Description of SIR model extension for our COVID-19 analysis

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1 Model populations and parameters

We define the following functions:

\[
\begin{align*}
S(t) & : \text{over 65 susceptible} \\
I(t) & : \text{over 65 infected} \\
R(t) & : \text{over 65 recovered} \\
M(t) & : \text{over 65 mortality} \\
s(t) & : \text{under 65 susceptible} \\
s(t) & : \text{under 65 infected} \\
r(t) & : \text{under 65 recovered} \\
m(t) & : \text{under 65 mortality}
\end{align*}
\]  

and the following constants:

\[
\begin{align*}
\beta_{uu} & : \text{under 65/under 65 infection rate} \\
\beta_{uo} & : \text{under 65/over 65 infection rate} \\
\beta_{oo} & : \text{over 65-over 65 infection rate} \\
\gamma_u & : \text{under 65 recovery rate} \\
\gamma_o & : \text{over 65 recovery rate} \\
\delta_u & : \text{under 65 mortality rate} \\
\delta_o & : \text{over 65 mortality rate}
\end{align*}
\]

\[
\begin{align*}
\alpha_u & := \gamma_u + \delta_u \\
\alpha_o & := \gamma_o + \delta_o
\end{align*}
\]

We let \( S_f \) the fraction of the total population which is susceptible to infection; \( N_u, N_o, \) and \( N \) are the under-65, over-65, and total populations, respectively.
2 Model dynamics

Our model, which is a simple extension of the SIR model, assumes that these populations satisfy the following differential equations:

$$\frac{dI}{dt} = \beta_{uo} \cdot S(t) \cdot i(t)/N + \beta_{oo} \cdot S(t)I(t)/N - \alpha_o I(t)$$  \hspace{1cm} (6)

$$\frac{dS}{dt} = -\beta_{uo} \cdot S(t) \cdot i(t)/N - \beta_{oo} \cdot S(t)I(t)/N$$  \hspace{1cm} (7)

$$\frac{dR}{dt} = \gamma_o I(t)$$  \hspace{1cm} (8)

$$\frac{dM}{dt} = \delta_o I(t)$$  \hspace{1cm} (9)

$$\frac{di}{dt} = \beta_{uu} \cdot s(t) \cdot i(t)/N + \beta_{uo} \cdot s(t)I(t)/N - \alpha_u i(t)$$  \hspace{1cm} (10)

$$\frac{ds}{dt} = -\beta_{uu} \cdot s(t) \cdot i(t)/N - \beta_{uo} \cdot s(t)I(t)/N$$  \hspace{1cm} (11)

$$\frac{dr}{dt} = \gamma_u i(t)$$  \hspace{1cm} (12)

$$\frac{dm}{dt} = \delta_u i(t)$$  \hspace{1cm} (13)

(14)

**Explanation:** Consider the change $\frac{dI}{dt}$ in the infected over-65 population $I(t)$. It has two terms corresponding to positive changes, let us consider the first. $i(t)/N$ captures the rate at which encounters with random members of the overall population are actually encounters with younger infected individuals. $\beta_{uo}$ is the constant which translates this rate to the transmission rate to a single over-65 individual, and $S(t)$ is the total susceptible population. The other differential equations have analogous simple interpretations.

3 Correspondence of ideal $R_0$ with $\beta_{uu}, \beta_{uo}, \beta_{oo}$

To make use of our model in connection with real-world estimates of the transmissibility of COVID-19 it is necessary to make choices of $\beta_{uu}, \beta_{uo}, \beta_{oo}$ which correspond to known ranges of the ideal $R_0$ value of COVID-19.

In the simple SIR model, there is only one transmission parameter $\beta$, one recovery/removal parameter $\alpha$, and $R_0$ is the ratio $\beta/\alpha$, which is the expected number of new infections which would occur from one initially infected individual in a completely susceptible population.
Our extension of this model collapses to the SIR model in the case that $\beta_{uo} = \beta_{uu} = \beta_{oo}$ and $\alpha_u = \alpha_o$.

Defining $\rho_o = N_o/N$ and $\rho_u = N_u/N$, we assume that at time 0 we have a single infected individual, which corresponds to having $I(0) = \rho_o$ and $i(0) = \rho_u$, while $S(0) = N_o$ and $s(0) = N_u$. In this case we write $\beta := \beta_{uu} = \beta_{uo} = \beta_{oo}$, and we have at $t = 0$ that

\[
\frac{dI}{dt}(0) + \frac{di}{dt}(0) = \beta_oo\rho_o^2 + 2\beta_uo\rho_u\rho_o + \beta_{uu}\rho_u^2 - \alpha_o\rho_o - \alpha_u\rho_u
\]

\[
= \beta(\rho_o + \rho_u)^2 - \alpha_o\rho_o - \alpha_u\rho_u = \beta - (\alpha_o\rho_o + \alpha_u\rho_u).
\]

(15)

In particular, the correspondence between $\beta = \beta_{uu} = \beta_{uo} = \beta_{oo}$, $\alpha = \alpha_u\rho_u + \alpha_o\rho_o$ and $R_0$ are the same for our model as for the SIR model. Thus given, e.g., a target $R_{0uu}$ for the $R_0$ value we wish to set for transmission within the under-65 population, we translate this to the $\beta_uu$ transmission coefficient via

\[
\beta_{uu} := \frac{R_{0uu}}{\alpha_o\rho_o + \alpha_u\rho_u},
\]

and similarly for the the coefficients $\beta_{uo}$ and $\beta_{oo}$. In particular, we translate $R_0$ values into each transmission coefficient by setting them at the value which would achieve the given $R_0$ value in the population as a whole if all transmission coefficients were set to that value, reproducing the same $R_0$ value in the SIR model.

**Correction for $S_f$**

Our model includes the parameter $S_f$, which is the fraction of the initial population which is susceptible to infection. (As discussed in our main document, our findings about the relative merits of mitigation strategies are not sensitive to this parameter, though it affects total mortality estimates.)

Using $S_f = 1$ corresponds to assuming an initial population which is entirely susceptible to infection. This is the value used when estimating the $R_0$ value for COVID-19 from observed infection rates. Thus if we assume that in fact $S_f$ is less than 1, we must correct the given $R_0$ value (we use 2.8) by a factor of $\frac{1}{S_f}$. 

3