Probability (graduate class)

Lecture Notes

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# Contents

1 Probability space 6  
1.1 Definitions ........................................... 6  
1.2 Basic examples ...................................... 7  
1.3 Conditioning ........................................... 11  
1.4 Exercises ............................................. 13  

2 Random variables 14  
2.1 Definitions and basic properties .......................... 14  
2.2 $\pi - \lambda$ systems .................................. 16  
2.3 Properties of distribution functions ....................... 16  
2.4 Examples: discrete and continuous random variables ....... 18  
2.5 Exercises ............................................. 21  

3 Independence 22  
3.1 Definitions ........................................... 22  
3.2 Product measures and independent random variables .... 24  
3.3 Examples .............................................. 25  
3.4 Borel-Cantelli lemmas .................................. 28  
3.5 Tail events and Kolmogorov’s 0 – 1 law ................... 29  
3.6 Exercises ............................................. 31  

4 Expectation 33  
4.1 Definitions and basic properties .......................... 33  
4.2 Variance and covariance .................................. 35  
4.3 Independence again, via product measures .................. 36  
4.4 Exercises ............................................. 39  

5 More on random variables 41  
5.1 Important distributions .................................... 41  
5.2 Gaussian vectors ......................................... 45  
5.3 Sums of independent random variables ...................... 46  
5.4 Density .................................................. 47  
5.5 Exercises ............................................. 49  

6 Important inequalities and notions of convergence 52  
6.1 Basic probabilistic inequalities ............................ 52  
6.2 $L_p$-spaces ............................................ 54  
6.3 Notions of convergence ................................... 59  
6.4 Exercises ............................................. 63
7 Laws of large numbers

7.1 Weak law of large numbers .............................................. 67
7.2 Strong law of large numbers ........................................... 73
7.3 Exercises ................................................................. 78

8 Weak convergence

8.1 Definition and equivalences ........................................... 81
8.2 Relations to other notions of convergence and basic algebraic properties 86
8.3 Compactness ............................................................. 87
8.4 Exercises ................................................................. 90

9 Characteristic functions

9.1 Definition and basic properties ........................................ 92
9.2 Inversion formulae ....................................................... 94
9.3 Relations to convergence in distribution ............................. 97
9.4 Exercises ................................................................. 100

10 Central limit theorem

10.1 Auxiliary elementary lemmas ....................................... 105
10.2 Vanilla Central Limit Theorem ..................................... 107
10.3 Lindeberg’s Central Limit Theorem ............................... 108
10.4 Multidimensional case ................................................ 110
10.5 Poisson limit theorem ................................................. 111
10.6 Exercises ................................................................. 114

11 Quantitative versions of the limit theorem*

11.1 Berry-Esseen theorem via Stein’s method ....................... 118
11.2 Local central limit theorem ......................................... 124
11.3 Poisson limit theorem ................................................. 128
11.4 Exercises ................................................................. 133

12 Conditional expectation

12.1 Construction ............................................................ 134
12.2 Important properties .................................................. 137
12.3 Basic examples .......................................................... 139
12.4 Exercises ................................................................. 141

13 Martingales I

13.1 Definitions and basic examples .................................... 142
13.2 Martingale transforms and stopping times ....................... 144
13.3 Convergence theorem .................................................. 148
1 Probability space

1.1 Definitions

Let $\Omega$ be a set. A collection $\mathcal{F}$ of its subsets is called a $\sigma$-algebra (sometimes also $\sigma$-field) if

(i) $\Omega \in \mathcal{F}$,

(ii) for every $A \in \mathcal{F}$, we have $A^c \in \mathcal{F}$, that is $\mathcal{F}$ is closed under taking complements,

(iii) for every sets $A_1, A_2, \ldots$ in $\mathcal{F}$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, that is $\mathcal{F}$ is closed under taking countable unions.

Note that these imply that $\emptyset \in \mathcal{F}$ and that $\mathcal{F}$ is also closed under taking set difference, countable intersections, etc. For instance, $\mathcal{F} = \{\emptyset, \Omega\}$ is the trivial $\sigma$-algebra and $\mathcal{F} = 2^\Omega$ (all the subsets of $\Omega$) is the largest possible $\sigma$-algebra.

Suppose $\mathcal{F}$ is a $\sigma$-algebra on the set $\Omega$. A function $\mu: \mathcal{F} \rightarrow [0, +\infty]$ is called a measure if

(i) $\mu(\emptyset) = 0$,

(ii) $\mu$ is countably-additive, that is for every pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathcal{F}$, we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The measure $\mu$ is finite if $\mu(\Omega) < \infty$, $\sigma$-finite if $\Omega$ is a countable union of sets in $\mathcal{F}$ of finite measure. The measure $\mu$ is a probability measure if $\mu(\Omega) = 1$.

A probability space is a triple $(\Omega, \mathcal{F}, P)$, where $\Omega$ is a set, $\mathcal{F}$ is a $\sigma$-algebra on $\Omega$ and $P$ is a probability measure on $\mathcal{F}$. The sets in $\mathcal{F}$ are called events. The empty set is called an impossible event because $P(\emptyset) = 0$. Set operations have natural interpretations, for instance for “$A \cap B$”, we say “$A$ and $B$ occur”, for “$A \cup B$”, we say “$A$ or $B$ occurs”, for “$A^c$”, we say “$A$ does not occur”, for “$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$”, we say “infinitely many of the events $A_k$ occur”, etc.

This definition is a starting point of modern probability theory. It was laid as foundations by Kolmogorov who presented his axiom system for probability theory in 1933.

We record some basic and useful properties of probability measures.

1.1 Theorem. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A, B, A_1, A_2, \ldots$ be events. Then

(i) $P(A^c) = 1 - P(A)$,
(ii) if $A \subset B$, then $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$ and $\mathbb{P}(A) \leq \mathbb{P}(B)$,

(iii) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B),$

(iv) $\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n-1} \mathbb{P}(A_1 \cap \cdots \cap A_n),$

(v) $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i),$

(vi) if $A_1, \ldots, A_n$ are pairwise disjoint, then $\mathbb{P}(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i).$

We omit proofs (which are rather standard). Part (iv) is the so-called inclusion-exclusion formula. Part (v) is the so-called union bound.

We also have the following continuity of measure-type results for monotone events.

1.2 Theorem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $A_1, A_2, \ldots$ be events.

(i) if the events $A_n$ are increasing, that is $A_1 \subset A_2 \subset \ldots$, then

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n \to \infty} \mathbb{P}(A_n),$$

(ii) if the events $A_n$ are decreasing, that is $A_1 \supset A_2 \supset \ldots$, then

$$\mathbb{P}\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{n \to \infty} \mathbb{P}(A_n),$$

Proof. (i) It helps consider the events

$$B_1 = A_1, \ B_2 = A_2 \setminus A_1, \ B_3 = A_3 \setminus A_2, \ldots$$

which are disjoint. We skip the details. Part (ii) can be obtained from (i) by using the complements.

1.3 Remark. Theorem 1.1 and Theorem 1.2 (i) hold for arbitrary measures (the proofs do not need the assumption $\mathbb{P}(\Omega) = 1$ of the measure $\mathbb{P}$ being probabilistic). Theorem 1.2 (ii) holds for arbitrary measures $\mathbb{P}$ as long as $\mathbb{P}(A_k) < \infty$ for some $k$.

1.2 Basic examples

1.4 Example. Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be a countable set and $\mathcal{F} = 2^{\Omega}$ (all subsets).

Defining a probability measure on $(\Omega, \mathcal{F})$ really amounts to specifying a nonnegative sequence $p_1, p_2, \ldots$ such that $\sum_{i} p_i = 1$ and defining $\mathbb{P}(\{\omega_i\}) = p_i$. Then for every subset $A$ of $\Omega$,

$$\mathbb{P}(A) = \sum_{i: \omega_i \in A} p_i.$$
Conversely, since
\[ 1 = \mathbb{P}(\Omega) = \sum_i \mathbb{P}(\{\omega_i\}), \]
every probability measure is of this form.

1.5 Example. Let \( \Omega \) be a finite nonempty set and \( \mathcal{F} = 2^\Omega \). The uniform probability measure on \((\Omega, \mathcal{F})\) (sometimes referred to as classical) is defined as
\[ \mathbb{P}(A) = \frac{|A|}{|\Omega|}, \]
for every subset \( A \) of \( \Omega \), where here \(|\cdot|\) denotes cardinality.

Our next two examples will require nontrivial constructions. We wish to define two probability spaces which will be reasonable models of
1) selecting a point uniformly at random on the interval \([0,1]\)
2) tossing a fair coin infinitely many times.

As much as choosing the ground set \( \Omega \) is fairly natural, say \( \Omega = [0,1] \) for 1), defining an appropriate \( \sigma \)-algebra and a probability measure on it poses certain challenges. Let us first try to illustrate possible subtleties.

Let \( \Omega = [0,1] \). If \((\Omega, \mathcal{F}, \mathbb{P})\) is meant to be a probability space modelling selecting a point uniformly at random on \([0,1]\), for \(0 \leq a < b \leq 1\), we should have \( \mathbb{P}((a,b)) = b - a \) (the probability that a point is in the interval \((a,b)\) equal its length), and more generally, \( \mathbb{P} \) should be translation-invariant. Thus \( \mathcal{F} \) should at the very least contain all intervals. Thus let \( \mathcal{F} \) be such a \( \sigma \)-algebra, that is the smallest \( \sigma \)-algebra containing all the intervals in \([0,1] \); we write
\[ \mathcal{F} = \sigma(\mathcal{I}), \]
where \( \mathcal{I} \) is the family of all the intervals in \([0,1] \) and in general
\[ \mathcal{F} = \sigma(\mathcal{A}) \]
denotes the \( \sigma \)-algebra generated by a family \( \mathcal{A} \) of subsets of \( \Omega \) (the smallest \( \sigma \)-algebra containing \( \mathcal{A} \), which makes sense because intersections \( \sigma \)-algebras are still \( \sigma \)-algebras).

1.6 Example. As a result, for every \( x \in [0,1] \), we have
\[ \mathbb{P}(\{x\}) = \mathbb{P}\left(\bigcap_{n \geq 1} (x - 1/n, x + 1/n)\right) = \lim_{n \to \infty} \mathbb{P}((x - 1/n, x + 1/n)) = \lim_{n \to \infty} \frac{2}{n} = 0 \]
(recall Theorem 1.2 (ii)), that is, of course, probability of selecting a fixed point is zero. This however indicates why probability measures are defined to be only countably additive as opposed to fully additive, because if the latter was the case, we would have
\[ 1 = \mathbb{P}([0,1]) = \mathbb{P}\left(\bigcup_{x \in [0,1]} \{x\}\right) = \sum \mathbb{P}(\{x\}) = 0, \]
a contradiction. \( \square \)
Moreover, we cannot just crudely take $F = 2^\Omega$ because of the following construction of the Vitali set.

