THE DIPOLE PROBLEM FOR $H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$–MAPS AND APPLICATION

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Abstract. In this note, we consider a dipole type problem related to the question of smooth approximation in the functional space $X = H^{1/2}(\mathbb{S}^2;\mathbb{S}^1)$. The results presented here are consequences of previous results obtained by A. Pisante and the author in [16]. Extensions to more general geometries for the starting manifold and open questions are also addressed.

Keywords: fractional Sobolev spaces; topological singularities; minimal connections.

Mathematics Subject Classification 2000: 46E35, 49Q15, 49Q20

1. Introduction

Recently there has been some interest in the study of maps between manifolds with $H^{1/2}$–regularity, especially in the context of the complex Ginzburg-Landau equations and their applications to superconductivity models (see [1], [2], [4], [6], [8], [9], [10],[13], [16], [17], [18], [19] and [20]). In this note, we shall consider one of the most common class of such maps, namely,

$$X = H^{1/2}(\mathbb{S}^2;\mathbb{S}^1) := \big\{ g \in L^2(\mathbb{S}^2;\mathbb{R}^2)\,;\ |g| = 1 \text{ a.e. in } \mathbb{R}^2 \text{ and } |g|_{1/2} < +\infty \ \big\}\,, \tag{1.1}$$

where $|\cdot|_{1/2}$ denotes the standard Gagliardo fractional seminorm, i.e.,

$$|g|^2_{1/2} = \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \frac{|g(x)-g(y)|^2}{|x-y|^3}\,dxdy\,.$$

One easily sees that X is a closed subset of the Sobolev space $H^{1/2}(\mathbb{S}^2;\mathbb{R}^2)$ and that it defines a complete metric space for the distance induced by the $H^{1/2}$–norm $\|\cdot\|_{1/2} = \|\cdot\|_{L^2(\mathbb{S}^2)} + |\cdot|_{1/2}$.

The main feature of the space X is that the subspace of smooth maps $X \cap C^\infty(\mathbb{S}^2)$ fails to be dense in X with respect to the strong topology. However density holds for the weak topology (see [19]). A typical example of a map in X which can not be strongly approximated by smooth maps is given by $g : \mathbb{S}^2 \to \mathbb{S}^1$,

$$g(x_1,x_2,x_3) = \frac{(x_1,x_2)}{\sqrt{x_1^2+x_2^2}}\,.$$

Indeed, it can be shown, using degree theory, that g cannot be smoothed near its singularities at the north and south poles without violating the constraint of being $\mathbb{S}^1$-valued. This simple example contains the essence of the obstruction and shows its topological nature.

For an arbitrary map $g \in X$, a characterization of the topologically relevant part of the singular set of g has been obtained in terms of a certain distribution $T(g)$ in [2], [11], [19] (we also refer to [10] for an alternative approach in terms of Cartesian currents and to [3], [11] for higher dimensional analogues). Given $\varphi \in \mathrm{Lip}(\mathbb{S}^2;\mathbb{R})$, an arbitrary extension $u \in H^1(B^3;\mathbb{R}^2)$ of g and an arbitrary extension $\Phi \in \mathrm{Lip}(B^3;\mathbb{R})$ of φ, the distribution $T(g)$ is defined through its action on φ by

$$\langle T(g), \varphi\rangle = \int_{B^3} H(u)\cdot\nabla\Phi \tag{1.2}$$

where

$$H(u) = 2\big(\partial_2 u\wedge\partial_3 u, \partial_3 u\wedge\partial_1 u, \partial_1 u\wedge\partial_2 u\big) \in L^1(B^3)\,. \tag{1.3}$$

One observes that $\operatorname{div} H(u) = 0$ in $\mathcal{D}'(B^3)$ so that the integral in (1.2) formally reduces to an integration on the plane. In particular, this integral only depends on g and φ and so $T(g)$ is well defined (see [2]). As shown in [19] (see also [2]), g can be approximated strongly by smooth maps if and only if $T(g) = 0$ and in particular $T(g) = 0$ whenever g is smooth.

If $g \in X \cap W^{1,1}(\mathbb{S}^2)$, the integration by parts in (1.2) can be rigorously performed (see [2]) and yields

$$\langle T(g), \varphi\rangle = -\int_{\mathbb{S}^2} (g\wedge\nabla_T g)\cdot\nabla_T^{\perp}\varphi \tag{1.4}$$

where ∇_T denotes the tangential gradient in a direct orthonormal frame (τ_1, τ_2, ν) and ν is the outward normal on $\mathbb{S}^2 = \partial B^3$. If in addition g is smooth except at a finite number of points $\{a_j\}_{j=1}^N$ then (1.4) leads to (see e.g. [7])

$$\langle T(g), \varphi\rangle = 2\pi\sum_{j=1}^N d_j\varphi(a_j) \tag{1.5}$$

where $d_j \in \mathbb{Z}$ is the topological degree of g around a_j and $\sum_{j=1}^N d_j = 0$ for obvious topological reasons.

Our main objective in this note is to measure the obstruction to smooth approximation in terms of energy. We shall rely on the method and techniques developed in [16] (see also [19] and [10] for an approach in terms of Cartesian currents). We consider the following energy for $g \in H^{1/2}(\mathbb{S}^2;\mathbb{R}^2)$,

$$E(g) = \int_{B^3} |\nabla u_g|^2$$

where $u_g \in H^1(B^3,\mathbb{R}^2)$ is the harmonic extension of g to the tridimensional unit ball $B^3 \subset \mathbb{R}^3$, i.e.,

$$\begin{cases} \Delta u_g = 0 & \text{in } H^{-1}(B^3), \\ u_{g|\mathbb{S}^2} = g\,. \end{cases}$$

Equivalently,

$$E(g) = \operatorname{Inf}\left\{ \int_{B^3} |\nabla u|^2 \; ; \; u \in H^1_g(B^3;\mathbb{R}^2) \right\}.$$

It is well known that $|g|^2_{1/2} \sim E(g)$ and in particular, $g \mapsto (E(g))^{1/2}$ provides a seminorm equivalent to $|\cdot|_{1/2}$.

