategies. Are there sets $G \subset \mathbb{R}^2$, $|G| = 4$ for which Player 1 cannot win?

3. Player 1 can easily be shown to have strong winning strategies for any goal set of size at most 3, and any 4-point goal set which consists of the vertices of a parallelogram. It is not difficult to give a 5 point goal set for which Player 1 can be shown to have a strong winning strategy. Are there arbitrarily large goal sets G for which Player 1 has a strong winning strategy?

4. In the general achievement game played on a hypergraph (in which the two players select vertices, and the goal sets are the edges) we define some stronger win types for Player 1:

Definition 3.1. In the achievement game played on a hypergraph $\mathcal{H}$, Player 1 has a *fair win* if he builds some $e \in E(\mathcal{H})$ on a turn which comes before any turn on which Player 2 builds some $f \in E(\mathcal{H})$.

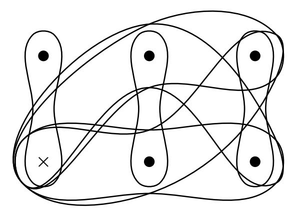

Figure 4: The hypergraph $\mathcal{F}_3$. There are four (in general 2^{n-1}) Type 1 edges, and three (in general n) Type 2 edges. (The vertex $(1,0)$ is marked with $\times$.)

Each 'turn' of the game consists of a move by Player 1 followed by a move by Player 2. Definition 3.1 requires simply that Player 1 builds a goal set in fewer turns than it takes Player 2 to do the same (if Player 2 can at all).

Definition 3.2. In the achievement game played on a hypergraph $\mathcal{H}$, Player 1 has an *early win* if he builds some $e \in E(\mathcal{H})$ (say in n moves) such that there is no $m \leq n$ for which Player 2 had $|e| - 1$ points of a set $e \in E(\mathcal{H})$ on his mth turn, and on which Player 1 had no point on his mth turn.

So every early win is a fair win, and every fair win is a strong win. In general, none of the win types we have defined are the same, and they all occur for Player 1 for some hypergraph: Already for K_4, Player 1 has a strong win but not a fair win. On the hypergraph $\mathcal{H}_T$, whose vertices are the vertices of some balanced binary directed tree T, and whose edges are the vertex-sets of longest directed paths in T (Figure 3), Player 1 has a fair win and an early win. Finally, let the hypergraph $\mathcal{F}_n$ have vertex set $[n] \times \{0,1\}$. Edges are of two types: Type 1 edges are the n-subsets $S \subset [n] \times \{0,1\}$ for which the $\pi_1(S) = [n]$ and $(1,0) \in S$, and Type 2 edges are all the pairs $\{(m,0),(m,1)\}$ where $m \in [n]$ (see Figure 4). Player 1 has a fair win in $\mathcal{F}_n$ for $n \geq 2$, but not an early win. Probably, however, the situation is not so rich in the plane:

Conjecture 3.3. *There is no finite point set $G \subset \mathbb{R}^2$ for which Player 1 has a strategy which ensures a fair win in the G-achievement game played in the plane.*

The conjecture may seem painfully obvious. If we play the achievement game in $\mathbb{R} \setminus \{c\}$ for any point $c \in \mathbb{R}^2$, for example, Player 2 can prevent a fair win by always choosing the point which is the central reflection across c of Player 1's last move. Annoyingly, even proving that Player 1 cannot have an early win for any G when playing in $\mathbb{R}^2$ may be very difficult.

For the sake of completeness, we note the situtation on the hypergraph $\mathcal{H}_T$ is in some way the worst possible for Player 2. It is easy to see that although Player 2 never occupies all but one vertex of an unblocked edge when playing on $\mathcal{H}_T$, it is easy for him to occupy all but one vertex of some edge which may be blocked. The natural strengthening of the 'early win' suggested here never occurs for Player 1:

Definition 3.4. In the achievement game played on a hypergraph $\mathcal{H}$, Player 1 has a *humiliating win* if he occupies some $e \in E(\mathcal{H})$ before Player 2 occupies all but one vertex of some edge $f \in E(\mathcal{H})$.

(So every humiliating win is an early win.) The fact that Player 1 never has a humiliating win will follow from the strategy stealing argument; we include the proof for completeness.

Lemma 3.5 (Strategy Stealing)**.** *On any hypergraph $\mathcal{H}$, a second player cannot have a strategy which ensures strong win in the achievement game.*

Proof. The proof of Lemma 3.5 is the strategy stealing argument; we include the proof for completeness. We argue by contradiction: if the second player has a strong win strategy σ (a funtion from game positions to vertices), the first player makes an arbitrary first move g (his ghost move). Now on each move, the first player mimicks the second player's strategy by ignoring his ghost move: formally, let G_n denote the game's position on the nth move, and let $G_n \setminus x$ denote the game position modified so that the vertex x is unchosen. Then on each turn, the first player chooses the point $\sigma(G_n \setminus g)$ if it is not equal to g (and thus must be unoccupied, since σ is a valid strategy), or, if $\sigma(G_n \setminus g) = g$, the first player chooses an arbitrary point $x \in V(\mathcal{H})$ and sets $g := x$. The fact that σ was a 'strong win' strategy for the second player implies that the first player will occupy all of an edge $e \in E(\mathcal{H})$ (even requiring $e \not\ni g$) before the second player occupies all some some edge $f \in E(\mathcal{H})$. In particular, the first player has a strong win, a contradiction. □

Fact 3.6. *On any hypergraph $\mathcal{H}$, Player 2 can prevent Player 1 from achieving a humiliating win.*

Proof. Denote by x the vertex Player 1 chooses on his first move. The hypergraph $\mathcal{H} \setminus x$ is the hypergraph with vertices $V \setminus \{x\}$ and edges $e \setminus \{x\}$ for each $e \in E(H)$. We see that Player 1 has a humiliating win on $\mathcal{H}$ only if he has a strong win on $\mathcal{H} \setminus \{x\}$ *as a second player*, and we are done by Lemma 3.5. □

Lemma 3.5 is deceptive in its simplicity. Keep in mind that the strategy stealing argument shows only the existence of a strategy for a first player to prevent a second player strong win. In general, we have no better way to find such a strategy than the naíve 'backwards labeling' method, which runs on the whole game tree. Thus, though Fact 3.6 tells us that Player 2 should never fall more than one behind Player 1 (in the sense of Definition 3.4), it is quite possible for this to happen in actual play between good (yet imperfect) players.

5. We restricted our attention here to the first ω moves, and indeed, our proof does not show that Player 1 can't force a strong win if transfinite move numbers are allowed. So we ask: are there finite sets G for which Player 1 cannot force a strong win, when the players make a move for each successor ordinal?

Acknowledgment

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