The Hales–Jewett number is exponential—game-theoretic consequences

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1. The Hales–Jewett number. This number is related to the n^d hypercube, and in this way it is closely related to the multidimensional Tic-Tac-Toe game. The d-dimensional solid hypercube of side n precisely means the d-fold Cartesian product of the solid interval $[0,n] = \{x : \ 0 \le x \le n\}$ with itself:

$$[0,n] \times [0,n] \times \cdots \times [0,n] = [0,n]^d.$$

The $[0,n]^d$ solid hypercube consists of n^d small unit cubes, which we call "cells". If each "cell" is identified with its "upper right corner", we obtain the $[n]^d$ grid ($[n] = \{1,2,\ldots,n\}$):

$$[n]^d = \Big\{\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d : \ 1 \le a_j \le n \text{ for each } 1 \le j \le d\Big\}.$$

When we talk about the "d-dimensional hypercube of side n", formally n^d, we either mean the solid cube $[0,n]^d$ or the grid $[n]^d$; usually there is no confusion.

The ordinary $3 \times 3 = 3^2$ Tic-Tac-Toe is played on a 3×3 board, and the "winning sets" are the eight 3-in-a-line's. The n^d Tic-Tac-Toe game is played on the $[0,n]^d$ solid hypercube. The two players alternately put their marks, X and O, in the previously unmarked "cells" (i.e., unit cubes) of $[0,n]^d$. Each player marks one cell per move. The winner is the player to occupy a whole winning set *first.* We define the winning sets in terms of the $[n]^d$ grid. Each cell is identified with its "upper right corner", and the winning sets are exactly the n-in-a-line's in $[n]^d$, that is, the n-element sequences

$$\Big(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots, \mathbf{a}^{(n)}\Big)$$

of the $[n]^d$ grid such that, for each j, the sequence $a_j^{(1)}, a_j^{(2)}, \ldots, a_j^{(n)}$ composed of the jth coordinates is either $1,2,3,\ldots,n$ ("increasing"), or $n, n-1, n-2, \ldots, 1$ ("decreasing"), or a *constant.*

If neither player gets n-in-a-line the play is a draw. The special case $n = 3, d = 2$ gives back ordinary Tic-Tac-Toe, and the case $n = 4, d = 3$ gives the $4 \times 4 \times 4 = 4^3$ grown-up version, which is a very interesting and difficult game. Note that in higher dimensions most of the n-in-a-line's are some kind of diagonal.

In the rest of the paper we often call the cells "points" (identifying a cell with a special point like the center or the "upper right corner").

The winning sets in the n^d Tic-Tac-Toe game are "lines", so we often call them "winning lines". The number of winning lines in the 3^2 and 4^3 Tic-Tac-Toe games are 8 and 76, respectively. In the general case there is an elegant short formula for the number of winning lines.

Simple Facts. *(a) The total number of winning lines in the n^d Tic-Tac-Toe game is $\big((n+2)^d - n^d\big)/2$.*

(b) If n is odd, there are at most $(3^d-1)/2$ winning lines through any point, and this is attained only at the center of the board. In other words, the maximum degree of the n^d line-hypergraph is $(3^d-1)/2$.

(c) If n is even ("when the board does not have a center"), the maximum degree drops to 2^d-1, and equality occurs if there is a common $c \in \{1,\dots,n\}$ such that every coordinate c_j equals either c or $n+1-c$ ($j=1,2,\dots,d$).

This is a folklore result (it was rediscovered so often); for the sake of completeness we include a proof.

Proof. To prove (a) note that for each $j \in \{1,2,\dots,d\}$, the sequence $a_j^{(1)}, a_j^{(2)}, \dots, a_j^{(n)}$ composed of the jth coordinates of the points on a winning line is either strictly *increasing* from 1 to n, or strictly *decreasing* from n to 1, or a *constant* $c = c_j \in \{1,2,\dots,n\}$. Since for each coordinate we have $(n+2)$ possibilities $\{1,2,\dots,n, \textit{increasing}, \textit{decreasing}\}$, this gives $(n+2)^d$, but we have to subtract n^d because at least one coordinate must change. Finally, we have to divide by 2, since every line has two orientations.

Next we prove (b): let n be odd. Given a point $\mathbf{c} = (c_1, c_2, \dots, c_d) \in n^d$, for each $j \in \{1,2,\dots,d\}$ there are three options: the jth coordinates of the points on an *oriented* line containing $\mathbf{c}$

(1) either increase from 1 to n,

(2) or decrease from n to 1,

(3) or remain constant c_j.

Since every line has two orientations, and it is impossible that all coordinates remain constant, the maximum degree is $\le (3^d-1)/2$, and we have equality for the center (only).

This suggests that the center of the board is probably the best opening move in the game (n is odd).

Finally, assume that n is even. Let $\mathbf{c} = (c_1, c_2, \dots, c_d) \in n^d$ be a point, and consider the family of those n-lines which contain $\mathbf{c}$. Fixing a proper subset index-set $I \subset \{1,2,\dots,d\}$, there is at most *one* n-in-a-line in this family for which the jth coordinates of the points on the line remain constant c_j for each $j \in I$, and increase or decrease for each $j \notin I$. So the maximum degree is $\le \sum_{i=0}^{d-1} \binom{d}{i} = 2^d - 1$, and equality occurs if for some fixed $c \in \{1,\dots,n\}$ every coordinate c_j equals c or $n+1-c$ ($j=1,2,\dots,d$). □

The well-known Hales–Jewett theorem, a cornerstone of Ramsey Theory, is about a combinatorial property of the family of n-in-a-line's in the n^d hypercube. The combinatorial property is the existence of *monochromatic* n-in-a-line's in an arbitrary k-coloring ($k \ge 2$) of the points of n^d. Actually, the Hales–Jewett proof gives more: it guarantees the existence of a monochromatic *combinatorial line.* A *combinatorial line* is basically a "1-parameter set"; to explain what it means, write $[n] = \{1,2,\dots,n\}$. An *x-string* is a finite word $a_1 a_2 a_3 \cdots a_d$ of the symbols $a_i \in [n] \cup \{x\}$ where at least one symbol a_i is x. An x-string is denoted by $\mathbf{w}(x)$. For every integer $i \in [n]$ and x-string $\mathbf{w}(x)$, let $\mathbf{w}(x;i)$ denote the string obtained from $\mathbf{w}(x)$ by replacing each x by i. A *combinatorial line* is a set of n strings $\{\mathbf{w}(x;i) : \ i \in [n]\}$ where $\mathbf{w}(x)$ is an x-string.