1.7 Example. For $x,y \in [0,1]$, let $x \sim y$ if and only if $x - y \in \mathbb{Q}$. This is an equivalence relation and let $V$ be the set of representatives of its abstract classes. Without loss of generality assume that $0 \not\in V$. Let $x \oplus y$ denote the addition modulo $1$, that is $x \oplus y = x + y$ if $x + y \leq 1$ and $x \oplus y = x + y - 1$ if $x + y > 1$. Consider the translations of $V$,

$$V \oplus r = \{ v \oplus r, \; v \in V \}, \quad r \in [0,1] \cap \mathbb{Q}.$$ 

Note that these sets are pairwise disjoint (because if $v_1 \oplus r_1 = v_2 \oplus r_2$ for some $v_1, v_2 \in V$ and $r_1, r_2 \in [0,1] \cap \mathbb{Q}$, then $v_1 - v_2 \in \mathbb{Q}$, hence $v_1 = v_2$, thus $r_1 = r_2$). Moreover,

$$\bigcup_{r \in [0,1] \cap \mathbb{Q}} V \oplus r = [0,1]$$

(because every point in $[0,1]$ is in a certain abstract class, hence differs from its representative by a rational). Thus, by countable-additivity

$$1 = \mathbb{P} \left( \bigcup_{r \in [0,1] \cap \mathbb{Q}} V \oplus r \right) = \sum_{r \in [0,1] \cap \mathbb{Q}} \mathbb{P} (V \oplus r).$$

If $\mathbb{P}$ is translation-invariant, we have $\mathbb{P} (V \oplus r) = \mathbb{P} (V)$ and then the right hand side is either $0$ or $+\infty$, a contradiction. $\square$

Summarising, to model a uniform random point on $[0,1]$, we take $\Omega = [0,1]$ and $F$ to be the $\sigma$-algebra generated by all the intervals. We know how to define $\mathbb{P}$ on the generators. Carathéodory’s theorem is an important abstract tool which allows to extend this definition from the generators to the whole $\sigma$-algebra $F$, provided that certain conditions are met.

A family $\mathcal{A}$ of subsets of a set $\Omega$ is called an algebra if

(i) $\Omega \in \mathcal{A}$,

(ii) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,

(iii) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.

1.8 Theorem (Carathéodory). Let $\Omega$ be a set and let $\mathcal{A}$ be an algebra on $\Omega$. Suppose a function $\mathbb{P}: \mathcal{A} \to [0, +\infty)$ satisfies

(i) $\mathbb{P} (\Omega) = 1$,

(ii) $\mathbb{P}$ is finitely additive, that is for every $A_1, \ldots, A_n \in \mathcal{A}$ which are pairwise disjoint, we have

$$\mathbb{P} \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} \mathbb{P} (A_i),$$
(iii) for every $A_1, A_2, \ldots \in \mathcal{A}$ with $A_1 \subset A_2 \subset \ldots$ such that $A = \bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{A}$, we have

$$\lim_{n \to \infty} \mathbb{P}(A_n) = \mathbb{P}(A).$$

Then $\mathbb{P}$ can be uniquely extended to a probability measure on the $\sigma$-algebra $\mathcal{F} = \sigma(\mathcal{A})$ generated by $\mathcal{A}$.

1.9 Remark. By considering $B_n = A \setminus A_n$, condition (iii) is equivalent to the following: if $B_1, B_2, \ldots \in \mathcal{F}_0$ such that $B_1 \supset B_2 \supset \ldots$ with $\bigcap B_n = \emptyset$, then $\mathbb{P}(B_n) \to 0$ as $n \to \infty$.

We defer the proof of Carathéodory’s theorem to Appendix A.

1.10 Example. We are ready to construct a probability space modelling a random point uniform on $[0, 1]$. Let $\Omega = [0, 1]$. Let

$$\mathcal{F}_0 = \left\{ (a_1, b_1] \cup \cdots \cup (a_n, b_n], \ n \geq 1, 0 \leq a_1 \leq b_1 \leq \cdots \leq a_n \leq b_n \leq 1 \right\}.$$

It is easy to check that $\mathcal{F}_0$ is an algebra on $\Omega_0 = (0, 1]$. For a set $F$ in $\mathcal{F}_0$, say $F = (a_1, b_1] \cup \cdots \cup (a_n, b_n]$, we define

$$\mathbb{P}(F) = \sum_{i=1}^{n} (b_i - a_i).$$

Clearly $\mathbb{P}$ satisfies conditions (i) and (ii) of Theorem 1.8. We now verify (iii) by means of Remark 1.9. Suppose $B_1 \supset B_2 \supset \ldots$ are in $\mathcal{F}_0$ with $\bigcap B_n = \emptyset$. If it is not the case that $\mathbb{P}(B_n) \to 0$, there is $\varepsilon > 0$ such that $\mathbb{P}(B_k) > \varepsilon$ for infinitely many $k$, say for simplicity for all $k \geq 1$. We show that $\bigcap B_n \neq \emptyset$. For every $k$, there is a set $C_k$ in $\mathcal{F}_0$ whose closure is a subset of $B_k \cap (0, 1]$ and $\mathbb{P}(B_k \setminus C_k) \leq \varepsilon 2^{-k-1}$. Then for every $n$, we have

$$\mathbb{P} \left( B_n \setminus \bigcap_{k \leq n} C_k \right) = \mathbb{P} \left( \bigcup_{k \leq n} B_k \setminus C_k \right) \leq \mathbb{P} \left( \bigcup_{k \leq n} B_k \setminus C_k \right) \leq \sum_{k \leq n} \mathbb{P}(B_k \setminus C_k) \leq \sum_{k \leq n} \varepsilon 2^{-k-1} < \varepsilon / 2.$$

This and $\mathbb{P}(B_n) > \varepsilon$ together give that $\mathbb{P} \left( \bigcap_{k \leq n} C_k \right) > \varepsilon / 2$. In particular, for every $n$, $\bigcap_{k \leq n} C_k$ is nonempty and consequently $K_n = \bigcap_{k \leq n} \text{cl}(C_k)$ is nonempty. Thus $\{K_n\}_{n=1}^{\infty}$ is a decreasing family $(K_1 \supset K_2 \supset \ldots)$ of nonempty compact sets. By Cantor’s intersection theorem, $\bigcap_{n} K_n = \bigcap_{n=1}^{\infty} \text{cl}(C_n)$ is nonempty (recall a simple argument: otherwise $\bigcup_n (\text{cl}(C_n))^c$ covers $[0, 1]$ without any finite subcover). Since $\bigcap B_n$ contains $\bigcap_n \text{cl}(C_k)$, the argument is finished.

Theorem 1.8 provides a unique extension of $\mathbb{P}$ onto the $\sigma$-algebra generated by $\mathcal{F}_0$. This extension is nothing but Lebesgue measure on $(0, 1]$, denoted Leb. We can trivially extend it onto $[0, 1]$ by assigning $\mathbb{P}(\{\emptyset\}) = 0$. $\Box$
Given a metric space \((E, \rho)\), the \(\sigma\)-algebra of subsets of \(E\) generated by all open sets in \(E\) is called the **Borel \(\sigma\)-algebra** on \(E\), denoted \(\mathcal{B}(E)\). For example, the \(\sigma\)-algebra constructed in the previous example is exactly \(\mathcal{B}([0,1])\).

**1.11 Example.** We construct a probability space modelling an infinite sequence of tosses of a fair coin. Let \(\Omega = \{\omega_1, \omega_2, \ldots\}\), \(\omega_1, \omega_2, \ldots \in \{0,1\}\) be the set of all infinite binary sequences. We can proceed as for the random point on \([0,1]\): we define an algebra of subsets of \(\Omega\) on which defining a finitely additive measure will be intuitive and easy.

Let \(\text{Cyl}\) be the family of all cylinders on \(\Omega\), that is sets of the form \(A_{\varepsilon_1, \ldots, \varepsilon_n} = \{\omega \in \Omega, \omega_j = \varepsilon_j, j = 1, \ldots, n\}\). We define the algebra of cylinders, that is the family of all finite unions of cylinders,

\[
\mathcal{F}_0 = \{A_1 \cup \cdots \cup A_k, k \geq 1, A_1, \ldots, A_k \in \text{Cyl}\}.
\]

For \(A_{\varepsilon_1, \ldots, \varepsilon_n} \in \text{Cyl}\), we set

\[
\mathbb{P}(A_{\varepsilon_1, \ldots, \varepsilon_n}) = \frac{1}{2^n}.
\]

It remains to apply Theorem 1.8. Checking (iii) proceeds similarly and eventually boils down to a topological argument (by Tikhonov’s theorem \(\Omega = \{0,1\} \times \{0,1\} \times \ldots\) is compact with the standard product topology).

Alternatively, a binary expansion of a random point \(x \in (0,1]\) gives a random sequence which intuitively does the job, too. Formally, let \(f : \Omega \to [0,1], f(\omega) = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}\).

We define

\[
\mathcal{F} = \{f^{-1}(B), B \in \mathcal{B}([0,1])\},
\]

which is a \(\sigma\)-algebra,

\[
\mathbb{P}(A) = \text{Leb}(f(A)), \quad A \in \mathcal{F},
\]

which is a probability measure (\(f\) is surjective, hence \(f(f^{-1}(B)) = B\) for every \(B\)). Note that for cylinders we have that \(f(A_{\varepsilon_1, \ldots, \varepsilon_n})\) is an interval of length \(\frac{1}{2^n}\). Thus \(\mathbb{P}(A_{\varepsilon_1, \ldots, \varepsilon_n}) = \frac{1}{2^n}\) and this construction also fulfils our intuitive basic requirement. We need get back to this example when we discuss independence. □

### 1.3 Conditioning

Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and an event \(B\) of positive probability, \(\mathbb{P}(B) > 0\), we can define

\[
\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, \quad A \in \mathcal{F}.
\]

It is natural to introduce a \(\sigma\)-algebra of events \(\mathcal{F}_B\) seen by \(B\), that is

\[
\mathcal{F}_B = \{A \cap B, A \in \mathcal{F}\}.
\]

**1.12 Theorem.** \(\mathbb{P}(\cdot|B)\) is a probability measure on \(\mathcal{F}\), thus also on \(\mathcal{F}_B\).
The new probability measure $P(\cdot|B)$ is referred to as the **conditional probability** given $B$. Introducing it often times makes computations more intuitive. We have several useful facts.

1.13 **Theorem** (Chain rule). Suppose that $A_1, \ldots, A_n$ are events which satisfy the condition $P(A_1 \cap \cdots \cap A_{n-1}) > 0$. Then

$$P(A_1 \cap \cdots \cap A_n) = P(A_1) P(A_2|A_1) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1}).$$

1.14 **Theorem** (Law of total probability). Suppose $\{B_n, n = 1, 2, \ldots\}$ is a finite or countable family of events which partition $\Omega$ and $P(B_n) > 0$ for each $n$. Then for every event $A$, we have

$$P(A) = \sum_n P(A|B_n) P(B_n).$$

1.15 **Theorem** (Bayes’ formula). Suppose $\{B_n, n = 1, 2, \ldots\}$ is a finite or countable family of events which partition $\Omega$ and $P(B_n) > 0$ for each $n$. Then for every event $A$ of positive probability and every $k$, we have

$$P(B_k|A) = \frac{P(A|B_k) P(B_k)}{\sum_n P(A|B_n) P(B_n)}.$$

We leave all the proofs as exercise to the dedicated reader.
1.4 Exercises

1. If $A$ and $B$ are events, then $\Pr(A \cap B) \geq \Pr(A) - \Pr(B^c)$.

2. If $A_1, \ldots, A_n$ are events, then we have
   a) $\Pr(\bigcup_{i=1}^n A_i) \leq \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Pr(A_{i_1} \cap \cdots \cap A_{i_k})$ for $m$ odd,
   b) $\Pr(\bigcup_{i=1}^n A_i) \geq \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \Pr(A_{i_1} \cap \cdots \cap A_{i_k})$ for $m$ even.