As a first step in our analysis, we introduce a suitable "dipole problem" motivated by the following remark. Let $g \in X$ and $\{h_n\}_{n\in\mathbb{N}} \subset X \cap C^\infty$ be such that $h_n \to g$ a.e. in $\mathbb{S}^2$. If one consider the sequence $\{\bar{h}_n g\}_{n\in\mathbb{N}} \subset X$, where $\bar{h}_n g$ denotes the complex product between g and the complex conjugate of h_n (we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$), it turns out that $T(\bar{h}_n g) = T(\bar{h}_n) + T(g) = T(g)$ for every $n \in \mathbb{N}$ since $T(\bar{h}_n) = 0$ (see [2]). And obviously, $\bar{h}_n g \to (1,0)$ a.e. in $\mathbb{S}^2$. Assuming that $T(g)$ takes the simple form $T(g) = 2\pi\big(\delta_P - \delta_Q\big)$ for some distinct points $P, Q \in \mathbb{S}^2$, it is therefore relevant to consider the quantity

$$m\big(2\pi(\delta_P - \delta_Q)\big) = \operatorname{Inf}\bigg\{ \liminf_{n\to+\infty} E(g)\,;\, \{g_n\}_{n\in\mathbb{N}} \subset X\,, \qquad T(g_n) = 2\pi\big(\delta_P - \delta_Q\big)\,,\, g_n \to (1,0) \text{ a.e.} \bigg\} \tag{1.6}$$

which heuristically gives the minimal $H^{1/2}$-energy necessary to remove the pair of singularities (P,Q). The explicit computation of $m\big(2\pi(\delta_P - \delta_Q)\big)$ is referred to as the "dipole problem" and is the object of our first result.

Theorem 1.1. *Let P and Q be two distinct points on $\mathbb{S}^2$. We have*

$$m\big(2\pi(\delta_P - \delta_Q)\big) = 2\pi\, d_{\mathbb{S}^2}(P,Q)\,,$$

where $d_{\mathbb{S}^2}$ denotes the geodesic distance on the sphere $\mathbb{S}^2$.

The result of Theorem 1.1 can be generalized to an arbitrary $T(g)$ (see [16]) but for our purposes it suffices to concentrate on the special case (1.5). Indeed, the class of maps which belong to $X \cap W^{1,1}(\mathbb{S}^2)$ and are smooth except for a finite number of points, is dense in X for the strong topology. This important result was first obtained in [19] (see also [2] and [16]). The statement of Theorem 1.1 for (1.5) should be modified as follows. Let $a_1, \ldots, a_N$ be N distinct points on $\mathbb{S}^2$, let $d_1, \ldots, d_N \in \mathbb{N}$ be such that $\sum_{j=1}^N d_j = 0$, and consider the quantity $m\big(2\pi \sum_j d_j \delta_{a_j}\big)$ defined as in (1.6) using the constraint $T(g_n) = 2\pi \sum_j d_j \delta_{a_j}$ (admissible sequences can be easily constructed, see e.g. Section 3). Since the sum of the d_j's equals zero, we can relabel the a_j's taking into account their multiplicity $|d_j|$ as two lists $(p_1, \dots, p_K)$ and $(q_1, \dots, q_K)$ in such a way that

$$2\pi \sum_{j=1}^{N} d_j \delta_{a_j} = 2\pi \sum_{i=1}^{K} \delta_{p_i} - \delta_{q_i} \,. \tag{1.7}$$

Then we have

$$m\big(2\pi \sum_{j=1}^{N} d_j \delta_{a_j}\big) = 2\pi \operatorname*{Min}_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d_{\mathbb{S}^2}(p_i, q_{\sigma(i)})\,, \tag{1.8}$$

where $\mathcal{S}_K$ denotes the set of all permutations of K indices. We shall omit the proof of (1.8) since the main ingredients are already contained in the proof of Theorem 1.1 (see also Section 3).

The quantity on the right hand side of (1.8) is often referred to as the length of a minimal connection between the singularities relative to the distance $d_{\mathbb{S}^2}$. It has been first introduced in [5] for similar questions involving $\mathbb{S}^2$-valued maps from tridimensional domains with H^1-regularity, and it can be written in the dual form (see [5]):

$$\operatorname*{Min}_{\sigma \in \mathcal{S}_K} \sum_{i=1}^{K} d_{\mathbb{S}^2}(p_i, q_{\sigma(i)}) = \mathrm{Sup}\bigg\{ \sum_{i=1}^{K} \varphi(p_i) - \varphi(q_i)\,;\ \varphi \in \mathrm{Lip}(\mathbb{S}^2; \mathbb{R})\,, \\ \varphi \text{ is 1-Lipschitz with respect to } d_{\mathbb{S}^2} \bigg\}\,. \tag{1.9}$$

The formula above motivates the following definition for the length of a minimal connection relative to $d_{\mathbb{S}^2}$ for an arbitrary map $g \in X$,

$$L(g) = \frac{1}{2\pi} \mathrm{Sup}\bigg\{ \langle T(g), \varphi \rangle\,;\ \varphi \in \mathrm{Lip}(\mathbb{S}^2; \mathbb{R})\,,\ \varphi \text{ is 1-Lipschitz with respect to } d_{\mathbb{S}^2} \bigg\}\,.$$

It turns out that $L(g)$ is the right quantity to quantify the obstruction to smooth approximation. If one wants to measure, for a given $g \in X$, the distance of g to the subspace of smooth maps, it is natural to consider the relaxed energy $\overline{E} : X \to \mathbb{R}$ defined by

$$\overline{E}(g) = \mathrm{Inf}\bigg\{ \liminf_{n \to +\infty} E(g_n)\,;\ \{g_n\}_{n \in \mathbb{N}} \subset X \cap C^\infty(\mathbb{S}^2)\,,\ g_n \to g \text{ a.e.} \bigg\}\,.$$

Obviously $\overline{E}(g) \geq E(g)$ and the gap between $\overline{E}(g)$ and $E(g)$ is precisely the quantity we are interested in. We have the following result.

Theorem 1.2. *For every $g \in X$, we have*

$$\overline{E}(g) = E(g) + 2\pi L(g)\,.$$

A similar result has first been obtained by H. Brezis, F. Bethuel and J.M. Coron [5] in the context of $\mathbb{S}^2$-valued maps. Since then many studies have been done on this type of relaxation problems for Sobolev maps between manifolds, see e.g. [7], [12], [14], [15], see also the recent works by M. Giaquinta, G. Modica, J. Souček [10] and M. Giaquinta, D. Mucci [9] for $H^{1/2}$-maps in the context of Cartesian currents.