Every combinatorial line is a geometric line (n-in-a-line), but the converse is not true. Before showing a counter-example note that a *geometric line* can be described as an *xx'-string* $a_1 a_2 a_3 \cdots a_d$ of the symbols $a_i \in [n] \cup \{x\} \cup \{x'\}$ where at least one symbol a_i is x or x'. An xx'-string is denoted by $\mathbf{w}(xx')$. For every integer $i \in [n]$ and xx'-string $\mathbf{w}(xx')$, let $\mathbf{w}(xx';i)$ denote the string obtained from $\mathbf{w}(xx')$ by replacing each x by i and each x' by $(n+1-i)$. A *directed geometric line* is a *sequence* $\mathbf{w}(xx';1)$, $\mathbf{w}(xx';2)$, $\mathbf{w}(xx';3)$,…, $\mathbf{w}(xx';n)$ of n strings where $\mathbf{w}(xx')$ is an xx'-string. Note that every geometric line has two orientations.

As we said before, it is *not* true that every geometric line is a combinatorial line. What is more, it is clear from the definition that there are substantially more geometric lines than combinatorial lines: in the n^d Tic-Tac-Toe game there are $((n+2)^d - n^d)/2$ geometric lines and $(n+1)^d - n^d$ combinatorial lines. Note that the maximum degree of the family of combinatorial lines is $2^d - 1$, and the maximum is attained in the points of the "main diagonal" $(j, j, \ldots, j)$ where j runs from 1 to n.

For example, in ordinary Tic-Tac-Toe:

$(1,3)$	$(2,3)$	$(3,3)$
$(1,2)$	$(2,2)$	$(3,2)$
$(1,1)$	$(2,1)$	$(3,1)$

the "main diagonal" $\{(1,1),(2,2),(3,3)\}$ is a combinatorial line defined by the x-string xx, $\{(1,1),(2,1),(3,1)\}$ is another combinatorial line defined by the x-string $x1$, but the "other diagonal"

$$\{(1,3),(2,2),(3,1)\}$$

is a geometric line defined by the xx'-string xx'. The "other diagonal" is the only geometric line of the 3^2 game which is *not* a combinatorial line.

The Hales–Jewett threshold $HJ(n,k)$ is the smallest integer d such that in each k-coloring of the points of n^d there is a monochromatic *geometric* line. The modified Hales–Jewett threshold $HJ^c(n,k)$ is the smallest integer d such that in each k-coloring of n^d there is a monochromatic *combinatorial* line ("c" stands for "combinatorial"). Trivially

$$HJ(n,k) \le HJ^c(n,k).$$

In the case of "two colors" ($k=2$) we write: $HJ(n) = HJ(n,2)$ and $HJ^c(n) = HJ^c(n,2)$; trivially $HJ(n) \le HJ^c(n)$.

In 1963 Hales and Jewett made the crucial observation that Van der Waerden's famous "double-induction proof" (to prove his well-known theorem about monochromatic arithmetic progressions, see later) can be adapted to the n^d hypercube. This way Hales and Jewett proved that $HJ^c(n,k) < \infty$ for all positive integers n and k. This, of course, implies $HJ(n,k) < \infty$ for all positive integers n and k.

The Hales–Jewett theorem has a wonderful application to the hypercube Tic-Tac-Toe: it implies that the d-dimensional n^d Tic-Tac-Toe is a first player win if the dimension d is large enough in terms of the winning size n. We will return to this later. The Hales–Jewett theorem is a deep qualitative result; unfortunately, the quantitative aspects are truly dreadful!

How large is $HJ(n) = HJ(n, 2)$? Well, this is a famous open problem. We have to admit that, in spite of all efforts, our present knowledge on the Hales–Jewett number $HJ(n)$ is still rather disappointing. The best known upper bound on $HJ(n)$ was proved by Shelah [1988]. It is a primitive recursive function (namely the *supertower* function), which is much-much better than the original Van der Waerden–Hales–Jewett bound. The original "double-induction" argument gave the (totally ridiculous) Ackermann function. The much better Shelah's bound is still far too large for "layman combinatorics".

For a precise discussion of Shelah's upper bound we have to introduce the so-called Grzegorczyk hierarchy of primitive recursive functions. In fact, we define the *representative* function for each class. (For a more detailed treatment of primitive recursive functions we refer the reader to any monograph in Mathematical Logic.)

Let $g_1(n) = 2n$, and for $i > 1$, let $g_i(n) = g_{i-1}\Big(g_{i-1}\big(\ldots g_{i-1}(1)\ldots\big)\Big)$, where g_{i-1} is taken n times. An equivalent definition is $g_i(n+1) = g_{i-1}\big(g_i(n)\big)$. For example, $g_2(n) = 2^n$ is the exponential function,

$$g_3(n) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$$

is the "tower function" of height n. The next function $g_4(n+1) = g_3\big(g_4(n)\big)$ is that we call the "Shelah's supertower function" because this is exactly what shows up in Shelah's proof. Note that $g_k(x)$ is the *representative* function of the $(k+1)$st Grzegorczyk class.

In 1988 Shelah proved the following remarkable upper bound.

Shelah's primitive recursive upper bound: *For every $n \geq 1$ and $k \geq 1$,*

$$HJ^c(n,k) \leq \frac{1}{(n+1)k} g_4(n+k+2).$$

That is, given any k-coloring of the hypercube n^d, where the dimension $d \geq \frac{1}{(n+1)k} g_4(n+k+2)$, there is always a monochromatic **combinatorial** *line.*

What can we say about the small values of $HJ(n) = HJ(n, 2)$ and $HJ^c(n) = HJ^c(n, 2)$? An easy case-study shows that $HJ(3) = HJ^c(3) = 3$, but the numerical value of $HJ(4)$ remains a mystery. We know that it is ≥ 5 (see Golomb–Hales [2002]), and also know that it is finite, but no one can prove a "reasonable" upper bound like $HJ(4) \leq 1000$ or even a much weaker bound like $HJ(4) \leq 10^{1000}$. Shelah's proof gives the explicit upper bound

$$HJ(4) \leq HJ^c(4) \leq g_3(24) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$$

where the "height" of the tower is 24. This upper bound is *absurdly large*; it is rather disappointing that Ramsey Theory is unable to provide a "reasonable" upper bound even for the first "non-trivial" value $HJ(4)$ of the Hales–Jewett function $HJ(n)$.

In general, it remains an open problem to decide whether or not $HJ(n)$ is less than the tower function $g_3(n)$; perhaps $HJ(n)$ is simply plain exponential.

2. A new lower bound. In their original paper Hales and Jewett [1963] proved a linear lower bound to the Hales–Jewett number: $HJ(n) \geq n$. Here we improve this to an *exponential* lower bound.