   These are called Bonferroni inequalities.

3. There are $n$ invitation cards with the names of $n$ different people and $n$ envelopes with their names. We put the cards at random into the envelopes, one card per envelope. What is the chance that not a single invitation landed in the correct envelope? What is the limit of this probability as $n$ goes to infinity?

4. Describe all $\sigma$-algebras on a countable set.

5. Is there an infinite $\sigma$-algebra which is countable?

6. Show that the number of $\sigma$-algebras on the $n$-element set equals $\frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$.

7. Prove Theorems 1.12 – 1.15.
2 Random variables

2.1 Definitions and basic properties

Central objects of study in probability theory are random variables. They are simply measurable functions. To put it formally, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. A function \(X : \Omega \to \mathbb{R}\) is called a random variable if for every Borel set \(B \in \mathcal{B}(\mathbb{R})\), we have \(X^{-1}(B) \in \mathcal{F}\). In other words, \(X\) is a measurable function on \((\Omega, \mathcal{F}, \mathbb{P})\). An \(\mathbb{R}^n\)-valued random variable, that is a measurable function \(X : \Omega \to \mathbb{R}^n\) is called a random vector.

2.1 Example. Let \(A\) be an event. We define

\[
1_A(\omega) = \begin{cases} 
1, & \text{if } \omega \in A, \\
0, & \text{if } \omega \not\in A.
\end{cases}
\]

This is a random variable called the indicator random variable of the event \(A\).

2.2 Example. Let \(\Omega = [0, 1], \mathcal{F} = \mathcal{B}([0, 1])\) and \(\mathbb{P} = \text{Leb}\). Define \(X : \Omega \to [0, 1]\) as \(X(\omega) = \omega\). This is a random variable which intuitively is uniform on \([0, 1]\). We will make this precise soon. On the other hand, if \(V\) is the Vitali set from Example 1.7, then \(X = 1_V\) is not a random variable because \(X^{-1}(\{1\}) = V \notin \mathcal{F}\).

We record several very basic facts. One piece of notation: we often write \(\{X \leq t\}\), or \(\{X \in B\}\), etc. meaning \(\{\omega \in \Omega, \ X(\omega) \leq t\} = X^{-1}((-\infty, t])\), or \(\{\omega \in \Omega, \ X(\omega) \in B\} = X^{-1}(B)\), etc. Moreover, \(\{X \in A, X \in B\}\) means \(\{X \in A\} \cap \{X \in B\}\).

2.3 Theorem. If \(X : \Omega \to \mathbb{R}\) satisfies: for every \(t \in \mathbb{R}\),

\[
\{X \leq t\} \in \mathcal{F},
\]

then \(X\) is a random variable.

Proof. Consider the family \(\{A \subset \mathbb{R}, \ X^{-1}(A) \in \mathcal{F}\}\). It is not difficult to check that this is a \(\sigma\)-algebra. By the assumption, it contains the intervals \((-\infty, t], t \in \mathbb{R}\), which generate \(\mathcal{B}(\mathbb{R})\).

2.4 Theorem. If \(X, Y\) are random variables (defined on the same probability space), then \(X + Y\) and \(XY\) are random variables.

Proof. We use Theorem 2.3. Note that

\[
\{X + Y > t\} = \bigcup_{q \in \mathbb{Q}} \{X > q, Y > t - q\}
\]

and the right hand side is in \(\mathcal{F}\) as a countable union of events. Thus \(X + Y\) is a random variable. Moreover, for \(t \geq 0\),

\[
\{X^2 \leq t\} = \{-\sqrt{t} \leq X \leq \sqrt{t}\} = \{X \leq \sqrt{t}\} \setminus \{X < -\sqrt{t}\} \in \mathcal{F}
\]
so \(X^2\) and \(Y^2\) are also random variables. Thus

\[
XY = \frac{1}{2} \left((X + Y)^2 - X^2 - Y^2\right)
\]

is a random variable.

2.5 Theorem. If \(X_1, X_2, \ldots\) are random variables (defined on the same probability space), then \(\inf_n X_n\), \(\liminf_n X_n\), \(\lim_n X_n\) (if exists, understood pointwise) are random variables.

Proof. For instance \(\{\inf_n X_n \geq t\} = \bigcap_n \{X_n \geq t\}\) justifies that \(\inf_n X_n\) is a random variable. We leave the rest as an exercise.

2.6 Theorem. Let \(X\) be a random variable. If \(f: \mathbb{R} \to \mathbb{R}\) is a (Borel) measurable function, that is \(f^{-1}(B) \in \mathcal{B}(\mathbb{R})\) for every \(B \in \mathcal{B}(\mathbb{R})\), then \(f(X)\) is a random variable.

Proof. We have \((f(X))^{-1}(B) = X^{-1}(f^{-1}(B))\).

2.7 Example. If \(X\) is a random variable, then \(|X|^p\), \(e^X\), etc. are random variables.

Given a random variable \(X\), we define the \(\sigma\)-algebra generated by \(X\), denoted \(\sigma(X)\), as the smallest \(\sigma\)-algebra with respect to which \(X\) is measurable, that is

\[
\sigma(X) = \sigma\left( X^{-1}(B) , \ B \in \mathcal{B}(\mathbb{R}) \right) = \{ X^{-1}(B) , \ B \in \mathcal{B}(\mathbb{R}) \}
\]

(the family of the right is a \(\sigma\)-algebra). Similarly, given a collection of random variables \(\{X_i\}_{i \in I}\) we define its \(\sigma\)-algebra as the smallest \(\sigma\)-algebra with respect to which every \(X_i\) is measurable, that is

\[
\sigma(X_i , i \in I) = \sigma\left( X_i^{-1}(B) , \ B \in \mathcal{B}(\mathbb{R}) , i \in I \right).
\]

Let \(X\) be a random variable. The law of \(X\), denoted \(\mu_X\), is the following probability measure on \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\),

\[
\mu_X(B) = \mathbb{P}(X \in B) , \quad B \in \mathcal{B}(\mathbb{R}) .
\]

2.8 Example. Let \(X\) be a constant random variable a.s., that is \(\mathbb{P}(X = a) = 1\) for some \(a \in \mathbb{R}\). Its law \(\mu_X\) is a very simple measure on \(\mathbb{R}\),

\[
\mu_X(A) = \begin{cases} 
1, & \text{if } a \in A, \\
0, & \text{if } a \notin A.
\end{cases}
\]

This measure on \(\mathbb{R}\) is called the Dirac delta at \(a\), denoted \(\delta_a\).

The cumulative distribution function of \(X\) (distribution function or CDF in short) is the following function \(F_X: \mathbb{R} \to [0,1]\),

\[
F_X(t) = \mathbb{P}(X \leq t) , \quad t \in \mathbb{R}.
\]

15
It is rather clear that for some two random variables $X$ and $Y$, $\mu_X = \mu_Y$ does not imply that $X = Y$ (the random variables may even be defined on different probability spaces). We say that $X$ and $Y$ have the same distribution (law) if $\mu_X = \mu_Y$. Is it clear that $F_X = F_Y$ implies that $X$ and $Y$ have the same distribution? In other words, do CDFs determine distribution? To answer this and many other similar questions, it is convenient to use an abstract tool from measure theory – Dynkin’s theorem on $\pi - \lambda$ systems.

2.2 $\pi - \lambda$ systems

A family $\mathcal{A}$ of subsets of a set $\Omega$ is a $\pi$-system if it is closed under finite intersections, that is for every $A, B \in \mathcal{A}$, we have $A \cap B \in \mathcal{A}$.

A family $\mathcal{L}$ of subsets of a set $\Omega$ is a $\lambda$-system if

(i) $\Omega \in \mathcal{L}$,

(ii) if $A, B \in \mathcal{L}$ and $A \subseteq B$, then $B \setminus A \in \mathcal{L}$,

(iii) for every $A_1, A_2, \ldots \in \mathcal{L}$ such that $A_1 \subseteq A_2 \subseteq \ldots$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$.

For example, the family of intervals $\{(−\infty, t], t \in \mathbb{R}\}$ is a $\pi$-system. The importance of this example is that this family generates $\mathcal{B}(\mathbb{R})$.

Note that if a family is a $\pi$-system and a $\lambda$-system, then it is a $\sigma$-algebra.

A fundamental and useful result is the following theorem (see Appendix B for the proof).

2.9 Theorem (Dynkin). If a $\lambda$-system $\mathcal{L}$ contains a $\pi$-system $\mathcal{A}$, then $\mathcal{L}$ contains $\sigma(\mathcal{A})$.

2.3 Properties of distribution functions

Equipped with Dynkin’s theorem, we are able to show that distribution functions indeed determine the distribution, which reverses the trivial implication that if $\mu_X = \mu_Y$, then $F_X = F_Y$.

2.10 Theorem. Let $X$ and $Y$ be random variables (possibly defined on different probability spaces). If $F_X = F_Y$, then $\mu_X = \mu_Y$.

Proof. Let $\mathcal{A} = \{(-\infty, t], t \in \mathbb{R}\}$. This is a $\pi$-system and $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. Consider

$$\mathcal{L} = \{A \in \mathcal{B}(\mathbb{R}), \mu_X(A) = \mu_Y(A)\}.$$

This is a $\lambda$-system (which easily follows from properties of probability measures). The assumption $F_X = F_Y$ gives $\mathcal{L} \supset \mathcal{A}$. Thus, by Theorem 2.9, we get $\mathcal{L} \supset \sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$. By the definition of $\mathcal{L}$, this gives $\mu_X = \mu_Y$. \qed
2.11 Remark. The same proof gives the following: if for some two probability measures $\mu, \nu$ defined on the same space, we have $\mu = \nu$ on a $\pi$-system generating a $\sigma$-algebra $F$, then $\mu = \nu$ on $F$.

We list 3 basic properties of distribution functions.

2.12 Theorem. Let $X$ be a random variable. Then its distribution function $F_X$ satisfies

(i) $F_X$ is nondecreasing, that is for every $s \leq t$, $F_X(s) \leq F_X(t)$,

(ii) $\lim_{t \to -\infty} F_X(t) = 0$ and $\lim_{t \to +\infty} F_X(t) = 1$,

(iii) $F_X$ is right-continuous, that is for every $t \in \mathbb{R}$, $\lim_{s \to t^+} F_X(s) = F_X(t)$.

Proof. Part (i) follows from the inclusion $\{X \leq s\} \subset \{X \leq t\}$ if $s \leq t$. Alternatively,

$$0 \leq P(X \in (s,t]) = P(X \leq t) - P(X \leq s) = F_X(t) - F_X(s).$$

Part (ii), (iii) follow from the continuity of probability measures (Theorem 1.2).

These properties in fact characterise distribution functions.

2.13 Theorem. If a function $F : \mathbb{R} \to [0,1]$ satisfies (i)-(iii) from Theorem 2.12, then $F = F_X$ for some random variable $X$.