The plan of the paper is as follows. Sections 2 and 3 are devoted to the proofs of Theorem 1.1 and Theorem 1.2 respectively. In Section 4, we present some generalizations of the previous results when $\mathbb{S}^2$ is replaced by a more general two dimensional manifold $\mathcal{M}$. Then we compare our dipole problem with the more classical notion where one simply computes the infimum of the energy over the class of maps having two prescribed singularities of degree $+1$ and -1.

2. The dipole problem

Proposition 2.1. *Let P and Q be two distinct points on $\mathbb{S}^2$. There exists a sequence $\{g_n\}_{n\in\mathbb{N}}\subset X$ such that $T(g_n)=2\pi\big(\delta_P-\delta_Q\big)$ for every n, $g_n\to(1,0)$ a.e. and*

$$\limsup_{n\to+\infty} E(g_n)\leq 2\pi\, d_{\mathbb{S}^2}(P,Q)\,.$$

Proof. We split the proof into several steps.
Step 1. Due to the invariance of the energy E with respect to rotations, we may assume without loss of generality that the points P and Q satisfy

$$P=(l,0,-h)\quad\text{and}\quad Q=(-l,0,-h)$$

for some $l>0$ and $h\geq 0$. In the sequel, we shall make use of the conformal transformation $\Phi:\overline{\mathbb{R}^3_+}=\mathbb{R}^2\times\overline{\mathbb{R}_+}\to\overline{B^3}\subset\mathbb{R}^3$ given by

$$\Phi(y_1,y_2,y_3)=\left(\frac{2y_1}{y_1^2+y_2^2+(y_3+1)^2}\,,\,\frac{2y_2}{y_1^2+y_2^2+(y_3+1)^2}\,,\,\frac{y_1^2+y_2^2+y_3^2-1}{y_1^2+y_2^2+(y_3+1)^2}\right).$$

We observe that $\Phi(\cdot,\cdot,0)\in C^\infty(\mathbb{R}^2;\mathbb{S}^2\setminus\{(0,0,1)\})$ is an inverse stereographic projection. Moreover

$$P=\Phi\big((\ell,0,0)\big)\quad\text{and}\quad Q=\Phi\big((-\ell,0,0)\big)$$

for some $0\leq\ell\leq 1$ and the image of the segment $\big[(-\ell,0,0),(\ell,0,0)\big]$ through Φ is a minimizing geodesic on $\mathbb{S}^2$ connecting Q to P.
Step 2. We introduce the second conformal transformation $\phi:\overline{\mathbb{R}^2_+}=\mathbb{R}\times\overline{\mathbb{R}_+}\to\mathbb{D}$ defined by

$$\phi(y_2,y_3)=\left(\frac{y_2^2+y_3^2-1}{y_2^2+(1+y_3)^2}\,,\,\frac{2y_2}{y_2^2+(1+y_3)^2}\right),$$

so that $\phi(\cdot,0)\in C^\infty(\mathbb{R};\mathbb{S}^1\setminus\{(1,0)\})$ is an inverse stereographic projection. Using the complex notation, for $y_2\neq 0$ we have

$$\phi(y_2,0)=e^{i\theta(y_2)}\qquad\text{with}\ \ \theta(y_2)=2\arctan(1/y_2)\,.$$

Next we define the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ by

$$\phi_n(y_2,y_3)=\begin{cases}\phi(ny_2,ny_3) & \text{if } |y_2|^2+|y_3|^2\leq 1\,,\\ (1,0) & \text{if } |y_2|^2+|y_3|^2\geq 4\,,\\ e^{i(\theta(ny_2))(2-|y_2|)} & \text{if } 1<|y_2|<2 \text{ and } y_3=0\,,\\ \text{harmonic} & \text{if } 1<|y_2|^2+|y_3|^2<4 \text{ and } y_3>0\,.\end{cases}$$

We may easily check that $\phi_n\in \mathrm{Lip}(\mathbb{R}^2_+)$, $\phi_n(\cdot,0)\in\mathrm{Lip}(\mathbb{R};\mathbb{S}^1)$ and

$$\int_{\mathbb{R}^2_+}|\nabla\phi_n|^2dy_2dy_3 \underset{n\to+\infty}{\longrightarrow} \int_{\mathbb{R}^2_+}|\nabla\phi|^2dy_2dy_3=2\pi\,. \tag{2.1}$$

Now we consider for $y=(y_1,y_2,y_3)\in\overline{\mathbb{R}^3_+}$,

$$v_n(y)=\begin{cases}\phi_n\left(\dfrac{y_2}{\eta_n(y_1)},\dfrac{y_3}{\eta_n(y_1)}\right) & \text{if } |y_1|<\ell\,,\\ (1,0) & \text{if } |y_1|\geq\ell\,,\end{cases}$$

where $\eta_n(y_1)=(\ell-|y_1|)/n$. Once more, one may check that v_n is locally Lipschitz in $\overline{\mathbb{R}^3_+}$ away from $\{(-\ell,0,0),(\ell,0,0)\}$ and has finite energy. Moreover, v_n is identically equal to $(1,0)$ outside a small neighborhood $\mathcal{U}\subset\overline{\mathbb{R}^3_+}$ of $\big[(-\ell,0,0),(\ell,0,0)\big]$ independent of n. Finally, we define $u_n:\overline{B^3}\subset\mathbb{R}^3\to\mathbb{R}^2$ by

$$u_n(x)=\begin{cases}v_n\big(\Phi^{-1}(x)\big) & \text{if } x\in\Phi(\mathcal{U})\,,\\ (1,0) & \text{otherwise}\,.\end{cases}$$

By construction $u_n\in H^1(B^3,\mathbb{R}^2)$, u_n is locally Lipschitz in $\overline{B^3}$ away from the points P and Q. In addition, $g_n:=u_{n|\mathbb{S}^2}\in X\cap W^{1,1}(\mathbb{S}^2)$, $g_n\to(1,0)$ a.e. as $n\to+\infty$ and $T(g_n)=2\pi(\delta_P-\delta_Q)$ for every n.