To illustrate the idea on a simpler example, we start the discussion with $HJ^c(n)$ ("combinatorial lines"), which is less interesting from our game-theoretic/geometric viewpoint, but more natural from a purely combinatorial viewpoint.

First we recall Van der Waerden's famous combinatorial theorem on arithmetic progressions.

Van der Waerden's theorem [1927] *For all positive integers n and k, there exists an integer W such that, if the set of integers $\{1, 2, \ldots, W\}$ is k-colored, then there always exists a monochromatic n-term arithmetic progression.*

Let $W(n, k)$ be the least such integer; we call it the Van der Waerden threshold. For $k = 2$ we simply write $W(n) = W(n, 2)$.

We claim the following one-sided inequality between the Van der Waerden threshold and the "combinatorial" Hales–Jewett threshold:

$$\frac{W(n,k) - 1}{n - 1} \leq HJ^c(n, k). \tag{1}$$

To prove (1) write $W = HJ^c(n, k) \cdot (n - 1)$, and let χ denote an arbitrary k-coloring of the interval $[0, W] = \{0, 1, 2, \ldots, W\}$; we want to show that there is a monochromatic n-term arithmetic progression in $[0, W]$. Consider the d-dimensional hypercube $[n]^d$ with $d = HJ^c(n)$, where, as usual, $[n] = \{1, 2, \cdots, n\}$. Let $\mathbf{w} = (a_1, a_2, \cdots, a_d) \in [n]^d$ be an arbitrary point in the hypercube. We can define the color of a point $\mathbf{w}$ as the χ-color of the coordinate-sum

$$g(\mathbf{w}) = (a_1 - 1) + (a_2 - 1) + (a_3 - 1) + \ldots + (a_d - 1). \tag{2}$$

We refer to this particular k-coloring of hypercube $[n]^d$ as the "lift-up of χ". Since the dimension of the hypercube is $d = HJ^c(n)$, there is a monochromatic combinatorial line in $[n]^d$ (monochromatic in the "lift-up of χ"). Thus the coordinate-sums of the n points on the line form a χ-monochromatic n-term arithmetic progression in $[0, W]$. This completes the proof of (1).

Remark. Inequality (1), using the linear mapping (2), is a short lemma in Shelah's paper [1988]. Is there an earlier appearance of this inequality?

In order to apply (1), we need a lower bound on the Van der Waerden number $W(n)$. If n is a prime, then Berlekamp [1968] proved the bound $W(n) > (n - 1)2^{n-1}$. Combined this with (1) gives

$$HJ^c(n) \geq 2^{n-1}. \tag{3'}$$

In general, for an arbitrary n (which is not necessarily a prime), the well-known Local Lemma (see Erdős-Lovász [1975]) gives the slightly weaker lower bound $W(n) \geq 2^{n-3}/n$; this implies

$$HJ^c(n) \geq \frac{2^{n-3}}{n^2}. \tag{3''}$$

Notice that lower bounds (3')-(3") are exponential.

How about the geometric threshold $HJ(n)$? Can we prove a similar exponential lower bound? The answer is yes, and we are going to employ the *quadratic* coordinate sum

$$Q(\mathbf{w}) = (a_1 - 1)^2 + (a_2 - 1)^2 + (a_3 - 1)^2 + \ldots + (a_d - 1)^2 \tag{4}$$

where $\mathbf{w} = (a_1, a_2, \cdots, a_d) \in [n]^d$. Notice that the old linear function g (see (2)) has a handicap: it may map a whole n-in-a-line into a single integer (as a "degenerate n-term arithmetic progression"). The quadratic function Q in (4) basically solves this kind of problem, but it leads to a minor technical difficulty: the Q-image of a geometric line is a *quadratic progression* (instead of an arithmetic progression). We pay a little price for this change: the set of n-term arithmetic progressions is a 2-parameter family, but the set of n-term quadratic progressions is a 3-parameter family. Also an n-term quadratic progression is a multiset with maximum multiplicity 2 (since a quadratic equation has 2 roots), representing at least $n/2$ distinct integers (another loss of a factor of 2). After this outline, we can easily work out the details as follows.

Any geometric line can be encoded as a string of length d over the alphabet $\Lambda = \{1, 2, \cdots, n, x, x'\}$ (where x' represents "reverse x") with at least one x or x'. The n points P_1, P_2,..., P_n constituting a geometric line can be obtained by substituting $x = 1, 2, \ldots, n$ and $x' = n + 1 - x = n, n - 1, \ldots, 1$. If the encoding of a geometric line L contains a occurences of symbol x and b occurences of symbol x', and $L = \{P_1, P_2, \ldots, P_n\}$ where P_i arises by the choice $x = i$, the sequence $Q(P_1)$, $Q(P_2)$, $Q(P_3)$,..., $Q(P_n)$ (see (4)) has the form

$$a(x-1)^2 + b(n-x)^2 + c = (a+b)x^2 - 2(a+bn)x + (c+a+bn^2) \quad \text{as} \quad x = 1, 2, \ldots, n. \tag{5}$$

Let $W = HJ(n) \cdot (n-1)^2$; the quadratic sequence (5) falls into the interval $[0, W]$. A quadratic sequence $Ax^2 + Bx + C$ with $x = 1, 2, \ldots, n$ is called an *n-term non-degenerate quadratic progression* if A, B, C are integers and $A \neq 0$.

Motivated by Van der Waerden's theorem, we define $W_q(n)$ to be the least integer such that any 2-coloring of $[0, W_q(n) - 1] = \{0, 1, 2, \ldots, W_q(n) - 1\}$ yields a monochromatic n-term non-degenerate quadratic progression. We prove the following inequality (an analog of (1))

$$\frac{W_q(n) - 1}{(n-1)^2} \leq HJ(n). \tag{6}$$

In order to prove (6), let $W = W_q(n) - 1$ and let χ be an arbitrary 2-coloring of the interval $[0, W] = \{0, 1, 2, \ldots, W\}$. We want to show that there is a monochromatic n-term non-degenerate quadratic progression in $[0, W]$. Consider the d-dimensional hypercube $[n]^d$ with $d = HJ(n)$, where $[n] = \{1, 2, \cdots, n\}$. Let $\mathbf{w} = (a_1, a_2, \cdots, a_d) \in [n]^d$ be an arbitrary point in the hypercube. We can define a color of point $\mathbf{w}$ as the χ-color of the quadratic coordinate sum (see (4))

$$Q(\mathbf{w}) = (a_1 - 1)^2 + (a_2 - 1)^2 + (a_3 - 1)^2 + \ldots + (a_d - 1)^2.$$

We refer to this particular 2-coloring of hypercube $[n]^d$ as the "lift-up of χ". Since the dimension of the hypercube is $d = HJ(n)$, there is a monochromatic geometric line in

$[n]^d$ (monochromatic in the "lift-up of χ"). Thus the quadratic coordinate sums of the n points on the line form a χ-monochromatic n-term non-degenerate quadratic progression in $[0, W]$. This completes the proof of (6).