Proof. Let $\Omega = [0,1]$, $F = B([0,1])$ and $P = \text{Leb}$. The idea is to define $X$ as the inverse of $F$. Formally, we set

$$X(\omega) = \inf \{y, F(y) \geq \omega\}, \quad \omega \in [0,1].$$

By the definition of infimum and (i)-(iii), $X(\omega) \leq t$ if and only if $\omega \leq F(t)$ (check!). Thus

$$F_X(t) = P(X \leq t) = \text{Leb}\{\omega \in [0,1], \omega \leq F(t)\} = F(t).$$

2.14 Remark. There is another construction, sometimes called canonical, based on Carathéodory’s theorem. We set $\Omega = \mathbb{R}$, $F = B(\mathbb{R})$, define $P((-\infty,t]) = F(t)$ and then extend $P$. With such $P$, the desired random variable is the canonical one, $X(x) = x$, $x \in \mathbb{R}$.

For a random vector $X = (X_1, \ldots, X_n)$ in $\mathbb{R}^n$, the cumulative distribution function of $X$ is the function $F_X : \mathbb{R}^n \to [0,1]$,

$$F_X(t_1, \ldots, t_n) = P(X_1 \leq t_1, \ldots, X_n \leq t_n).$$

As before, for random vectors $X, Y$ in $\mathbb{R}^n$, $F_X = F_Y$ implies that $\mu_X = \mu_Y$. The characterising properties are almost the same – the monotonicity statement is strengthened.
2.15 Theorem. Let $X$ be a random vector in $\mathbb{R}^n$. Then its distribution function $F_X$ satisfies

(i) $F_X$ is nondecreasing, that is for every $s,t \in \mathbb{R}^n$ with $s_i \leq t_i$, $i \leq n$, we have $F_X(s) \leq F_X(t)$. Moreover,

$$\sum_{\varepsilon \in \{0,1\}^n} (-1)^{\sum_{k=1}^n \varepsilon_k} F_X(\varepsilon_1 s_1 + (1 - \varepsilon_1)t_1, \ldots, \varepsilon_n s_n + (1 - \varepsilon_n)t_n) \geq 0,$$

(ii) $F_X(t^{(m)}_1, \ldots, t^{(m)}_n) \xrightarrow{m \to \infty} 0$ provided that $\inf_{k \leq n} t^{(m)}_k \xrightarrow{m \to \infty} -\infty$,

(iii) $F_X(t^{(m)}_1, \ldots, t^{(m)}_n) \xrightarrow{m \to \infty} 1$ provided that $\inf_{k \leq n} t^{(m)}_k \xrightarrow{m \to \infty} +\infty$,

(iv) $F_X$ is right-continuous.

Proof. We only show (i) as the rest is proved in much the same way as in one dimension. The inequality is nothing but the statement that the probability of $X$ being in the box $\prod_{k=1}^n (s_k, t_k]$ is nonnegative (cf. the proof of Theorem 2.13 (i)). To see this, let $A = \bigcap_{k=1}^n \{X_k \leq t_k\}$ and $B = \bigcup_{k=1}^n \{X_k \leq s_k\}$. Then

$$0 \leq \mathbb{P} \left( X \in \prod_{k=1}^n (s_k, t_k] \right) = \mathbb{P}(A \cap B^c) = \mathbb{P}(A) - \mathbb{P}(A \cap B).$$

Note that $\mathbb{P}(A) = F_X(t)$ and $\mathbb{P}(A \cap B) = \bigcup_{k=1}^n \{X_k \leq s_k\} \cap B$. Applying the inclusion-exclusion formula finishes the proof. \qed

Again, these properties characterise distribution functions of random vectors. The proof follows a canonical construction sketched in Remark 2.14. We leave the details as an exercise.

2.16 Theorem. If a function $F: \mathbb{R}^n \to [0,1]$ satisfies (i)-(iv) from Theorem 2.15, then $F = F_X$ for some random vector $X$ in $\mathbb{R}^n$.

We end with a simple remark which follows from the right-continuity.

2.17 Remark. For a random variable $X$ and $a \in \mathbb{R}$, we have

$$\mathbb{P}(X = a) = \mathbb{P}(\{X \leq a\} \setminus \{X < a\}) = \mathbb{P}(X \leq a) - \mathbb{P}(X < a) = F_X(a) - F_X(a^{-}).$$

Now, $\mathbb{P}(X = a) > 0$ if and only if $F_X$ is discontinuous at $a$ (has a jump) and the value of the jump is precisely $\mathbb{P}(X = a)$. In this case, we say that $X$ has an atom at $a$.

2.4 Examples: discrete and continuous random variables

2.18 Example. We say that a random variable $X$ is discrete if there is a countable subset $A$ of $\mathbb{R}$ such that $\mathbb{P}(X \in A) = 1$. Say $A = \{a_1, a_2, \ldots\}$ and denote $p_k = \mathbb{P}(X = a_k)$. 

18
We can assume that the $p_k$ are all positive (otherwise, we just do not list $a_k$ in $A$). We have $\sum_k p_k = 1$. The $a_k$ are then the atoms of $X$. The law of $X$ is a mixture of Dirac deltas at the atoms,

$$\mu_X = \sum p_k \delta_{a_k}.$$ 

The CDF of $X$ is a piecewise constant function with jumps at the atoms with the values being the $p_k$.

2.19 Example. We say that a random variable $X$ is **continuous** if there is an integrable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\mu_X(A) = \int_A f, \quad A \in \mathcal{B}(\mathbb{R}).$$

Then $f$ is called the density of $X$ (note it is not unique – we can modify $f$ on a set of Lebesgue measure zero without changing the above). In particular,

$$F_X(t) = \mu_X((-\infty, t]) = \int_{-\infty}^t f(x)dx$$

and necessarily $F_X$ is continuous. We collect basic characterising properties of density functions.

2.20 Theorem. Let $X$ be a continuous random variable and let $f$ be its density function. Then

(i) $\int f = 1$

(ii) $f \geq 0$ a.e.

(iii) $f$ is determined by $X$ uniquely up to sets of measure 0.

Proof. Plainly, $\int f = \mu_X(\mathbb{R}) = 1$, so we have (i). To see (ii), let $A_n = \{ f < -1/n \}$ and $A = \{ f < 0 \} = \bigcup A_n$. We have

$$0 \leq \mu_X(A_n) = \int_{A_n} f \leq -\frac{1}{n} \text{Leb}(A_n),$$

so $\text{Leb}(A_n) = 0$ and thus $\text{Leb}(A) = 0$. The proof of (iii) is similar. \hfill $\square$

2.21 Theorem. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies properties (i)-(ii) of Theorem 2.20. Then there is a continuous random variable $X$ with density $f$.

Proof. We set $F(x) = \int_{-\infty}^x f, \ x \in \mathbb{R}$ and use Theorem 2.13. \hfill $\square$

Of course, it is easy to give examples of random variables which are neither discrete nor continuous, say $F(x) = \frac{x}{2} \mathbf{1}_{[0,1)}(x) + \mathbf{1}_{[1,\infty)}(x)$ is a distribution function of such a random variable (it is not continuous because $F$ is not continuous and it is not discrete because $F$ is not piecewise constant). We finish off this chapter with an interesting strong example of this sort.
2.22 Example. Let $F : [0, 1] \to [0, 1]$ be the Cantor’s devil’s staircase function (a continuous nondecreasing function which is piecewise constant outside the Cantor set $\mathcal{C} \subset [0, 1]$). Extend $F$ on $\mathbb{R}$ by simply putting 0 on $(-\infty, 0]$ and 1 on $[1, +\infty)$. Then $F$ is a distribution function of some random variable. It is not discrete because $F$ is continuous and if it was continuous, we would have

$$F(x) = \int_{-\infty}^{x} f$$

for some integrable function $f$, but since $f(x) = F'(x) = 0$ for $x \notin \mathcal{C}$ ($F$ is constant on $\mathcal{C}^c$), we would also have

$$1 = \int_{\mathbb{R}} f = \int_{\mathcal{C}} f + \int_{\mathcal{C}^c} = 0$$

(the first integral vanishes because $\mathcal{C}$ is of measure 0 and the second integral vanishes because as we just saw $f$ vanishes on $\mathcal{C}^c$), a contradiction. What is this random variable?
2.5 Exercises

1. Give an example of two different random variables \(X\) and \(Y\) with \(\mu_X = \mu_Y\).

2. Fill out the details in the proof of Theorem 2.13.

3. Prove Theorem 2.16.

4. Show that every random variable has at most countably many atoms.

5. Suppose that a random vector \((X, Y)\) is such that both \(X\) and \(Y\) are continuous random variables. Does the random vector \((X, Y)\) have to be continuous?

6. Is there a random vector \((X, Y, Z)\) in \(\mathbb{R}^3\) such that \(aX + bY + cZ\) is a uniform random variable on \([-1, 1]\) for every reals \(a, b, c\) with \(a^2 + b^2 + c^2 = 1\)?
   
   \textit{Hint:} Archimedes’ Hat-Box Theorem.

7. Let \(X\) be a random variable uniform on \([0, 2]\). Find the distribution function of random variables \(Y = \max\{1, X\}, Z = \min\{X, X^2\}\).

8. Give an example of an uncountable family of random variables \(\{X_i\}_{i \in I}\) such that \(\sup_{i \in I} X_i\) is not a random variable.

9. Is there a random variable such that the set of the discontinuity points of its distribution function is dense in \(\mathbb{R}\)?
3 Independence

Recall that two events $A, B$ are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \mathbb{P}(B)$, which is equivalent to $\mathbb{P}(A^c \cap B) = \mathbb{P}(A^c) \mathbb{P}(B)$. A good way to generalise this is via $\sigma$-algebras.

3.1 Definitions

For an event $A \in \mathcal{F}$, we define the $\sigma$-algebra generated by it as

$$\sigma(A) = \{\emptyset, \Omega, A, A^c\},$$

that is $\sigma(A)$ is $\sigma(1_A)$, the $\sigma$-algebra generated by the indicator random variable $1_A$.

The crucial general definition of independence is as follows.

A family $\{F_i\}_{i \in I}$ of collections of subsets of $\Omega$ (typically $\sigma$-algebras, $\pi$-systems, etc.) with each $F_i$ being a subset of $\mathcal{F}$ is called independent if for every $n, i_1, \ldots, i_n \in I$ and every $A_1 \in F_{i_1}, \ldots, A_n \in F_{i_n}$, we have

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1) \cdots \mathbb{P}(A_n).$$

A family of events $\{A_i\}_{i \in I}$ is independent (or simply the events $A_i$, $i \in I$, are independent) if the family of the $\sigma$-algebras generated by them, $\{\sigma(A_i)\}_{i \in I}$ is independent. A family of random variables $\{X_i\}_{i \in I}$ is independent (or simply the random variables $X_i$, $i \in I$, are independent) if the family of the $\sigma$-algebras generated by them, $\{\sigma(X_i)\}_{i \in I}$ is independent. Note that since $\sigma(A) = \sigma(1_A)$, the events $A_i$ are independent if and only if the random variables $1_{A_i}$ are independent.