Step 3. Using the conformal invariance of Φ, a straightforward computation yields

$$\int_{B^3}|\nabla u_n(x)|^2dx=\int_{\mathbb{R}^3_+}|\nabla v_n(y)|^2a(y)dy\,, \tag{2.2}$$

where the weight function $a(y)$ is given by

$$a(y)=\frac{2}{y_1^2+y_2^2+(y_3^2+1)^2}\,.$$

Then we estimate

$$\int_{\mathbb{R}^3_+}|\nabla v_n(y)|^2a(y)dy\leq\int_{-\ell}^{\ell}\left(\int_{\mathbb{R}^2_+}|\nabla_{2,3}\,v_n|^2a(y)dy_2dy_3+2\int_{\mathbb{R}^2_+}|\partial_1v_n|^2dy_2dy_3\right)dy_1\,.$$

Changing variables, $y_i=\eta_n(y_1)z_i$ for $i=2,3$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2_+}|\nabla_{2,3}\,v_n|^2a(y)dy_2dy_3&=\int_{\mathbb{R}^2_+}|\nabla\phi_n(z_2,z_3)|^2a\big(y_1,\eta_n(y_1)z_2,\eta_n(y_1)z_3\big)dz_2dz_3\\&=a(y_1,0,0)\int_{\mathbb{R}^2_+}|\nabla\phi_n(z_2,z_3)|^2dz_2dz_3+O(1/n)\end{aligned}$$

and

$$\int_{\mathbb{R}^2_+} |\partial_1 v_n|^2 dy_2 dy_3 \leq |\dot\eta_n(y_1)|^2 \int_{\mathbb{R}^2_+} |\nabla\phi_n(z_2,z_3)|^2 (z_2^2+z_3^2) dz_2 dz_3$$
$$\leq 4|\dot\eta_n(y_1)|^2 \int_{\mathbb{R}^2_+} |\nabla\phi_n(z_2,z_3)|^2 dz_2 dz_3 \,.$$

Since $\int_{-\ell}^{\ell} |\dot\eta_n(y_1)|^2 dy_1 = O(1/n^2)$, we deduce that

$$\int_{\mathbb{R}^3_+} |\nabla v_n(y)|^2 a(y) dy \leq \left(\int_{-\ell}^{\ell} a(y_1,0,0) dy_1 \right) \int_{\mathbb{R}^2_+} |\nabla\phi_n|^2 \; + O(1/n) \,.$$

which leads by (2.1) to

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^3_+} |\nabla v_n(y)|^2 a(y) dy \leq 2\pi \int_{-\ell}^{\ell} a(y_1,0,0) dy_1 \,.$$

In view of (2.2) and the definition of E, we conclude

$$\limsup_{n\to+\infty} E(g_n) \leq \limsup_{n\to+\infty} \int_{B^3} |\nabla u_n|^2 dx \leq 2\pi \int_{-\ell}^{\ell} a(y_1,0,0) dy_1 \,.$$

On the other hand, $\gamma : [-\ell,\ell] \to \mathbb{S}^2$ defined by $\gamma(t) = \Phi((t,0,0))$ is a minimizing geodesic on $\mathbb{S}^2$ joining P and Q so that

$$d_{\mathbb{S}^2}(P,Q) = \int_{-\ell}^{\ell} |\dot\gamma(t)| dt = \int_{-\ell}^{\ell} a((t,0,0)) dt \,.$$

Therefore

$$\limsup_{n\to+\infty} E(g_n) \leq 2\pi \, d_{\mathbb{S}^2}(P,Q)$$

and the proof is complete. ■

Proposition 2.2. *Let P and Q be two distinct points on $\mathbb{S}^2$. Then, for any sequence $\{g_n\}_{n\in\mathbb{N}} \subset X$ such that $g_n \to (1,0)$ a.e. and $T(g_n) = 2\pi(\delta_P - \delta_Q)$ for every n, we have*

$$\liminf_{n\to+\infty} E(g_n) \geq 2\pi \, d_{\mathbb{S}^2}(P,Q) \,.$$

Proof. Without loss of generality, we may assume that

$$\liminf_{n\to+\infty} E(g_n) = \lim_{n\to+\infty} E(g_n) < +\infty \,.$$

Let $u_n \in H^1(B^3)$ be the harmonic extension of g_n to B^3. Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ remains bounded in H^1. Since $g_n \to (1,0)$ a.e. on $\mathbb{S}^2$, extracting a subsequence, if necessary, we have $u_n \rightharpoonup (1,0)$ weakly in H^1. Next we observe that $|\nabla u_n|^2 \geq |H(u_n)|$ a.e. in B^3. We fix $0 < r \ll 1$ and we consider a cut-off function $\chi \in$

$C^\infty(\mathbb{R};\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(t) \equiv 1$ if $|t| \ge 1-r$ and $\chi(t) \equiv 0$ if $|t| \le 1-2r$. Then

$$E(g_n) = \int_{B^3} |\nabla u_n|^2 dx \ge \int_{B^3} \chi(|x|)|H(u_n)|dx = \int_{\mathcal{A}_{2r}} \chi(|x|)|H(u_n)|dx\,,$$

where $\mathcal{A}_{2r} = \overline{B^3} \setminus B(0,1-2r)$. Setting d_{2r} to be the geodesic distance in $\mathcal{A}_{2r}$, we define for $x \in \mathcal{A}_r$

$$\Psi(x) = d_{2r}(x,Q)\,,$$

and for $x \in B^3$

$$\zeta(x) = \chi(|x|)\Psi(x) \text{ if } x \in \mathcal{A}_{2r}\,, \quad \zeta(x) = 0 \text{ otherwise}\,,$$

so that $\zeta \in \mathrm{Lip}(B^3)$. Since $|\nabla\Psi| \le 1$ a.e. in $\mathcal{A}_{2r}$, we may estimate