Next we need a lower bound for $W_q(n)$; the following simple bound suffices for our purposes:

$$W_q(n) \geq \frac{2^{n/4}}{3n^2}. \tag{7}$$

Lower bound (7) is an easy application of the Local Lemma. In fact, we apply the following well-known corollary of the Local Lemma.

Erdős–Lovász 2-Coloring Theorem. *If $\mathcal{F}$ is an n-uniform hypergraph, and its Maximum Degree is at most $2^{n-3}/n$ (i.e., every hyperedge intersects at most $2^{n-3}/n$ other hyperedges), then the hypergraph has a Proper 2-Coloring (i.e., the points can be colored by two colors so that no hyperedge $A \in \mathcal{F}$ is monochromatic).*

Remark. The very surprising message of the Erdős–Lovász theorem is that the "global size" of hypergraph $\mathcal{F}$ is irrelevant—it can even be infinite!—only the "local size" matters.

We apply the Erdős-Lovász 2-coloring theorem as follows. First note that an n-term non-degenerate quadratic progression Ax^2+Bx+C represents at least $n/2$ different integers (since a quadratic polynomial has at most 2 real roots). Three different terms "almost" determine an n-term quadratic progression; more precisely, they determine less than n^3 n-term quadratic progressions. Thus, any n-term non-degenerate quadratic progression contained in $[1, W]$, where $W = W_q(n)$, intersects fewer than $n^4 \cdot W^2$ other n-term non-degenerate quadratic progressions in $[1, W]$. It follows that

$$8n^4 \cdot W^2 > 2^{n/2}; \tag{8}$$

indeed, otherwise the Erdős-Lovász 2-coloring theorem applies, and yields the existence of a 2-coloring of $[1, W]$ with no monochromatic n-term non-degenerate quadratic progression, which contradicts the choice $W = W_q(n)$. Now (8) implies (7).

Combining (6) and (7) we obtain

$$HJ(n) \geq \frac{2^{n/4}}{3n^4}. \tag{9}$$

(9) is somewhat weaker than (3')-(3"), but it is still exponential, representing a big improvement on the original linear lower bound $HJ(n) \geq n$ of Hales and Jewett.

Note that Berlekamp's explicit algebraic construction—$W(n) > (n-1)2^{n-1}$ if n is a prime—is a Proper *Halving* 2-Coloring, where the two color classes have exactly the same size (or differ by one). The proof of the Erdős–Lovász 2-Coloring Theorem, on the other hand, does not provide a Proper *Halving* 2-Coloring. It is not clear at all how to modify the standard proof of the Local Lemma to get a proper halving 2-coloring. Also the "lift-up" destroys the halving property. This raises the following natural question. Let $HJ_{1/2}(n)$ denote the "halving" version of the Hales–Jewett number: let $HJ_{1/2}(n)$ be the least integer

d such that in each *halving* 2-coloring of n^d there is a monochromatic geometric line (i.e., n-in-a-line).

Is it true that the "halving" threshold $HJ_{1/2}(n)$ is also (at least) exponentially large? The answer is, once again, "yes". One possible way to prove it is to involve "rainbow" 3-colorings.

A *rainbow* k-coloring of hypergraph $\mathcal{F}$ means that each hyperedge $A \in \mathcal{F}$ contains all k different colors.

Of course, the concepts of proper k-coloring and rainbow k-coloring are identical for $k = 2$, but become very different for $k \geq 3$.

Here is a trivial, but still useful observation.

Rainbow Fact: If $\mathcal{F}$ is an arbitrary finite hypergraph such that it has a rainbow 3-coloring, then it also has a proper halving 2-coloring (i.e., the two color classes have equal size, or differ by one).

Indeed, let C_1, C_2, C_3 be the 3 color classes of the vertex-set in a rainbow 3-coloring of hypergraph $\mathcal{F}$, and assume that $|C_1| \leq |C_2| \leq |C_3|$. Since C_3 is the largest color class, one can always divide it into two parts $C_3 = C_{3,1} \cup C_{3,2}$ such that the two sums $|C_1| + |C_{3,1}|$ and $|C_2| + |C_{3,2}|$ become equal (or differ by at most one). Coloring $C_1 \cup C_{3,1}$ red and $C_2 \cup C_{3,2}$ blue gives a proper halving 2-coloring of hypergraph $\mathcal{F}$.

One possible way to prove that $HJ_{1/2}(n)$ is also (at least) exponentially large is to repeat the proof of (9) with *rainbow 3-colorings* instead of proper 2-colorings, and to apply the Rainbow Fact.

Another—more direct and much better—way to prove an exponential lower bound for the halving Hales–Jewett number is to apply the following inequality:

$$HJ_{1/2}(n) \geq HJ(n-2). \tag{10}$$

Note that inequality (10) is "hypercube-specific"; it does not extend to a general hypergraph result like the Rainbow Fact above.

In fact, the following slightly stronger version of (10) holds:

$$HJ^*_{1/2}(n) \geq HJ(n-2), \tag{11}$$

where $HJ^*_{1/2}(n)$ is the largest dimension d_0 such that for any $d < d_0$ the n^d hypercube has a proper halving 2-coloring (*proper* means that there is no monochromatic geometric line).

Since $HJ_{1/2}(n)$ denotes the least integer d such that in each *halving* 2-coloring of n^d there is a monochromatic geometric line (i.e., n-in-a-line), we trivially have $HJ_{1/2}(n) \geq HJ^*_{1/2}(n)$, and we cannot exclude the possibility of a strict inequality $HJ_{1/2}(n) > HJ^*_{1/2}(n)$ for some n. This means the halving Hales–Jewett number is possibly(!) a "fuzzy threshold"; unlike the ordinary Hales–Jewett number $HJ(n)$, where there is a critical dimension d_0 such that for every 2-coloring of n^d with $d \geq d_0$ there is always a monochromatic geometric line, and for every n^d with $d < d_0$ there is a 2-coloring with no monochromatic geometric line. In the halving case we cannot prove the existence of such a critical dimension.