As is stated now, to check the independence of say 3 events $\{A_1, A_2, A_3\}$, we have to verify $4^3$ identities of the form $\mathbb{P}(B_1 \cap B_2 \cap B_3) = \mathbb{P}(B_1) \mathbb{P}(B_2) \mathbb{P}(B_3)$, where each $B_i$ is one of the sets $\emptyset, \Omega, A_i, A_i^c$. Of course, many of these identities are either trivial of follow from the other. It turns out that $\pi$-systems can help and we have the following useful general lemma.

3.1 Lemma. Let $\{A_i\}_{i \in I}$ be a family of $\pi$-systems. Then the family $\{\sigma(A_i)\}_{i \in I}$ is independent if and only if the family $\{A_i\}_{i \in I}$ is independent.

Proof. Since the definition of independence involves only finite sub-families, we can assume that $I = \{1, \ldots, n\}$. One implication is clear, so we assume that the $\pi$-systems are independent and want to deduce the independence of the $\sigma$-algebras generated by them. To this end, we shall use Dynkin’s theorem. We define the class

$$\mathcal{L}_1 = \{B_1 \in \mathcal{F} : \forall A_2 \in A_2, \ldots, A_n \in A_n, \mathbb{P}(B_1 \cap A_2 \cap \cdots \cap A_n) = \mathbb{P}(B_1) \mathbb{P}(A_2) \cdots \mathbb{P}(A_n)\}$$

22
By the assumption \( \mathcal{L}_1 \) contains \( A_1 \). By properties of probability measures, \( \mathcal{L}_1 \) is a \( \lambda \)-system. Hence, by Dynkin’s theorem (Theorem 2.9), \( \mathcal{L}_1 \) contains \( \sigma(A_1) \). It remains to inductively repeat the same argument: suppose we know for some \( k < n \) that

\[
P(B_1 \cap \cdots \cap B_k \cap A_{k+1} \cap \cdots \cap A_n) = P(B_1) \cdots P(B_k) \cdot P(A_{k+1}) \cdots P(A_n) \tag{3.1}
\]

for every \( B_i \in \sigma(A_i), i \leq k \) and \( A_j \in A_j, j > k \). Fix some \( B_i \in \sigma(A_i), i \leq k \) and \( A_j \in A_j, j > k \). As above, considering \( \mathcal{L}_{k+1} = \{B_{k+1} \in \mathcal{F} : \forall A_{k+2} \in A_{k+2}, \ldots, A_n \in A_n \}
\]

\[
P(B_1 \cap \cdots \cap B_k \cap B_{k+1} \cap A_{k+2} \cap \cdots \cap A_n) = P(B_1) \cdots P(B_k) \cdot P(B_{k+1}) \cdot P(A_{k+2}) \cdots P(A_n)
\]

shows that (3.1) holds for \( k + 1 \). Thus this holds for \( k = n \).

We note two useful results about packaging independence.

**3.2 Theorem.** Let \( \{F_i\}_{i \in I} \) be a family of independent \( \sigma \)-algebras. Suppose the index set \( I \) is partitioned into nonempty sets \( \{I_j, j \in J\} \). Then the \( \sigma \)-algebras

\[
G_j = \sigma(\{F_i, i \in I_j\}), \quad j \in J
\]

are independent.

**Proof.** For each \( j \in J \), define \( A_j \) to be the \( \pi \)-system of all finite intersections of the form \( B_{i_1} \cap \cdots \cap B_{i_m} \), where \( i_1, \ldots, i_m \in I_j \) and \( B_i \in F_i, k = 1, \ldots, m \). We have \( \sigma(A_j) = G_j \).

By the assumption, it follows that the families \( A_j, j \in J \) are independent (check!), so by Lemma 3.1, the \( G_j \) are independent.

**3.3 Theorem.** Suppose

\[
X_{1,1}, \ldots, X_{1,n_1},
X_{2,1}, \ldots, X_{2,n_2},
\vdots \quad \vdots \quad \vdots
X_{k,1}, \ldots, X_{k,n_k}
\]

are independent random variables and

\[
f_1 : \mathbb{R}^{n_1} \to \mathbb{R},
\]

\[
f_k : \mathbb{R}^{n_k} \to \mathbb{R}
\]

are measurable functions. Then the random variables

\[
Y_1 = f_1(X_{1,1}, \ldots, X_{1,n_1}),
\]

\[
\vdots
\]

\[
Y_k = f_k(X_{k,1}, \ldots, X_{k,n_k})
\]

are independent.
Proof. By Theorem 3.2, the $\sigma$-algebras $G_i = \sigma(\sigma(X_{i,1}), \ldots, \sigma(X_{i,n_i})), i \leq k$, are independent. The result follows because $\{Y_i \leq t\} \in G_i$, so $\sigma(Y_i) \subset G_i$. \qed

3.2 Product measures and independent random variables

Given two probability measures $\mu, \nu$ on ($\mathbb{R}, \mathcal{B}(\mathbb{R})$), recall that their product, denoted $\mu \otimes \nu$ is the unique measure on ($\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)$) such that

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B) \quad \text{for all } A, B \in \mathcal{B}(\mathbb{R})$$

(see Appendix C for the details). Exploiting Lemma 3.1, we derive convenient and important equivalent conditions for independence of random variables. For simplicity we state it just for two random variables, but of course it can be easily generalised to arbitrary many of them.

3.4 Theorem. The following are equivalent

(i) random variables $X, Y$ are independent,

(ii) $F_{(X,Y)}(s,t) = F_X(s)F_Y(t)$, for all $s, t \in \mathbb{R},$

(iii) $\mu(X,Y) = \mu_X \otimes \mu_Y$.

Proof. (i)⇒(ii) follows from the definition since $\{X \leq s\} \in \sigma(X)$ and $\{Y \leq y\} \in \sigma(Y)$.

(ii)⇒(i) follows from Lemma 3.1 ($\{X \leq s\}_{s \in \mathbb{R}}$ is a $\pi$-system generating $\sigma(X)$).

(i)⇒(iii) from the definition, $\mu(X,Y) = \mu \otimes \nu$ on the $\pi$-system of the product sets $A \times B$, $A, B \in \mathcal{B}(\mathbb{R})$ which generate $\mathcal{B}(\mathbb{R}^2)$, thus, by Remark 2.11, $\mu(X,Y) = \mu \otimes \nu$ on $\mathcal{B}(\mathbb{R}^2)$.

(iii)⇒(ii) follows by applying (iii) to $A = \{X \leq s\}, B = \{Y \leq t\}$. \qed

For continuous random variables, we have another convenient criterion in terms of densities.

3.5 Theorem. If $X_1, \ldots, X_n$ are continuous random variables with densities $f_1, \ldots, f_n$ respectively, then they are independent if and only if the random vector $(X_1, \ldots, X_n)$ is continuous with density

$$f(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n).$$
Proof. Suppose we have independence. Then, by Theorem 3.4 (ii) and Fubini’s theorem, for every Borel sets \( A_i \) in \( \mathbb{R} \), we have

\[
\mathbb{P} \left( (X_1, \ldots, X_n) \in A_1 \times \cdots \times A_n \right) = \prod_{i=1}^{n} \mathbb{P} (X_i \in A_i) = \prod_{i=1}^{n} \int_{A_i} f_i \, dx = \int_{A_1 \times \cdots \times A_n} f_1(x_1) \cdots f_n(x_n) \, dx_1 \cdots dx_n.
\]

This means that \( f_1(x_1) \cdots f_n(x_n) \) is the density of \((X_1, \ldots, X_n)\). To see the opposite implication, simply backtrack the above equalities. \(\square\)

We leave it as an exercise to prove a discrete analogue.

3.6 Theorem. If \( X_1, \ldots, X_n \) are discrete random variables with the atoms in some sets \( A_1, \ldots, A_n \) respectively, then they are independent if and only if for every sequence \( a_1, \ldots, a_n \) with \( a_i \in A_i \) for each \( i \), we have

\[
\mathbb{P} (X_1 = a_1, \ldots, X_n = a_n) = \mathbb{P} (X_1 = a_1) \cdots \mathbb{P} (X_n = a_n).
\]

3.3 Examples

3.7 Example. Let \( \Omega = \{0, 1\}^n \), \( \mathcal{F} = 2^\Omega \), \( \mathbb{P} \) is uniform, that is \( \mathbb{P} (\{\omega\}) = 2^{-n} \) for every \( \omega \in \Omega \) (the probability space of \( n \) tosses of a fair coin). For \( k = 1, \ldots, n \) consider the events

\[
A_k = \{\omega \in \Omega : \omega_k = 0\} \quad (k\text{th toss is } 0).
\]

We claim that the events \( A_1, \ldots, A_n \) are independent. For \( 1 \leq i_1 < \ldots < i_k \leq n \), we have

\[
\mathbb{P} (A_{i_1} \cap \ldots \cap A_{i_k}) = \mathbb{P} (\{\omega \in \Omega : \omega_{i_1} = \ldots = \omega_{i_k} = 0\}) = \frac{2^{n-k}}{2^n} = 2^{-k} = \prod_{j=1}^{k} \mathbb{P} (A_{i_j}).
\]

Lemma 3.1 finishes the argument.

3.8 Example. Let \( \Omega = \{1, 2, 3, 4\} \), \( \mathcal{F} = 2^\Omega \), \( \mathbb{P} \) is uniform, that is \( \mathbb{P} (\{\omega\}) = 1/4 \) for every \( \omega \in \Omega \) (4 sided fair die). Let \( A_i = \{1, i+1\}, i = 1, 2, 3 \). Then

\[
\mathbb{P} (A_i) = \frac{1}{2}, \quad i = 1, 2, 3,
\]

\[
\mathbb{P} (A_i \cap A_j) = \mathbb{P} (\{1\}) = \frac{1}{4} = \mathbb{P} (A_i) \mathbb{P} (A_j), \quad i \neq j
\]

\[
\mathbb{P} (A_1 \cap A_2 \cap A_3) = \mathbb{P} (\{1\}) = \frac{1}{4} \neq \mathbb{P} (A_1) \mathbb{P} (A_2) \mathbb{P} (A_3),
\]

so the events \( A_1, A_2, A_3 \) are pairwise independent but not independent.
3.9 Example. Let \( \Omega = [0, 1]^2 \), \( \mathcal{F} = \mathcal{B}([0, 1]^2) \), \( \mathbb{P} = \text{Leb} \) (a random point uniformly selected from the unit square \([0, 1]^2\)). Let \( A = B = \{(x, y) \in [0, 1]^2, \ x > y\} \) and \( C = \{(x, y) \in [0, 1]^2, \ x < \frac{1}{2}\} \). Then
\[
\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \mathbb{P}(B) \mathbb{P}(C),
\]
but
\[
\mathbb{P}(A \cap B) \neq \mathbb{P}(A) \mathbb{P}(B), \quad \mathbb{P}(A \cap C) \neq \mathbb{P}(A) \mathbb{P}(C), \quad \mathbb{P}(B \cap C) \neq \mathbb{P}(B) \mathbb{P}(C).
\]

3.10 Example. If for events \( A_1, \ldots, A_n \) and every \( \varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\} \), we have
\[
\mathbb{P}(A_1^{\varepsilon_1} \cap \cdots \cap A_n^{\varepsilon_n}) = \mathbb{P}(A_1^{\varepsilon_1}) \cdots \mathbb{P}(A_n^{\varepsilon_n}),
\]
where \( A^\varepsilon = A \) if \( \varepsilon = 0 \) and \( A^\varepsilon = A^c \) if \( \varepsilon = 1 \), then the family \( \{A_i\}_{i \leq n} \) is independent. A simple explanation relies on algebraic manipulations like this one
\[
\mathbb{P}(A_2 \cap \cdots \cap A_n) = \mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) + \mathbb{P}(A_1^c \cap A_2 \cap \cdots \cap A_n).
\]

We skip the details.