$$\begin{aligned}\int_{\mathcal{A}_{2r}} \chi(|x|)|H(u_n)|dx \ge \int_{\mathcal{A}_{2r}} \chi(|x|)\, H(u_n)\cdot\nabla\Psi\, dx = \\ &= \int_{B^3} H(u_n)\cdot\nabla\zeta\, dx - \int_K \dot\chi(|x|)\Psi(x)\, H(u_n)\cdot\frac{x}{|x|}\, dx \\ &= \langle T(g_n), \zeta_{|\mathbb{S}^2}\rangle - \int_K \dot\chi(|x|)\Psi(x)\, H(u_n)\cdot\frac{x}{|x|}\, dx \\ &= 2\pi d_{2r}(P,Q) - \int_K \dot\chi(|x|)\Psi(x)\, H(u_n)\cdot\frac{x}{|x|}\, dx \qquad (2.3)\end{aligned}$$

with $K = \mathcal{A}_{2r} \setminus \mathcal{A}_r$. By classical results, $\nabla u_n \to 0$ in $L^2_{\mathrm{loc}}(B^3)$ so that $H(u_n) \to 0$ in $L^1(K)$ and the integral on the right handside of (2.3) vanishes as $n \to +\infty$. Therefore

$$\lim_{n\to+\infty} E(g_n) \ge 2\pi d_{2r}(P,Q)\,.$$

Now we let $r \to 0$ and since $d_{2r}(P,Q) \to d_{\mathbb{S}^2}(P,Q)$ as $r \to 0$, we recover the announced result. ∎

3. Relaxed energy and minimal connections

We begin with the proof of the upper bound $\overline{E} \le E + 2\pi L$ in Theorem 1.2.

Proposition 3.1. *For every $g \in X$, there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset X \cap C^\infty(\mathbb{S}^2)$ such that $g_n \to g$ a.e. in $\mathbb{S}^2$ and*

$$\limsup_{n\to+\infty} E(g_n) \le E(g) + 2\pi L(g)\,.$$

Proof. *Step 1.* First we assume that $g \in X \cap W^{1,1}(\mathbb{S}^2)$ and g is smooth except for a finite number of points. Then the distribution $T(g)$ may be written as $T(g) = 2\pi\sum_{i=1}^K \delta_{p_i} - \delta_{q_i}$ and without loss of generality, we may assume that

$$\sum_{i=1}^K d_{\mathbb{S}^2}(p_i,q_i) = \operatorname*{Min}_{\sigma\in\mathcal{S}_K} \sum_{i=1}^K d_{\mathbb{S}^2}(p_i,q_{\sigma(i)}) = L(g)$$

by (1.9). We consider for each $i = 1,\dots,K$, the sequence $\{g_{i,n}\}_{n\in\mathbb{N}} \subset X$ associated to the pair (p_i, q_i) given by Proposition 2.1. We denote by u_g, resp. $u_{i,n}$, the harmonic extension of g, resp. of $g_{i,n}$, to the unit ball B^3. Next we consider the sequence

$$g_n = \big(\Pi_{i=1}^K \bar{g}_{i,n}\big) g \in X\,,$$

using the complex product and complex conjugation. Obviously $g_n \to g$ a.e. since $g_{i,n} \to (1,0)$ a.e. for every i. By [2], we have

$$T(g_n) = T(g) - \sum_{i=1}^K T(g_{i,n}) = 0\,.$$

By an easy application of the maximum principle, one obtains $|u_g| \leq 1$ and $|u_{i,n}| \leq 1$ in B^3 so that a straightforward computation yields

$$E(g_n) \leq \int_{B^3} |\nabla u_g|^2 + \sum_{i=1}^K \int_{B^3} |\nabla u_{i,n}|^2 = E(g) + \sum_{i=1}^K E(g_{i,n})\,.$$

In view of Proposition 2.1, we derive that

$$\limsup_{n\to+\infty} E(g_n) \leq E(g) + 2\pi \sum_{i=1}^K d_{\mathbb{S}^2}(p_i,q_i) = E(g) + 2\pi L(g)\,.$$

Since $T(g_n) = 0$, g_n can be strongly approximated by smooth maps in X (see [19]). Hence the required sequence may be obtained by approximating each g_n by smooth maps and then applying a standard diagonalization argument.

Step 2. Now we consider an arbitrary map $g \in X$. First we observe that

$$|L(g_1) - L(g_2)| \leq C\big(|g_1|_{1/2} + |g_2|_{1/2}\big)|g_1 - g_2|_{1/2} \qquad \forall g_1, g_2 \in X\,, \tag{3.1}$$

for some universal constant $C > 0$ (see e.g. [16] for more details). By the results in [19], we may find a sequence $\{g_n\}_{n\in\mathbb{N}} \subset X$ such that $g_n \to g$ strongly in $H^{1/2}$ and g_n satisfies the assumption of Step 1 for every n. Obviously, we may assume that $g_n \to g$ a.e. extracting a subsequence if necessary. By (3.1), we have

$$E(g_n) + 2\pi L(g_n) \underset{n\to+\infty}{\longrightarrow} E(g) + 2\pi L(g)\,.$$

Hence we may apply Step 1 to each g_n and then obtain the required sequence by a diagonalization argument. ∎

To obtain the lower bound $\overline{E} \geq E + 2\pi L$, we shall use an argument based on the lower semicontinuity result below. We introduce the expected lower bound $F : X \to \mathbb{R}_+$ defined by

$$F(g) = E(g) + 2\pi L(g)\,.$$

We have the following.

Proposition 3.2. *The functional F is sequentially lower semicontinuous on X with respect to the a.e. convergence.*

Proof. *Step 1.* First we introduce for a given $0 < r < 1$, the functional $F_r : X \to \mathbb{R}_+$ defined by

$$F_r(g) = E(g) + 2\pi L_r(g)\,,$$

where $L_r(g)$ denotes the length of a minimal connection relative to the geodesic distance d_r in the annulus $\mathcal{A}_r = \overline{B^3} \setminus B(0, 1-r)$, ie.,

$$L_r(g) = \frac{1}{2\pi} \mathrm{Sup} \Big\{ \langle T(g), \Phi_{|\mathbb{S}^2} \rangle \,;\ \Phi \in \mathrm{Lip}(\mathcal{A}_r; \mathbb{R}),\ \Phi \text{ is 1-Lipschitz with respect to } d_r \Big\}\,.$$

In the remaining of the proof we shall extend any $\Phi \in \mathrm{Lip}(\mathcal{A}_r; \mathbb{R})$ to the whole $\overline{B^3}$ by setting

$$\Phi(x) = \chi(|x|) \Phi\left(\frac{(1-r)x}{|x|} \right) \qquad \forall x \in B(0, 1-r)$$

for a fixed function $\chi \in C^\infty(\mathbb{R})$ satisfying $0 \leq \chi \leq 1$, $\chi(t) = 1$ if $|t| \geq 1 - r$ and $\chi(t) = 0$ if $|t| \leq 1 - 2r$. Then for any $g \in X$ and any $\Phi \in \mathrm{Lip}(\mathcal{A}_r; \mathbb{R})$, we have

$$\langle T(g), \Phi_{|\mathbb{S}^2} \rangle = \int_{B^3} H(u_g) \cdot \nabla \Phi$$

where u_g is the harmonic extension of g to B^3.