By adding a trivial upper bound to (10)-(11), we have

$$HJ(n) \geq HJ_{1/2}(n) \geq HJ^*_{1/2}(n) \geq HJ(n-2). \tag{12}$$

Here is a proof of (11). The idea is to divide the $n^{HJ(n-2)-1}$ hypercube into subcubes of the form $(n-2)^j$, $j \leq HJ(n-2)-1$, and color them independently. We make use of the Hales–Jewett linear lower bound

$$HJ(n) \geq n. \tag{13}$$

The "large dimension" in (13) guarantees that most of the volume of the hypercube $n^{HJ(n-2)-1}$ lies on the "boundary"; this is why we can combine the proper 2-colorings of the subcubes $(n-2)^j$, $j \leq HJ(n-2)-1$ to obtain a proper *halving* 2-coloring of the whole.

The exact details go as follows. Let $H = [n]^d$ where $d = HJ(n-2)-1$ and $[n] = \{1, 2, \ldots, n\}$; so there is a proper 2-coloring for the "center" $(n-2)^d \subset H$. We need to show that there is a proper halving 2-coloring of H. We divide H into subcubes of the form $(n-2)^j$, $0 \leq j \leq d$: for each "formal vector"

$$\mathbf{v} = (v_1, v_2, \ldots, v_d) \in \{1, c, n\}^d$$

(here "c" stands for "center") we define the subhypercube $H_{\mathbf{v}}$ as the set of of all $(a_1, a_2, \ldots, a_d) \in H$ satisfying the following two requirements:

(1) $a_i = 1$ if and only if $v_i = 1$;

(2) $a_i = n$ if and only if $v_i = n$.

Then $H_{\mathbf{v}}$ is of size $(n-2)^j$, where the dimension $j = dim(H_{\mathbf{v}})$ is the number of coordinates of $\mathbf{v}$ equal to c, and the $H_{\mathbf{v}}$'s form a partition of H by mimicking the binomial formula

$$n^d = ((n-2)+2)^d = (n-2)^d + \binom{d}{1} 2 \cdot (n-2)^{d-1} + \binom{d}{2} 2^2 \cdot (n-2)^{d-2} + \ldots + 2^d. \tag{14}$$

Notice that $H_{(c,\ldots,c)}$ is the "center" of H; by assumption $H_{(c,\ldots,c)}$ has a proper 2-coloring.

Call $H_{\mathbf{v}}$ *degenerate* if its dimension $j = 0$; these are the 2^d "corners" of hypercube H.

The following fact is readily apparent.

Apparent Proposition: For any geometric line L (n-in-a-line) in $H = n^d$, there is some nondegenerate subhypercube $H_{\mathbf{v}} \subset H$ such that the intersection $L \cap H_{\mathbf{v}}$ is a geometric line ($(n-2)$-in-a-line) in $H_{\mathbf{v}}$ (considering $H_{\mathbf{v}}$ as an $(n-2)^j$ hypercube).

The Apparent Proposition implies that, any 2-coloring of H which is *improper* (i.e., there is a monochromatic line) must be improper in its restriction to a nondegenerate subhypercube $H_{\mathbf{v}}$.

As we said before, the "center" $H_{(c,\ldots,c)} \subset H$ has a proper 2-coloring; we use the colors X and O (like in an ordinary Tic-Tac-Toe play). Let the proportion of X's in the coloring be α_0; we can assume that $\alpha_0 \geq 1/2$. Considering all $(d-1)$-dimensional "slices" of the "center" $H_{(c,\dots,c)}$, the average proportion of X's is α_0, so the maximum proportion of X's, denoted by α_1, is greater or equal to α_0. It follows that the $(n-2)^{d-1}$ subhypercubes of H can be properly 2-colored with an α_1 fraction of X's (or O's; we can always flip a coloring!). Thus, inductively, we find a nondecreasing sequence

$$1/2 \le \alpha_0 \le \alpha_1 \le \ldots \le \alpha_d = 1$$

of "proportions" so that for $0 \le j \le d$, each $(n-2)^j$ subhypercube can be properly 2-colored with an α_{d-j} fraction of X's. For each $(n-2)^j$ subhypercube we have two options: either we keep this proper 2-coloring or we flip. By using this freedom, we can easily extend the proper 2-coloring of the "center" $H_{(c,\dots,c)} \subset H$ to a proper halving 2-coloring of H as follows. Let

$$A_k = \bigcup_{dim(H_{\mathbf{v}}) \ge d-k} H_{\mathbf{v}}.$$

By induction on k (as $k = 0, 1, 2, \dots, d$), we give a 2-coloring χ of H which is proper on each of the subhypercubes $H_{\mathbf{v}} \subset A_k$ and

$$disc(\chi, A_k) \le (2\alpha_k - 1) \cdot (n-2)^{d-k}, \tag{15}$$

where $disc(\chi, A_k)$ denotes the *discrepancy*, i.e., the absolute value of the difference between the sizes of the color classes. Notice that $2\alpha_k - 1 = \alpha_k - (1 - \alpha_k)$ and $(n-2)^j$ is the volume of a j-dimensional $H_{\mathbf{v}}$.

At the end, when $k = d$, coloring χ will be a proper *halving* 2-coloring (indeed, the color classes on all of H will differ by at most $(2\alpha_d - 1) \cdot (n-2)^{d-d} = 1$).

(15) is trivial for $k = 0$: on $A_0 = H_{(c,\dots,c)}$ (the "center") our 2-coloring χ is the above-mentioned proper 2-coloring of the "center" with X-fraction α_0.

Next comes the general induction step: let (15) be satisfied for some $(k-1) \ge 0$. The number N_{d-k} of subhypercubes $H_{\mathbf{v}}$ of dimension $(d-k)$ is $\binom{d}{k}2^k$ (binomial theorem: see (14)), so by $d = HJ(n-2) - 1$ and (13), $N_{d-k} \ge 2d \ge n-2$. Thus since $\alpha_k \ge \alpha_{k-1}$, we have

$$N_{d-k} \cdot (2\alpha_k - 1) \cdot (n-2)^{d-k} \ge (2\alpha_{k-1} - 1) \cdot (n-2)^{d-(k-1)} \ge disc(\chi, A_{k-1}). \tag{16}$$

In view of inequality (16) we have room enough to extend χ from A_{k-1} to A_k by coloring a suitable number of the $H_{\mathbf{v}}$ of dimension $(d-k)$ with an α_k-fraction of X's, and the rest with an α_k-fraction of O's ("we flip the coloring"). This completes the induction proof of (15), and (11) follows.

We summarize these results in a single statement.

Theorem 1. *We have*

$$HJ(n) \ge HJ_{1/2}(n) \ge HJ^*_{1/2}(n) \ge HJ(n-2) \ge \frac{2^{(n-2)/4}}{3n^4}. \tag{17}$$

There remains a huge gap between (17) and Shelah's supertower upper bound. Which one is closer to the truth?