3.11 Example. Let \( \Omega = (0, 1) \), \( \mathcal{F} = \mathcal{B}((0, 1]) \), \( \mathbb{P} = \text{Leb} \) (a random point uniformly selected from the unit interval \((0, 1])\). For every point \( x \in (0, 1] \), we write its binary expansion,
\[
x = \sum_{n=1}^{\infty} \frac{d_n(x)}{2^n},
\]
where \( d_n(x) \in \{0, 1\} \) is the \( n \)th digit of \( x \). For uniqueness, say we always write the expansion that has infinitely many 1’s, e.g. \( \frac{1}{2} = 0.0111 \ldots \). Consider the events
\[
A_k = \{x \in [0, 1], \ d_k(x) = 0\}, \quad k = 1, 2, \ldots
\]

Claim. \( \mathbb{P}(A_k) = \frac{1}{2} \) and \( \{A_k\}_{k \geq 1} \) are independent.

In other words, the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is also a good model for infinitely many tosses of a fair coin with the events \( A_k \) being “\( k \)th toss is heads”. To prove that \( \mathbb{P}(A_k) = \frac{1}{2} \), just note that \( A_k \) is the union of \( 2^k \) intervals \( \left(\sum_{i=1}^{k} \varepsilon_i 2^{-i}, \sum_{i=1}^{k} \varepsilon_i 2^{-i} + 2^{-k}\right] \), \( \varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\} \), each of length \( 2^{-k} \). To prove the independence, note that for fixed \( \varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\} \),
\[
\mathbb{P}(A_1^{\varepsilon_1} \cap \cdots \cap A_n^{\varepsilon_n}) = \text{Leb}\{x \in (0, 1], \ d_1(x) = \varepsilon_1, \ldots, d_n(x) = \varepsilon_n\}
\]
\[
= \text{Leb}\left(\frac{1}{2^n} \sum_{i=1}^{n} \varepsilon_i \frac{1}{2^{\frac{1}{2}}} \right)
\]
\[
= \frac{1}{2^n}
\]
and use Example 3.10.
To put this important example a bit differently, we have constructed the sequence \(d_1, d_2, \ldots\) of **independent, identically distributed** random variables (i.i.d. for short), each one having equal probability of taking the value 0 and 1 (\(d_k\) tells us the outcome of the \(k\)th toss).

### 3.12 Example

We construct a sequence \(X_1, X_2, \ldots\) of i.i.d. random variables uniform on \([0, 1]\). Let as before \(\Omega = (0, 1], \mathcal{F} = \mathcal{B}((0, 1]), \mathbb{P} = \text{Leb}\). For every \(\omega \in \Omega\) we write as before its (unique) binary expansion

\[
\omega = \sum_{i=1}^{\infty} \omega_i 2^{-i} = \omega_1 \omega_2 \ldots,
\]

where \(\omega_1, \omega_2, \ldots \in \{0, 1\}\) are the consecutive digits of \(\omega\). We define new functions

\[
\begin{align*}
X_1(\omega) &= 0.\omega_1 \omega_3 \omega_6 \omega_{10} \ldots \\
X_2(\omega) &= 0.\omega_2 \omega_5 \omega_9 \ldots \\
X_3(\omega) &= 0.\omega_4 \omega_8 \ldots \\
X_4(\omega) &= 0.\omega_7 \ldots \\
\end{align*}
\]

(we put the consecutive indices on the diagonals: 1, then 2, 3, then 4, 5, 6 then 7, 8, 9, 10 and so on). Intuitively

1) \(X_1, X_2, \ldots\) are independent random variables
2) each \(X_i\) is uniform on \([0, 1]\).

The intuition for 1) is that the rows are built on disjoint sequences of the \(\omega_i\). The formal proof follows instantly from Theorem 3.2 about packaging independence.

The intuition for 2) is that each \(\omega_i\) is just a random digit. The formal proof follows from the observation that for every \(k \geq 1\) and \(j = 0, 1, \ldots, 2^k - 1\), we have

\[
\mathbb{P} \left( \frac{j}{2^k} < X_i \leq \frac{j+1}{2^k} \right) = \frac{1}{2^k},
\]

so by the continuity of \(\mathbb{P}\), we have \(\mathbb{P}(a < X_i \leq b) = b - a\) for every interval \((a, b] \subset (0, 1]\).

### 3.13 Example

Given probability distribution functions \(F_1, F_2, \ldots\), we construct a sequence of independent random variables \(Y_1, Y_2, \ldots\) such that \(F_{Y_i} = F_i\) for each \(i\). We take the sequence \(X_1, X_2, \ldots\) of i.i.d. uniform random variables uniform on \([0, 1]\) from Example 3.12. We set

\[
Y_i = G_i(X_i),
\]

where \(G_i : [0, 1] \to \mathbb{R}\) is the **inverse** function of \(F_i\) defined in the proof of Theorem 2.13, that is

\[
G_i(x) = \inf\{y \in \mathbb{R}, \ F_i(y) \geq x\}.
\]

Then (see the proof of Theorem 2.13), we have

\[
F_{Y_i}(t) = \mathbb{P}(Y_i \leq t) = \mathbb{P}(G_i(X_i) \leq t) = \mathbb{P}(X_i \leq G_i(t)) = F_i(t)
\]

27
and the $Y_i$ are independent because the $X_i$ are independent.

### 3.4 Borel-Cantelli lemmas

Recall that for an infinite sequence of events $A_1, A_2, \ldots$, we define

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

(infinitely many $A_k$ occur) and

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

(eventually all the $A_k$ occur, that is only finitely many $A_k$ do not occur). Plainly,

$$(\limsup A_n)^c = \liminf A_n^c.$$ 

The notation is explained by the following identities involving the indicator functions $1_{A_n}$,

$$\limsup A_n = \{\omega \in \Omega, \limsup_{n \to \infty} 1_{A_n} (\omega) = 1\}$$

and

$$\liminf A_n = \{\omega \in \Omega, \liminf_{n \to \infty} 1_{A_n} (\omega) = 1\}.$$

#### 3.14 Lemma (The first Borel-Cantelli lemma).

If $A_1, A_2, \ldots$ are events such that

$$\sum_n \mathbb{P}(A_n) < \infty,$$

then

$$\mathbb{P}(\limsup A_n) = 0.$$

**Proof.** By the monotonicity of the events $B_k = \bigcup_{n \geq k} A_n$ and the union bound, we get

$$\mathbb{P}(\limsup A_n) = \mathbb{P}\left(\bigcap_k B_k\right) = \lim_{k \to \infty} \mathbb{P}(B_k) \leq \lim_{k \to \infty} \sum_{n \geq k} \mathbb{P}(A_n) = 0.$$

\[ \square \]

#### 3.15 Lemma (The second Borel-Cantelli lemma).

If $A_1, A_2, \ldots$ are independent events such that

$$\sum_n \mathbb{P}(A_n) = \infty,$$

then

$$\mathbb{P}(\limsup A_n) = 1.$$
Proof. By the monotonicity of the events $B_k = \bigcap_{n \geq k} A_n^c$, we get
\[ \mathbb{P} \left( (\limsup A_n)^c \right) = \mathbb{P} \left( \bigcup_k B_k \right) = \lim_{k \to \infty} \mathbb{P} (B_k). \]
so it is enough to show that $\mathbb{P} (B_k) = 0$. By independence, for $l \geq k$,
\[ \mathbb{P} (B_k) \leq \mathbb{P} \left( \bigcap_{l \geq n \geq k} A_n^c \right) = \prod_{k \leq n \leq l} \mathbb{P} (A_n^c) = \prod_{k \leq n \leq l} (1 - \mathbb{P} (A_n)). \]
Thus, by the inequality $1 - x \leq e^{-x}$,
\[ \mathbb{P} (B_k) \leq e^{-\sum_{k \leq n \leq l} \mathbb{P} (A_n)}. \]
Letting $l \to \infty$ and using that $\sum_{n \geq k} \mathbb{P} (A_n) = \infty$ finishes the proof.

3.16 Example. Let $X_1, X_2, \ldots$ be i.i.d. random variable with the distribution function specified by the condition $\mathbb{P} (X_k > t) = e^{-t}$, $t > 0$ for each $k$. Fix $\alpha > 0$ and consider the events $A_n = \{ X_n > \alpha \log n \}$. Since $\mathbb{P} (A_n) = e^{-\alpha \log n} = n^{-\alpha}$, by the Borel-Cantelli lemmas, we get
\[ \mathbb{P} (X_n > \alpha \log n \text{ for infinitely many } n) = \begin{cases} 0, & \text{if } \alpha > 1, \\ 1, & \text{if } \alpha \leq 1. \end{cases} \]
Let
\[ L = \limsup_{n \to \infty} \frac{X_n}{\log n}. \]
Thus,
\[ \mathbb{P} (L \geq 1) = \mathbb{P} \left( \frac{X_n}{\log n} \text{ for infinitely many } n \right) = 1 \]
and
\[ \mathbb{P} (L > 1) = \mathbb{P} \left( \bigcup_{k \geq 1} \left\{ L > 1 + \frac{1}{k} \right\} \right) \leq \sum_{k \geq 1} \mathbb{P} \left( \frac{X_n}{\log n} > 1 + \frac{1}{2k} \text{ for infinitely many } n \right) = 0. \]
Therefore, $L = 1$ a.s.

3.5 Tail events and Kolmogorov’s 0–1 law

For a sequence of random variables $X_1, X_2, \ldots$, we define its tail $\sigma$-algebra by
\[ \mathcal{T} = \bigcap_{n \geq 1} \sigma(X_{n+1}, X_{n+2}, \ldots). \]
For example, very natural events such as
\[ \{ \lim_n X_n \text{ exists} \}, \quad \{ \sum_n X_n \text{ converges} \}, \quad \{ \limsup_n X_n > 1 \} \]
belong to \( \mathcal{T} \). If we have independence, the tail \( \sigma \)-algebra carries only trivial events.

**3.17 Theorem (Kolmogorov's 0-1 law).** If \( X_1, X_2, \ldots \) is a sequence of independent random variables and \( \mathcal{T} \) is its tail \( \sigma \)-algebra, then for every \( A \in \mathcal{T} \), we have that either \( \mathbb{P}(A) = 0 \) or \( \mathbb{P}(A) = 1 \).

**Proof.** Define the \( \sigma \)-algebras
\[ X_n = \sigma(X_1, \ldots, X_n) \]
and
\[ T_n = \sigma(X_{n+1}, X_{n+2}, \ldots). \]

We prove the theorem by establishing the following 4 simple claims.