We claim that F_r is sequentially lower semicontinuous on X with respect to the a.e. convergence. Obviously, it suffices to prove that for any $\Phi \in \mathrm{Lip}(\mathcal{A}_r; \mathbb{R})$ with Φ 1-Lipschitz with respect to d_r,

$$g \in X \mapsto E(g) + \left| \int_{B^3} H(u_g) \cdot \nabla \Phi \right|$$

is sequentially lower semicontinuous on X with respect to the a.e. convergence since

$$F_r(g) = \mathrm{Sup} \bigg\{ E(g) + \left| \int_{B^3} H(u_g) \cdot \nabla \Phi \right| \,;\ \Phi \in \mathrm{Lip}(\mathcal{A}_r; \mathbb{R}), \\ \Phi \text{ is 1-Lipschitz with respect to } d_r \bigg\}\,.$$

Let $\{g_n\}_{n \in \mathbb{N}} \subset X$ be such that $g_n \to g \in X$ a.e. on $\mathbb{S}^2$. Without loss of generality, we may assume that

$$\liminf_{n \to +\infty} F_r(g_n) = \lim_{n \to +\infty} F_r(g_n) < +\infty\,.$$

Since $\sup_n E(g_n) \leq \sup_n F_r(g_n) < +\infty$, we can extract a subsequence (not relabeled) such that $g_n \rightharpoonup g$ weakly in $H^{1/2}$. Setting u_n to be the harmonic extension of g_n to B^3, it is well known that u_n converges to u_g weakly in $H^1(B^3)$ and strongly in $H^1(K)$ for any compact set $K \subset B^3$. Writing $u_n = u + v_n$, v_n converges to 0 weakly in $H^1(B^3)$ and strongly in $H^1(K)$ for any compact set $K \subset B^3$. Therefore

$$E(g_n) = E(g) + \int_{B^3} |\nabla v_n|^2 + o(1)\,.$$

Next we observe that

$$H(u_n) = H(v_n) + 2(\partial_2 v_n \wedge \partial_3 u_g, \partial_3 v_n \wedge \partial_1 u_g, \partial_1 v_n \wedge \partial_2 u_g)+ \\ + 2(\partial_2 u_g \wedge \partial_3 u_n, \partial_3 u_g \wedge \partial_1 u_n, \partial_1 u_g \wedge \partial_2 u_n)\,.$$

Hence $H(u_n) \to H(u)$ in $L^1(K)$ for any compact set $K \subset B^3$, so that

$$\int_{B^3} H(u_n)\cdot\nabla\Phi = \int_{B^3} H(u_g)\cdot\nabla\Phi + \int_{\mathcal{A}_r} H(v_n)\cdot\nabla\Phi + o(1)\,.$$

Since $|\nabla\Phi| \le 1$ a.e. in $\mathcal{A}_r$, we deduce

$$E(g_n)+\left|\int_{B^3} H(u_n)\cdot\nabla\Phi\right| \ge E(g)+\left|\int_{B^3} H(u_g)\cdot\nabla\Phi\right|+\int_{B^3}|\nabla v_n|^2-\int_{\mathcal{A}_r}|H(v_n)|+o(1)$$

and the conclusion follows because $|\nabla v_n|^2 \ge |H(v_n)|$ a.e. in $\mathcal{A}_r$.

Step 2. We claim that

$$F(g) = \sup_{r>0} F_r(g) \qquad \forall g \in X\,. \tag{3.2}$$

In view of Step 1, the proof of (3.2) will complete the proof of Proposition 3.2. Obviously it suffices to show that $L(g) = \sup_{r>0} L_r(g)$.

First we consider the case of a map $g \in X \cap W^{1,1}(\mathbb{S}^2)$ smooth except for a finite number of points. In this case, we may write $T(g) = 2\pi\sum_{i=1}^K \delta_{p_i} - \delta_{q_i}$ and by the results in [5],

$$L(g) = \operatorname*{Min}_{\sigma\in\mathcal{S}_K}\sum_{i=1}^K d_{\mathbb{S}^2}(p_i, q_{\sigma(i)}) \quad\text{and}\quad L_r(g) = \operatorname*{Min}_{\sigma\in\mathcal{S}_K}\sum_{i=1}^K d_r(p_i, q_{\sigma(i)}) \;\; \forall\, 0<r<1\,.$$

Now using the uniform convergence of d_r to $d_{\mathbb{S}^2}$ on $\mathbb{S}^2\times\mathbb{S}^2$ as $r\to 0$, one easily obtains $L_r(g) \to L(g)$ (see [16] for details). Then we conclude by observing that $L_r(g) \le L(g)$ since $d_r \le d_{\mathbb{S}^2}$ as distances on $\mathbb{S}^2$.