3. Game-theoretic aspects. Consider the n^d Tic-Tac-Toe game with $d = HJ(n)$. Then draw play is impossible (i.e., every play has a winner), so we have only two options: (1) either the first player has a winning strategy, (2) or the second player has a winning strategy. (This follows from a general theorem of Zermelo: every finite 2-player game of complete information is *determined*).

Next we apply the so-called

Strategy Stealing Argument. *Let* $(V, \mathcal{F})$ *be an arbitrary finite hypergraph. Playing the Generalized Tic-Tac-Toe Game on* $(V, \mathcal{F})$*, the first player can always force at least a draw, i.e., a draw or possibly a win.*

Remark. The Strategy Stealing Argument was used by J. Nash in the late 1940s in his "existential" solution of the game Hex. Here the *Generalized Tic-Tac-Toe Game* on $(V, \mathcal{F})$ simply means that the two players alternately take the points of V, and that player declared the winner who first occupies a whole winning set $A \in \mathcal{F}$; otherwise the play ends in a draw.

For completeness we outline the simple **proof**. It is not constructive! Assume that the second player (=**II**) has a winning strategy *STR*, we want to obtain a contradiction. The idea is to see what happens if the first player (=**I**) steals and uses *STR*. A winning strategy for a player is a list of instructions telling the player that if the opponent does this, then he does that, so if the player follows the instructions, he will always win. Now **I** can use **II**'s winning strategy *STR* to win as follows. **I** takes an arbitrary first move, and *pretends* to be the second player (he ignores his own first move). After **II**'s each move, **I**, as a fake second player, reads the instruction in *STR* to take action. If **I** is told to take a move that is still available, he takes it. If this move was taken by him before as his ignored "arbitrary" first move, then he takes another "arbitrary move". The crucial point here is that an extra move, namely the last "arbitrary move", only *benefits* **I** in a generalized Tic-Tac-Toe game. □

Corollary: ("Winning by Ramsey Theory") *Let* $(V, \mathcal{F})$ *be an arbitrary finite hypergraph. Suppose that the family* $\mathcal{F}$ *of winning sets has the property that there is no proper halving 2-coloring. Then, playing the Generalized Tic-Tac-Toe Game on* $(V, \mathcal{F})$*, the first player can always force a win.* □

The Corollary describes a subclass of Generalized Tic-Tac-Toe Games with the remarkable property that one can easily determine the winner without being able to say how one wins.

This is a "soft" existential criterion. Since the main objective of Game Theory is to find an explicit winning or drawing strategy, we have to conclude that *winning* is far more complex than Ramsey Theory!

For example, it is hugely disappointing that we know only two(!) explicit winning strategies in the whole class of n^d Tic-Tac-Toe games: the 3^3 version, which has an easy winning strategy, and the 4^3 version, which has an extremely complicated winning strategy.

It seems to be highly unlikely that the game-theoretic "phase transition" between win and draw for the n^d Tic-Tac-Toe games is anywhere close to the Hales–Jewett number $HJ(n)$, but no method is known for handling this problem.

To be precise, we introduce the so-called *Win Number*. Let $\mathbf{w}(n\text{−line})$ denote the least threshold such that for every $d \geq \mathbf{w}(n\text{−line})$ the n^d Tic-Tac-Toe game is a first player win ("**w**" stands for "win"). The Corollary above yields the inequality $\mathbf{w}(n\text{−line}) \leq HJ(n)$. By Patashnik's computer-assisted work we know that $\mathbf{w}(4\text{−line}) = 3$ (see Patashnik [1980]); luckily we don't really need to know the value of the extremely difficult threshold $HJ(4)$. Unfortunately we are not so lucky with $\mathbf{w}(5\text{−line})$, which remains a total mystery. The upper bound $\mathbf{w}(5\text{−line}) \leq HJ(5)$ is "useless": Shelah's proof gives a totally ridiculous upper bound for $HJ(5)$.

Open Problem 1. *Is it true that* $\mathbf{w}(n\text{−line}) < HJ(n)$ *for all sufficiently large values of* n*? Is it true that*

$$\frac{\mathbf{w}(n\text{−line})}{HJ(n)} \longrightarrow 0 \quad \text{as} \quad n \to \infty?$$

4. Weak Win. We know very little about the Hales–Jewett number $HJ(n)$, and know very little about the Win Number $\mathbf{w}(n\text{−line})$, but we know a lot about the Weak Win number $\mathbf{ww}(n\text{−line})$. What is the Weak Win number, and what is the motivation behind it?

First we try to pinpoint the reason why the n^d Tic-Tac-Toe games are so difficult. Well, these are all Who-Does-It-First games (which player gets the first n-in-a-line). Who-Does-It-First reflects competition, a key ingredient of every game playing, but it is not the most fundamental question. The most fundamental question is "What are the achievable configurations, achievable but not necessarily first?", and the complementary question "What are the impossible configurations?". Drawing the line between "doable" and "impossible" (doable but not necessarily first!) is the primary question. First we have to clearly understand "what is doable"; "what is doable first" is a secondary question. If "doing-it-first" is ordinary win, then we may call "doing it, but not necessarily first" a Weak Win. A failure to achieve a Weak Win is called a Strong Draw.

Here is the formal definition. On a given finite hypergraph $(V, \mathcal{F})$ (where V is the board and $\mathcal{F}$ is the family of winning sets) one can play the "symmetric" Generalized Tic-Tac-Toe Game and also the "one-sided" *Maker–Breaker* game, where the only difference is in the goals: (1) Maker's goal is to occupy a whole winning set $A \in \mathcal{F}$, but not necessarily first, and (2) Breaker's goal is simply to stop Maker (Breaker does not want to occupy any winning set). The player who achieves his goal is declared the winner—so a draw is impossible by definition. Of course, there are two versions: Maker can be the first or second player.

There is a trivial implication: if the first player can force a win in the Generalized Tic-Tac-Toe Game on $(V, \mathcal{F})$, then the same play gives him, as Maker, a win in the Maker-Breaker game on $(V, \mathcal{F})$. The converse is not true: ordinary 3^2 Tic-Tac-Toe is a simple counter-example.

We refer to Maker's win as a *Weak Win*. Weak Win is easier than ordinary win. While playing the Generalized Tic-Tac-Toe Game on a hypergraph, both players have their own threats, and either of them, fending off the other's, may build his own winning set. Therefore, a play is a delicate balancing between threats and counter-threats and can be of very intricate structure even if the hypergraph itself is simple.