**Claim 1.** For every \( n \), \( X_n \) and \( T_n \) are independent.
Indeed, consider the family \( \mathcal{A} \) of the events of the form \( \{ \forall i \leq n, \ X_i \leq s_i \} \), \( s_i \in \mathbb{R} \) and the family \( \mathcal{B} \) of the events of the form \( \{ \forall n < i < n + m, X_i \leq t_i \} \), \( m \geq 1, t_i \in \mathbb{R} \). These are \( \pi \)-systems which generate \( X_n \) and \( T_n \) respectively. Clearly \( \mathcal{A} \) and \( \mathcal{B} \) are independent, hence \( X_n \) and \( T_n \) are independent (Lemma 3.1).

**Claim 2.** For every \( n \), \( X_n \) and \( \mathcal{T} \) are independent.
This follows instantly because \( \mathcal{T} \subseteq T_n \).

**Claim 3.** Let \( \mathcal{X} = \sigma(X_1, X_2, \ldots) \). Then \( \mathcal{X} \) and \( T_n \) are independent.
Let \( \mathcal{A} = \bigcup_{n=1}^{\infty} X_n \). This is a \( \pi \)-system generating \( \mathcal{X} \). By Claim 2, \( \mathcal{A} \) and \( \mathcal{T} \) are independent, so \( \mathcal{X} \) and \( \mathcal{T} \) are independent (Lemma 3.1).

**Claim 4.** For every \( A \in \mathcal{T} \), \( \mathbb{P}(A) \in \{0, 1\} \).
Since \( \mathcal{T} \subseteq \mathcal{X} \), by Claim 3, \( \mathcal{T} \) is independent of \( \mathcal{T} \), thus
\[ \mathbb{P}(A) = \mathbb{P}(A \cap \mathcal{A}) = \mathbb{P}(A) \mathbb{P}(\mathcal{A}), \]
hence the result. \( \square \)
3.6 Exercises

1. Define the Rademacher functions $r_1, r_2, \ldots : [0, 1] \to \{-1, 0, 1\}$ by
   
   $$r_n(x) = \text{sgn} \left( \cos(2^n \pi x) \right), \quad x \in [0, 1],$$
   
   where sgn is the usual signum function. Consider these functions as random variables on the probability space $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$. What is the distribution of $r_n$? Show that the family $\{r_n\}_{n \geq 1}$ is independent.

2. Define the Walsh functions $w_A : \{-1, 1\}^n \to \{-1, 1\}$ indexed by all subsets $A$ of $\{1, \ldots, n\}$,
   
   $$w_A(x_1, \ldots, x_n) = \prod_{i \in A} x_i, \quad x = (x_1, \ldots, x_n) \in \{-1, 1\}^n$$
   
   and $w_{\emptyset} = 1$ (a constant function). Consider these functions as random variables on $\{-1, 1\}^n$ equipped with the uniform probability measure. What is the distribution of $w_A$? Show that the $w_A$ are pairwise independent. Are they independent?

3. For independent events $A_1, \ldots, A_n$,
   
   $$\left(1 - e^{-1}\right) \min \left\{ 1, \sum_{i=1}^n \mathbb{P}(A_i) \right\} \leq \mathbb{P} \left( \bigcup_{i=1}^n A_i \right) \leq \min \left\{ 1, \sum_{i=1}^n \mathbb{P}(A_i) \right\}.$$

4. Prove the so-called infinite monkey theorem: when we toss a fair coin infinitely many times then the event that “every given finite sequence of heads/tails occurs infinitely many times” is certain.

5. Suppose events $A_1, A_2, \ldots$ are independent and all have equal probabilities. What is the probability that infinitely many $A_i$’s occur?

6. Suppose events $A_1, A_2, \ldots$ are independent and $\mathbb{P}(A_n) \in (0, 1)$ for every $n$. Then infinitely many $A_n$ occur with probability 1 if and only if at least one $A_n$ occurs with probability 1.

7. Let $\Omega$ be the set of positive integers and let $A_k$ be the set of positive integers divisible by $k$, $k \geq 1$. Is there a probability measure $\mathbb{P}$ defined on all the subsets of $\Omega$ such that $\mathbb{P}(A_k) = \frac{1}{k}$ for every $k = 1, 2, \ldots$?

8. Prove Theorem 3.6.

9. Fill out the details in Example 3.10.

10. Let $X_1, X_2, \ldots$ be a sequence of independent random variables and let $\mathcal{T}$ be its tail $\sigma$-algebra. If a random variable $Y$ is $\mathcal{T}$-measurable, then $Y$ is a.s. constant.
11. Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} X_n z^n$ is a.s. constant.

12. Are there two nonconstant continuous functions $f, g: [0, 1] \to \mathbb{R}$ which, when viewed as random variables on the probability space $([0, 1], \mathcal{B}([0, 1]), \text{Leb})$, are independent?
4 Expectation

4.1 Definitions and basic properties

Let $X$ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$. We say that $X$ is integrable, if

$$\int_\Omega |X(\omega)| dP(\omega) < \infty$$

and then define its expectation (also called its mean) as

$$E X = \int_\Omega X(\omega) dP(\omega).$$

Both integrals are Lebesgue integrals (see Appendix E for a construction and basic properties). For a random vector $X$ in $\mathbb{R}^n$, its expectation is defined as the following vector in $\mathbb{R}^n$,

$$EX = \begin{bmatrix} E X_1 \\ \vdots \\ E X_n \end{bmatrix}.$$

We list the most important basic properties

(i) monotonicity: if $X$ is a nonnegative random variable, then $EX \geq 0$,

(ii) the triangle inequality: $|EX| \leq E|X|$, 

(iii) linearity: if $X, Y$ are integrable, then for every $a, b \in \mathbb{R}$, $aX + bY$ is integrable and

$$E(ax + bY) = aEX + bEY.$$

We also list the most important limit theorems.

4.1 Theorem (Lebesgue’s monotone convergence theorem). If $X_1, X_2, \ldots$ are nonnegative random variables such that for every $n$, $X_{n+1} \geq X_n$, then

$$E \left( \lim_{n \to \infty} X_n \right) = \lim_{n \to \infty} EX_n$$

(with the provision that the left hand side is $+\infty$ if and only if the right hand side is $+\infty$)

4.2 Theorem (Fatou’s lemma). If $X_1, X_2, \ldots$ are nonnegative random variables, then

$$E \left( \liminf_{n \to \infty} X_n \right) \leq \liminf_{n \to \infty} EX_n.$$

4.3 Theorem (Lebesgue’s dominated convergence theorem). If $X, X_1, X_2, \ldots$ are random variables such that $X_n \xrightarrow{n \to \infty} X$ a.s. and for every $n$, $|X_n| \leq Y$ for some integrable random variable $Y$, then

$$E|X_n - X| \xrightarrow{n \to \infty} 0.$$

In particular,

$$EX = E \left( \lim_{n \to \infty} X_n \right) = \lim_{n \to \infty} EX_n.$$
The proofs are in Appendix E.

We turn to the relation between the expectation of a random variable and the integral against its law.

4.4 Theorem. Let \( h: \mathbb{R} \to \mathbb{R} \) be a Borel measurable function. Let \( X \) be a random variable. Then
\[
Eh(X) = \int_{\mathbb{R}} h(x) d\mu_X(x).
\]
(The identity should be understood as follows: if the integral on one side exists, then the other one does and they are equal.)

Proof. We leverage the linearity of both sides in \( h \) and use a standard method from measure theory of complicating \( h \).

I. If \( h = 1_A \), for some \( A \in \mathcal{B}(\mathbb{R}) \), then
\[
Eh(X) = \int_{\Omega} 1_A(\omega) dP(\omega) = P(A) = \mu_X(A) = \int_{\mathbb{R}} 1_A(x) d\mu_X(x) = \int_{\mathbb{R}} h(x) d\mu_X(x).
\]

II. If \( h \) is a simple function, that is \( h = \sum_{i=1}^{N} x_i 1_{A_i} \) for some \( x_1, \ldots, x_N \in \mathbb{R} \) and \( A_1, \ldots, A_N \in \mathcal{B}(\mathbb{R}) \), then the identity follows from the previous step by linearity.

III. If \( h \) is a nonnegative function, then there is a sequence of nonnegative simple functions \( h_1, h_2, \ldots \) such that for every \( n \), \( h_{n+1} \geq h_n \) and \( h_n \to h \) (pointwise). Thus, the identity follows in this case from the previous step by Lebesgue’s monotone convergence theorem.

IV. If \( h \) is arbitrary, we decompose it into its positive and negative part, \( h^+ = \max\{0, h\} \), \( h^- = \max\{0, -h\} \),
\[
h = h^+ - h^-\]
and the identity follows from the previous step by linearity and the definition of Lebesgue integral. \( \square \)

4.5 Remark. Note that the identity we proved is linear in \( h \). The above argument of gradually complicating \( h \) is standard is such situations.

4.6 Corollary. If \( X \) is a discrete random variable with \( p_i = P(X = x_i) > 0 \), \( \sum_i p_i = 1 \), then since \( \mu_X = \sum_i p_i \delta_{x_i} \), we get
\[
Eh(X) = \sum_i p_i h(x_i).
\]
If \( X \) is a continuous random variable with density \( f \), then since
\[
\int_{\mathbb{R}} h(x) d\mu_X(x) = \int_{\mathbb{R}} h(x) f(x) dx
\]
(which can be justified exactly as in the proof of Theorem 4.4), we get
\[
Eh(X) = \int_{\mathbb{R}} h(x) f(x) dx.
\]
4.2 Variance and covariance

For a random variable $X$ with $E X^2 < \infty$ (as we say, square-integrable), we define its variance as

$$\text{Var}(X) = E(X - EX)^2.$$ 

Since $(X - EX)^2 = X^2 - 2(EX)X + (EX)^2$, we have the convenient formula,

$$\text{Var}(X) = EX^2 - (EX)^2.$$ 

Note that by its definition, the variance is shift invariant,

$$\text{Var}(X + c) = \text{Var}(X),$$

for every constant $c \in \mathbb{R}$, and scales quadratically,

$$\text{Var}(\lambda X) = \lambda^2 \text{Var}(X),$$

for every constant $\lambda \in \mathbb{R}$. Moreover, if $X$ and $Y$ are random variables with $EX^2, EY^2 < \infty$, then because $(X + Y)^2 \leq 2X^2 + 2Y^2$, we have $E(X + Y)^2 < \infty$ and denoting $\bar{X} = X - EX, \bar{Y} = Y - EY$, we obtain

$$\text{Var}(X + Y) = E(\bar{X} + \bar{Y})^2 = E\bar{X}^2 + E\bar{Y}^2 + 2E\bar{X}\bar{Y} = \text{Var}(X) + \text{Var}(Y) + 2E\bar{X}\bar{Y}.$$ 

This motives the following definition of the covariance between such two random variables,

$$\text{Cov}(X, Y) = E((X - EX)(Y - EY)) = EXY - (EX)(EY).$$

By the above identity we also obtain the following formula for the variance of the sum.