Now we consider a general map $g \in X$. By the results in [19], given $\varepsilon > 0$ small, we may find $g_\varepsilon \in X \cap W^{1,1}(\mathbb{S}^2)$ smooth except for a finite number of points such that $\|g - g_\varepsilon\|_{1/2} \le \varepsilon$. Using (3.1) and its equivalent form for L_r, i.e.,

$$|L_r(g_1) - L_r(g_2)| \le C\big(|g_1|_{1/2} + |g_2|_{1/2}\big)|g_1 - g_2|_{1/2} \qquad \forall g_1, g_2 \in X\,,$$

for a constant $C > 0$ independent of r (see [16] for details), we derive

$$\liminf_{r\to 0} L_r(g) \ge \lim_{r\to 0} L_r(g_\varepsilon) - C\varepsilon = L(g_\varepsilon) - C\varepsilon \ge L(g) - C\varepsilon$$

and

$$\limsup_{r\to 0} L_r(g) \le \lim_{r\to 0} L_r(g_\varepsilon) + C\varepsilon = L(g_\varepsilon) + C\varepsilon \le L(g) + C\varepsilon\,.$$

Letting $\varepsilon \to 0$, we recover $L_r(g) \to L(g)$ as $r \to 0$ and the conclusion follows since $L_r(g) \le L(g)$. ∎

Corollary 3.1. *Let $g \in X$ and let $\{g_n\}_{n\in\mathbb{N}} \subset X \cap C^\infty(\mathbb{S}^2)$ be an arbitrary sequence such that $g_n \to g$ a.e. in $\mathbb{S}^2$. We have*

$$\liminf_{n\to+\infty} E(g_n) \ge E(g) + 2\pi L(g)\,.$$

Proof. Since $\{g_n\}_{n\in\mathbb{N}} \subset C^\infty(\mathbb{S}^2)$, we have $T(g_n) = 0$ and hence $L(g_n) = 0$. Next we infer from Proposition 3.2 that

$$\liminf_{n\to+\infty} E(g_n) = \liminf_{n\to+\infty} F(g_n) \geq F(g) = E(g) + 2\pi L(g)$$

which completes the proof. ■

4. Some extensions and open problem

In this section, we extend the results of Theorem 1.1 and Theorem 1.2 to a more general starting manifold. We consider a smooth open bounded domain $\Omega \subset \mathbb{R}^3$ diffeomorphic to the unit ball B^3 and we set $\mathcal{M} = \partial\Omega$ so that the manifold $\mathcal{M}$ is diffeomorphic to the unit sphere $\mathbb{S}^2$. We are interested in the space of $\mathbb{S}^1$-valued maps

$$X_{\mathcal{M}} = H^{1/2}(\mathcal{M};\mathbb{S}^1) = \big\{g \in H^{1/2}(\mathcal{M};\mathbb{R}^2)\ ;\ |g| = 1 \text{ a.e. on } \mathcal{M}\big\}$$

endowed with the metric induced by the $H^{1/2}$-norm.

For $g \in X_{\mathcal{M}}$, we define the distribution $T(g)$ as in (1.2) by integrating over Ω and taking extensions in Ω instead of B^3, and the energy of g is still given by the energy of its harmonic extension, i.e.,

$$E_{\mathcal{M}}(g) = \int_\Omega |\nabla u_g|^2\,,$$

with $u_g \in H^1_g(\Omega;\mathbb{R}^2)$ satisfying $\Delta u_g = 0$ in $H^{-1}(\Omega)$.

For two distinct points P and Q on $\mathcal{M}$, we define the dipole problem on $\mathcal{M}$ as

$$m_{\mathcal{M}}\big(2\pi(\delta_P - \delta_Q)\big) = \operatorname{Inf}\bigg\{\liminf_{n\to+\infty} E_{\mathcal{M}}(g)\,;\ \{g_n\}_{n\in\mathbb{N}} \subset X_{\mathcal{M}}\,,\quad T(g_n) = 2\pi\big(\delta_P - \delta_Q\big)\,,\ g_n \to (1,0) \text{ a.e.}\bigg\}\,. \tag{4.1}$$

Using a suitable diffeomorphism between Ω and B^3 and applying the method in Section 2 with minor modifications, we obtain the following variant of Theorem 1.1.

Theorem 1.1.bis. *Let P and Q be two distinct points on $\mathcal{M}$. We have*

$$m_{\mathcal{M}}\big(2\pi(\delta_P - \delta_Q)\big) = 2\pi d_{\mathcal{M}}(P,Q)$$

where $d_{\mathcal{M}}$ denotes the geodesic distance on the manifold $\mathcal{M}$.

To characterize the obstruction to smooth approximation in $X_{\mathcal{M}}$, we proceed as in the case $\mathcal{M} = \mathbb{S}^2$. We introduce the relaxed energy $\overline{E}_{\mathcal{M}} : X_{\mathcal{M}} \to \mathbb{R}_+$ defined by

$$\overline{E}_{\mathcal{M}}(g) = \operatorname{Inf}\bigg\{\liminf_{n\to+\infty} E_{\mathcal{M}}(g_n)\,;\ \{g_n\}_{n\in\mathbb{N}} \subset X_{\mathcal{M}} \cap C^\infty(\mathcal{M})\,,\ g_n \to g \text{ a.e.}\bigg\}\,,$$

and we readily obtain from Section 3 and Theorem 1.1.bis the following result.

Theorem 2.2.bis. *For every $g \in X_{\mathcal{M}}$, we have*

$$\overline{E}_{\mathcal{M}}(g) = E_{\mathcal{M}}(g) + 2\pi L_{\mathcal{M}}(g)\,,$$

where $L_{\mathcal{M}}(g)$ is the length of a minimal connection relative to $d_{\mathcal{M}}$, i.e.,

$$L_{\mathcal{M}}(g) = \frac{1}{2\pi}\operatorname{Sup}\left\{\langle T(g),\varphi\rangle\,;\ \varphi \in \operatorname{Lip}(\mathcal{M};\mathbb{R})\,,\ \varphi \text{ is 1-Lipschitz with respect to } d_{\mathcal{M}}\right\}.$$

To conclude this section, we present a more classical notion of "dipole problem" where one minimizes the energy over all maps in $X_{\mathcal{M}}$ having exactly two prescribed singularities (see e.g. [7] in the context of $W^{1,1}$-maps with values into $\mathbb{S}^1$). More precisely, given two distinct points P and Q on $\mathcal{M}$, we consider

$$\tilde{m}_{\mathcal{M}}\big(2\pi(\delta_P-\delta_Q)\big) = \operatorname{Inf}\left\{E_{\mathcal{M}}(g)\,;\ g \in X_{\mathcal{M}}\,,\ T(g) = 2\pi(\delta_P-\delta_Q)\right\}. \tag{4.2}$$

The next result compares the dipole problems $m_{\mathcal{M}}$ and $\tilde{m}_{\mathcal{M}}$.