The Maker–Breaker version is usually somewhat simpler. Maker doesn't have to waste valuable moves fending off his opponent's threats. Maker can simply concentrate on his own goal of *building*, and Breaker can concentrate on *blocking* the opponent (unlike the Generalized Tic-Tac-Toe game in which either player has to *build and block* at the same time). Doing one job at a time is definitely simpler.

As we said before, Weak Win is obviously easier than ordinary win, *but* "easier" does not mean "easy". Absolutely not! For example, the well-known and notoriously difficult *Hex* is equivalent to a Maker–Breaker game, but this fact doesn't help to find an explicit winning strategy. Indeed, let *WeakHex* be the Maker–Breaker game in which the board is the $n \times n$ hexagonal Hex board, Maker=White, Breaker=Black, and the winning sets are the connecting chains of White. We claim that Hex and *WeakHex* are equivalent. To prove it, first note that in Hex a draw is impossible. Indeed, in order to prevent the opponent from making a connecting chain, one must build a "river" separating the opponent's sides, and a "river" itself must contains a chain connecting the *other* pair of opposite sides. (This fact seems plausible, but the precise proof is not completely trivial, see Gale [1979].) This means that Breaker's goal in *WeakHex* (i.e. "blocking") is identical to Black's goal in Hex (i.e. "building first"). Here "identical" means that if Breaker has a winning strategy in *WeakHex* then Black has a winning strategy in Hex, and vice versa—in fact, the same strategy works. Since a draw is impossible, Hex and *WeakHex* are equivalent.

Now we are ready to define the Weak Win number $\mathbf{ww}(n\text{−line})$. Let $\mathbf{ww}(n\text{−line})$ denote the least threshold such that for every $d \geq \mathbf{ww}(n\text{−line})$ the first player can force a Weak Win in the n^d game ("**ww**" stands for "weak win"). In other words, playing on n^d the first player can always occupy an n-in-a-line (but not necessarily first!).

We have the trivial inequality

$$HJ(n) \geq HJ_{1/2}(n) \geq HJ^*_{1/2}(n) \geq \mathbf{w}(n\text{−line}) \geq \mathbf{ww}(n\text{−line}). \qquad (18)$$

(18) is trivial because a Strong Draw strategy of the second player—in fact, any drawing strategy!—yields the existence of a drawing terminal position, i.e., a proper halving 2-coloring; indeed, the first player can "steal" the second player's strategy.

A simple study of ordinary Tic-Tac-Toe yields $\mathbf{ww}(3\text{−line}) = 2 < 3 = \mathbf{w}(3\text{−line})$, and Patashnik's computer-assisted study of the 4^3 Tic-Tac-Toe yields $\mathbf{ww}(4\text{−line}) = \mathbf{w}(4\text{−line}) = 3$ (see Patashnik [1980]).

Open Problem 2. *Is it true that* $\mathbf{ww}(n\text{−line}) < \mathbf{w}(n\text{−line})$ *for all sufficiently large values of* n*? Is it true that*

$$\frac{\mathbf{ww}(n\text{−line})}{\mathbf{w}(n\text{−line})} \longrightarrow 0 \quad \text{as} \quad n \to \infty?$$

An equally natural problem is the following: *Is it true that*

$$\frac{\mathbf{ww}(n\text{−line})}{HJ(n)} \longrightarrow 0 \quad \text{as} \quad n \to \infty?$$

Here comes the good news: unlike Open Problems 1 and 2, which remain wide open, for the last question we have a positive solution. We just need to apply the following Weak Win Criterion (see Beck [1981]).

Weak Win Criterion. *Let $(V, \mathcal{F})$ be a finite hypergraph: V is an arbitrary finite set, and $\mathcal{F}$ is an arbitrary family of subsets of V. The Maker-Breaker Game on $(V, \mathcal{F})$ is defined as follows: the two players, called Maker and Breaker, alternately occupy previously unoccupied elements of the board V ; Maker's goal is to occupy a whole winning set $A \in \mathcal{F}$, Breaker's goal is to stop Maker. If $\mathcal{F}$ is n-uniform and*

$$\frac{|\mathcal{F}|}{|V|} > 2^{n-3} \cdot \Delta_2(\mathcal{F}),$$

where $\Delta_2(\mathcal{F})$ is the Max Pair-Degree, then Maker, as the first player, has a winning strategy in the Maker-Breaker Game on $(V, \mathcal{F})$.

The Max Pair-Degree is defined as follows: assume that, fixing any two distinct points of board V, there are $\leq \Delta_2(\mathcal{F})$ winning sets $A \in \mathcal{F}$ containing both points, and equality occurs for some point pair. Then we call $\Delta_2(\mathcal{F})$ the Max Pair-Degree of $\mathcal{F}$.

In particular, for **Almost Disjoint** *hypergraphs, where any two hyperedges have at most one common point (like a family of "lines"), the condition simplifies to $|\mathcal{F}| > 2^{n-3}|V|$.*

Remark. If $\mathcal{F}$ is n-uniform, then $\frac{|\mathcal{F}|}{|V|}$ is $\frac{1}{n}$ times the *Average Degree.* Indeed, this equality follows from the easy identity $n|\mathcal{F}| = \text{AverDeg}(\mathcal{F})|V|$.

The hypothesis of the Weak Win Criterion is a simple Density Condition: in a "dense" hypergraph Maker can always occupy a whole winning set.

The proof of the Criterion is based on a Power-of-Two Scoring System (similar to the proof of the well-known Erdős–Selfridge theorem [1973]).

We recall that the total number of winning lines in the n^d Tic-Tac-Toe is $\frac{(n+2)^d - n^d}{2}$ (see Simple Facts (a) at the beginning). Because the line-hypergraph is Almost Disjoint, the Weak Win Criterion applies and yields a Weak Win if

$$\frac{(n+2)^d - n^d}{2} > 2^{n-3} n^d.$$

This is equivalent to

$$\left(1 + \frac{2}{n}\right)^d > 2^{n-2} + 1. \tag{19}$$

Inequality (19) holds if

$$d > \frac{1}{2}(\log 2) \cdot n^2. \tag{20}$$

Consider small values of n. Inequality (19) holds for the 3^3, 4^4, 5^7, 6^{10}, 7^{14}, 8^{19}, 9^{25}, 10^{31}, ... Tic-Tac-Toe games, so in these games the first player can force a Weak Win. Note that 3^3 and 4^4 on the list can be replaced by 3^2 and 4^3.

The list of small Weak Win n^d games: 3^2, 4^3, 5^7, 6^{10}, 7^{14}, 8^{19}, ... is complemented by the following list of known small Strong Draw games: 4^2, 8^3, 14^4, 20^5, 24^6, 26^7, ... There is a big gap between the two lists, proving that our knowledge of the small n^d games is very unsatisfactory.

Let's return to (20); it implies the bound

$$\frac{\log 2}{2} n^2 \geq \mathbf{ww}(n\text{−line}). \tag{21}$$

Thus comparing (17), (18), and (21) we have (assume that n is large)

$$HJ^*_{1/2}(n) \geq \frac{2^{(n-2)/4}}{3n^4} > \frac{\log 2}{2} n^2 \geq \mathbf{ww}(n\text{−line}), \tag{22}$$

that is, asymptotically the Hales–Jewett threshold $HJ^*_{1/2}(n)$ is (at least) exponential and the weak win threshold $\mathbf{ww}(n\text{−line})$ is (at most) quadratic. Roughly speaking, Ramsey Theory has nothing to do with Weak Win!

Inequality (22) leads to a very interesting problem as follows.

Delicate win or delicate draw? A wonderful question! We mention two particularly interesting subclasses of the family of all finite hypergraphs.

Delicate Win Class ("Forced win but Drawing Position still exists"): It contains those hypergraphs $\mathcal{F}$ which have a Drawing Terminal Position (=proper halving 2-coloring), but playing the Generalized Tic-Tac-Toe Game on $\mathcal{F}$ the first player can nevertheless force a win.

Delicate Draw Class: It contains those hypergraphs $\mathcal{F}$ for which the Generalized Tic-Tac-Toe Game is a Draw, but the first player can still force a Weak Win.

The 4^3 Tic-Tac-Toe is the only n^d game in the Delicate Win Class that we know, and the ordinary 3^2 Tic-Tac-Toe is the only n^d game in the Delicate Draw Class that we know. Are there other examples? This is an open problem!

What (22) implies is that the *union* of the Delicate Win Class and the Delicate Draw Class is *infinite*. Indeed, each n^d Tic-Tac-Toe with dimension d satisfying the inequality

$$HJ^*_{1/2}(n) > d \geq \mathbf{ww}(n\text{−line}) \tag{23}$$

belongs to one of these classes: if it is a first player win, the game belongs to the Delicate Win Class, if it is a draw game, then it goes to the Delicate Draw Class. Of course, (22) implies that the range (23) is nonempty, in fact, it is a very large range (if n is large). Unfortunately, we cannot decide which class (Delicate Win or Delicate Draw) for any single game in the large range (23). To decide which one is a truly wonderful open problem!

We conclude with a remark about inequality (22). It shows that the "halving" Hales–Jewett number $HJ_{1/2}(n)$ is at least exponential, but, unfortunately, we don't have a clue about the true order of magnitude (the best known upper bound is the enormous super-tower function). The (at least) exponential Ramsey Theory threshold $HJ_{1/2}(n)$ (about drawing terminal positions) is well-separated from the (at most) quadratic Weak Win threshold $\mathbf{ww}(n\text{−line})$. The major difference between the two thresholds is that the latter is known to be (roughly) quadratic. Indeed, we have the lower bound

$$\mathbf{ww}(n\text{−line}) > \frac{cn^2}{\log n}, \tag{24}$$

where $c = (\log 2)/16 - o(1)$.

We say a few words about the proof of (24). The proof consists of two steps.
Step One: Degree Reduction by Partial Truncation.

A serious technical difficulty is that the n^d line-hypergraph is very far from being degree-regular. Indeed, the Average Degree of the family of winning lines in n^d is

$$\text{AverageDegree}(n^d) = \frac{n \cdot \text{familysize}}{\text{boardsize}} = \frac{n\left((n+2)^d - n^d\right)/2}{n^d} \approx \frac{n}{2}\left(e^{2d/n} - 1\right).$$

This is *much* smaller than the Maximum Degree, which is either $(3^d - 1)/2$ (n odd) or $2^d - 1$ (n even), see Simple Facts (b)-(c) at the beginning. In fact, the Average Degree is about (very roughly speaking) the nth root of the Maximum Degree. It is natural, therefore, to ask the following

Question A: Can one reduce the Maximum Degree of an arbitrary n-uniform hypergraph close to the order of the Average Degree?

The answer is an easy *yes* if one is allowed to throw out *whole* winning sets. But throwing out a whole winning set means that Breaker loses control over that set, and Maker might completely occupy it. So we cannot throw out whole sets, but we may throw out a few points from each winning set. In other words, we can *partially truncate* the winning sets, but we cannot throw them out entirely. So the right question is

Question B: Can one reduce the Maximum Degree of an arbitrary n-uniform hypergraph, by *partially* truncating the winning sets, close to the order of the Average Degree?

Well, the answer to Question B is *no* for general n-uniform hypergraphs (we leave it to the reader to construct an example), but it is *yes* for the special case of the n^d line-hypergraphs.

Degree Reduction Lemma. *Let $\mathcal{F}_{n,d}$ denote the family of n-in-a-line's (i.e., geometric lines) in the n^d board; $\mathcal{F}_{n,d}$ is an n-uniform Almost Disjoint hypergraph. Let $0 < \alpha < 1/2$ be an arbitrary real number. Then for each geometric line $L \in \mathcal{F}_{n,d}$ there is a $2\lfloor(\frac{1}{2} - \alpha)n\rfloor$-element subset $\widetilde{L} \subset L$ such that the truncated family $\widetilde{\mathcal{F}_{n,d}} = \{\widetilde{L} : L \in \mathcal{F}_{n,d}\}$ has Maximum Degree*

$$\text{MaxDegree}\left(\widetilde{\mathcal{F}_{n,d}}\right) < d + d^{\lceil d/\alpha n\rceil - 1}.$$

Step Two: Game-theoretic Local Lemma.

We combine the Degree Reduction Lemma with the following

Ugly Blocking Criterion for Almost Disjoint Hypergraphs: *Assume that hypergraph $\mathcal{F}$ is n-uniform and Almost Disjoint. If the global size $|\mathcal{F}|$ and the Max Degree satisfy the upper bounds*

$$|\mathcal{F}| < 2^{n^{1.1}} \quad \text{and} \quad \text{MaxDegree}(\mathcal{F}) < 2^{n - 4n^{2/5}},$$

then the second player can put his mark in every $A \in \mathcal{F}$, that is, he can force a Strong Draw.

If the hypergraph is (nearly) degree-regular, then the Average Degree and the Max Degree are (nearly) equal, and so the Weak Win Criterion above and this Blocking Criterion nearly complement each other.

We stop here. For more about Weak Win, we refer the reader to the new book of Beck titled "Tic-Tac-Toe Theory" (to appear in Cambridge University Press). For example, the detailed proof of (24) is in this book.

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