Theorem 4.1. *Let P and Q be two distinct points on $\mathcal{M}$. We have*

$$2\pi d_\Omega(P,Q) \le \tilde{m}_{\mathcal{M}}\big(2\pi(\delta_P-\delta_Q)\big) \le 2\pi d_{\mathcal{M}}(P,Q)\,, \tag{4.3}$$

where d_Ω denotes the geodesic distance in $\overline{\Omega}$. Moreover, if $d_\Omega(P,Q) = d_{\mathcal{M}}(P,Q)$ and $d_\Omega(P,Q)$ is small enough, we can extract from any minimizing sequence $\{g_n\}_{n\in\mathbb{N}} \subset X_{\mathcal{M}}$, a subsequence $\{g_{n_k}\}_{k\in\mathbb{N}}$ such that $g_{n_k} \to \alpha$ a.e. on $\mathcal{M}$ for some constant $\alpha \in \mathbb{S}^1$. In particular, the infimum defining $\tilde{m}_{\mathcal{M}}\big(2\pi(\delta_P-\delta_Q)\big)$ is not achieved whenever $d_\Omega(P,Q) = d_{\mathcal{M}}(P,Q)$ and $d_\Omega(P,Q)$ is small enough.

Observe that Theorem 4.1 gives a sufficient condition for which the two dipole problems coincide. We believe that in the case $d_\Omega(P,Q) < d_{\mathcal{M}}(P,Q)$, the two problems completly differ in nature. For instance, it seems that the lower bound in (4.3) is sharp but we do not have a proof of this fact. Beyond this quantitative problem, one can ask the much more interesting question:

Open problem. *Let P and Q be two distinct points on the manifold $\mathcal{M}$ such that $d_\Omega(P,Q) < d_{\mathcal{M}}(P,Q)$. Is $\tilde{m}_{\mathcal{M}}\big(2\pi(\delta_P-\delta_Q)\big)$ achieved ?*

We emphasize that the main difficulty of this problem comes from the constraint $T(g) = 2\pi(\delta_P-\delta_Q)$ which is obviously noncompact. The reader should certainely relate this question to recent results by P. Mironescu and A. Pisante [17] concerning $H^{1/2}$-maps from the circle into itself.

Proof of Theorem 4.1. The upper bound in (4.3) is a trivial consequence of Theorem 1.1.bis and the lower bound can be easily deduced from the discussion below.

Let P and Q be two distinct points on $\mathcal{M}$ such that $d_\Omega(P,Q) = d_{\mathcal{M}}(P,Q)$, and let $\{g_n\}_{n\in\mathbb{N}} \subset X_{\mathcal{M}}$ be a minimizing sequence for $\tilde{m}_{\mathcal{M}}\big(2\pi(\delta_P-\delta_Q)\big)$. Set u_n to be the harmonic extension of g_n to Ω. Since $\sup_n E_{\mathcal{M}}(g_n) < +\infty$, by classial results, we can extract a subsequence (not relabeled) such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ with $\Delta u = 0$ in $H^{-1}(\Omega)$, $g_n \rightharpoonup g$ weakly in $H^{1/2}(\mathcal{M})$ with $g = u_{|\mathcal{M}} \in X_{\mathcal{M}}$ and $g_n \to g$ a.e. on $\mathcal{M}$.

Next we consider the function $\Psi : \overline{\Omega} \to \mathbb{R}$ defined by

$$\Psi(x) = \min\left\{d_\Omega(x,Q)\,,\, d_\Omega(P,Q)\right\}.$$

Obviously, Ψ is 1-Lipschitz in Ω with respect d_Ω so that $|\nabla\Psi| \leq 1$ a.e. in Ω. Since $|\nabla u_n|^2 \geq |H(u_n)|$ a.e. in Ω and $\nabla\Psi = 0$ a.e. in $K := \{x \in \overline{\Omega}\,;\, \Psi(x) = d_\Omega(P,Q)\}$, we easily estimate

$$E_{\mathcal{M}}(g_n) = \int_\Omega |\nabla u_n|^2 \geq \int_K |\nabla u_n|^2 + \int_{\Omega\setminus K} |H(u_n)| \geq \int_K |\nabla u_n|^2 + \int_\Omega H(u_n)\cdot\nabla\Psi\,.$$

Then we observe that $\int_\Omega H(u_n)\cdot\nabla\Psi = \langle T(g_n), \Psi_{|\mathcal{M}}\rangle = 2\pi d_\Omega(P,Q)$ and hence

$$E_{\mathcal{M}}(g_n) \geq \int_K |\nabla u_n|^2 + 2\pi d_\Omega(P,Q)\,.$$

Now we infer from (4.3) and the assumption $d_\Omega(P,Q) = d_{\mathcal{M}}(P,Q)$ that $E_{\mathcal{M}}(g_n) \to 2\pi d_\Omega(P,Q)$. Therefore, $\int_K |\nabla u_n|^2 \to 0$. On the other hand, if $d_\Omega(P,Q)$ is small enough, we may find a ball $B(x_0,\varepsilon) \subset \mathbb{R}^3$ such that $x_0 \in \Omega$, $B(x_0,\varepsilon)\cap\mathcal{M} \neq \emptyset$ and $B(x_0,\varepsilon)\cap(\Omega\setminus K) = \emptyset$. By lower semicontinuity, one obtains

$$\int_{B(x_0,\varepsilon)\cap\Omega} |\nabla u|^2 \leq \int_K |\nabla u|^2 \leq \lim_{n\to+\infty}\int_K |\nabla u_n|^2 = 0\,,$$

so that u is constant in $B(x_0,\varepsilon)\cap\Omega$. Since $u_{|\mathcal{M}} = g \in X_{\mathcal{M}}$, we deduce that $|u| = 1$ in $B(x_0,\varepsilon)\cap\Omega$. Now, by an easy application of the maximum principle, one deduces that u is constant in Ω and hence $u \equiv \alpha$ for some constant $\alpha \in \mathbb{S}^1$. In particular, $g \equiv \alpha$ and the proof is complete. ∎

Aknowledgements

The author would like to thank S. Alama, L. Bronsard and P. Sternberg for giving the opportunity to deliver a lecture on the material presented in this note at a Worshop that took place at the Centre de Recherche Mathématiques (CRM) in Montreal in July 2006. The author was partially supported by the Center for Nonlinear Analysis (CNA) under the National Science Fundation Grant No. 0405343.

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