4.7 Theorem. Let $X_1, \ldots, X_n$ be square-integrable random variables. Then

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j).$$

In particular, if the $X_i$ are uncorrelated, that is $\text{Cov}(X_i, X_j) = 0$ for all $i \neq j$, we have

$$\text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i).$$

For a random vector $X$ in $\mathbb{R}^n$ with square-integrable components, we define its covariance matrix as

$$\text{Cov}(X) = [\text{Cov}(X_i, X_j)]_{i,j \leq n}.$$ 

It is convenient to write

$$\text{Cov}(X) = \bar{X}\bar{X}^\top,$$

with $\bar{X} = X - EX$ (here the expectation of the $n \times n$ matrix $\bar{X}\bar{X}^\top$ is understood entry-wise). From this and the linearity of expectation, we quickly obtain the following basic properties of covariance matrices.
### 4.8 Theorem

Let $X$ be a random vector in $\mathbb{R}^n$ with square-integrable components. Then

(i) $\text{Cov}(X)$ is a symmetric positive semi-definite matrix

(ii) $\text{Cov}(X + b) = \text{Cov}(X)$, for every (deterministic) vector $b \in \mathbb{R}^n$,

(iii) $\text{Cov}(AX) = A \text{Cov}(X) A^\top$, for every $n \times n$ matrix $A$,

(iv) if $r = \text{rank}(\text{Cov}(X))$, then $\mathbb{P}(X \in H) = 1$ for some $r$-dimensional affine subspace of $\mathbb{R}^n$.

**Proof.** We show (iv) and leave the rest as an exercise. Let $M = \text{Cov}(X)$. If $M$ has rank $r$, then there are $n - r$ linearly independent vectors in its kernel, say $v_1, \ldots, v_{n-r}$. Since $M v_i = 0$, we have

$$0 = v_i^\top M v_i = \mathbb{E}(v_i^\top \bar{X} \bar{X}^\top v_i) = \mathbb{E}(\bar{X}^\top v_i)^2,$$

so the nonnegative random variable $(\bar{X}^\top v_i)^2$ whose expectation is 0 therefore has to be 0 a.s. This holds for every $i$, thus $\mathbb{P}(\forall i \leq n - r \bar{X}^\top v_i = 0) = 1$ and we can take $H = \{x \in \mathbb{R}^n, \forall i \leq n - r (x - \mathbb{E}X)^\top v_i = 0\}$. \qed

### 4.3 Independence again, via product measures

Given two probability spaces $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i = 1, 2$, we define their product by taking, of course,

$$\Omega = \Omega_1 \times \Omega_2$$

and

$$\mathcal{F} = \sigma(A_1 \times A_2, A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2)$$

which is called the **product $\sigma$-algebra**, denoted

$$\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2.$$ 

Then the **product measure** $\mathbb{P}$, denoted

$$\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2,$$

is the unique probability measure on $\mathcal{F}$ such that for all $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$,

$$\mathbb{P}(A_1 \times A_2) = \mathbb{P}_1(A_1) \mathbb{P}_2(A_2).$$

Its existence is related to Fubini’s theorem (see Appendix C). It plainly generalises to finite products.
4.9 Example. Thanks to separability, we have

\[ B(\mathbb{R}^n) = B(\mathbb{R}) \otimes \cdots \otimes B(\mathbb{R}) \]

(with the right hand side usually denoted \( B(\mathbb{R})^\otimes n \)). One inclusion relies only on the definition of the product topology, that is

\[ B(\mathbb{R})^\otimes n \subset B(\mathbb{R}^n) \]

holds because if \( A_1, \ldots, A_n \) are open, then \( A_1 \times \cdots \times A_n \) is open, so the generators of the left hand side belong to \( B(\mathbb{R}^n) \). The opposite inclusion,

\[ B(\mathbb{R}^n) \subset B(\mathbb{R})^\otimes n \]

holds because an open set in \( \mathbb{R}^n \) is a countable union of the sets of the form \( \prod_{i=1}^{n} (a_i, b_i) \), by separability, thus the generators of the left hand side belong to \( B(\mathbb{R})^\otimes n \).

For infinite products, we have the following result, which also gives a canonical construction of an infinite sequence of i.i.d. random variables with specified arbitrary laws.

4.10 Theorem. Let \( \mu_1, \mu_2, \ldots \) be probability measures on \((\mathbb{R}, B(\mathbb{R}))\). We set

\[ \Omega = \prod_{i=1}^{\infty} \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \ldots, \]

\[ X_n(\omega_1, \omega_2, \ldots) = \omega_n, \quad (\omega_1, \omega_2, \ldots) \in \Omega, \]

and

\[ \mathcal{F} = \sigma(X_1, X_2, \ldots). \]

There is a unique probability measure \( P \) on \((\Omega, \mathcal{F})\) such that for every \( k \geq 1 \) and \( A_1, \ldots, A_k \in B(\mathbb{R}) \), we have

\[ P(A_1 \times \cdots \times A_k \times \mathbb{R} \times \ldots) = \mu_1(A_1) \cdots \mu_k(A_k). \]

Moreover, \( X_1, X_2, \ldots \) are independent random variables on \((\Omega, \mathcal{F}, P)\) with \( \mu_{X_i} = \mu_i \).

We defer its proof to Appendix D. It is based on Carathéodory’s theorem.

Recall that random variables \( X_1, \ldots, X_n \) are independent if and only if its joint law \( \mu_{(X_1, \ldots, X_n)} \) is the product measure \( \mu_{X_1} \otimes \cdots \otimes \mu_{X_n} \) (Theorem 3.4). Using this, we prove one of the most significant consequences of independence: the expectation of the product is the product of the expectations.

4.11 Theorem. Let \( X_1, \ldots, X_n \) be integrable random variables. If they are independent, then \( X_1 \cdot \cdots \cdot X_n \) is integrable and

\[ \mathbb{E}(X_1 \cdot \cdots \cdot X_n) = \mathbb{E}X_1 \cdot \cdots \cdot \mathbb{E}X_n. \]
Proof. We have,

$$E|X_1 \cdot \ldots \cdot X_n| = \int_{\mathbb{R}^n} |x_1 \cdot \ldots \cdot x_n| d\mu_{(X_1,\ldots,X_n)}(x_1,\ldots,x_n)$$

$$= \int_{\mathbb{R}^n} |x_1 \cdot \ldots \cdot x_n| d\mu_{X_1}(x_1) \ldots d\mu_{X_n}(x_n)$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} |x_i| d\mu_{X_i}(x_i),$$

where in the second equality we use independence and in the last one – Fubini’s theorem. This shows that $X_1 \cdot \ldots \cdot X_n$ is integrable. The proof of the identity then follows exactly the same lines.

Of course, the converse statement is not true. Take for instance a uniform random variable $X$ on $\{-1,0,1\}$ and $Y = |X|$. Then $E(XY) = 0 = EX \cdot EY$, but $X$ and $Y$ are not independent.

As a useful corollary, independent random variables are uncorrelated, so we also have that the variance of the sum of independent random variables is the sum of their variances (recall Theorem 4.7).

4.12 Corollary. If $X_1, X_2$ are independent, then $\text{Cov}(X_1, X_2) = 0$. In particular, if $X_1, \ldots, X_n$ are independent square-integrable, then

$$\text{Var} \left( \sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i).$$

Since $F_X$ determines the law of $X$, it should be possible to express $EX$ using it. We finish this chapter with such a formula which is obtained from a simple trick that $x = \int_0^x dt, \ x \geq 0$, combined with Fubini’s theorem.

4.13 Theorem. If $X$ is a nonnegative random variable, then

$$EX = \int_0^\infty P(X > t) \, dt.$$

Proof. We have,

$$EX = E \left( \int_0^X dt \right) = E \int_0^\infty 1_{X > t} \, dt = \int_0^\infty E 1_{X > t} \, dt = \int_0^\infty P(X > t) \, dt,$$

where the usage of Fubini’s theorem is justified because $1_{X > t}$ is a nonnegative function.
4.4 Exercises

1. An urn contains $N$ balls among which exactly $b$ are yellow. We pick uniformly at random $n$ ($n \leq N$) balls without replacement. Let $X$ be the number of yellow balls picked. Find the expectation and variance of $X$.

2. Show that a nonnegative random variable $X$ is integrable if and only if

\[ \sum_{n=1}^{\infty} P(X > n) < \infty. \]

3. Let $p > 0$. If $X$ is a nonnegative random variable, then

\[ \mathbb{E}X^p = p \int_{0}^{\infty} t^{p-1} P(X > t) \, dt. \]

Give an analogous formula for $\mathbb{E}f(X)$ for an arbitrary increasing and differentiable function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$.

4. Let $p > 0$ and $X$ be a random variable with $\mathbb{E}|X|^p < \infty$. Then

\[ \lim_{t \to \infty} t^p P(|X| > t) = 0. \]

5. Let $X$ be a random variable satisfying $\lim_{t \to \infty} t^p P(|X| > t) = 0$. Show that for every $0 < \delta < 1$, we have $\mathbb{E}|X|^{1-\delta} < \infty$. Give an example of such a random variable for which $\mathbb{E}|X| = +\infty$.

6. Suppose $X$ and $Y$ are independent random variables and the distribution function of $X$ is continuous. Then $P(X = Y) = 0$.

7. Let $X$ and $Y$ be independent random variables taking values in $S = \{z \in \mathbb{C}, |z| = 1\}$. If $X$ is uniform, then $XY$ is also uniform.

8. Suppose $X$ and $Y$ are positive random variables with the same distribution. Does it follow that $\mathbb{E}\frac{X}{X+Y} = \mathbb{E}\frac{Y}{X+Y}$?

9. Let $X$ and $Y$ be bounded random variables. Show that $X$ and $Y$ are independent if and only if for every positive integers $m, n$, we have $\mathbb{E}(X^mY^n) = \mathbb{E}X^m\mathbb{E}Y^n$.

10. Let $X$ be a square-integrable random variable. Find $\min_{x \in \mathbb{R}} \mathbb{E}(X - x)^2$.

11. Let $X$ be an integrable random variable. Show that $\min_{x \in \mathbb{R}} \mathbb{E}|X - x|$ is attained at $x = \text{Med}(X)$, the median of $X$, that is any number $m$ for which $P(X \geq m) \geq \frac{1}{2}$ and $P(X \leq m) \geq \frac{1}{2}$.

12. Let $X$ be a square-integrable random variable. We have, $|\mathbb{E}X - \text{Med}(X)| \leq \sqrt{\text{Var}(X)}$.

13. Prove properties (i)-(iii) of covariance matrices from Theorem 4.8.
14. Suppose there is a countable family of disjoint open disks with radii $r_1, r_2, \ldots$, all contained in the unit square $[0, 1]^2$ on the plane. If the family covers $[0, 1]^2$ up to a set of (Lebesgue) measure 0, then $\sum_i r_i = \infty$. 


5 More on random variables

5.1 Important distributions

We list several discrete and continuous laws of random variables that appear very often in probability theory.

1) The Dirac delta distribution. For $a \in \mathbb{R}$, let $X$ be an a.s. constant random variable,

$$\mathbb{P}(X = a) = 1.$$ 

Then

$$\mu_X = \delta_a$$

is the Dirac delta distribution at $a$. We have,

$$\mathbb{E}X = a, \quad \text{Var}(X) = 0.$$ 

2) The Bernoulli distribution. For $p \in [0, 1]$, let $X$ be a random variable taking two values $0$ and $1$ with probabilities

$$\mathbb{P}(X = 1) = p, \quad \mathbb{P}(X = 0) = 1 - p.$$ 

Then

$$\mu_X = (1 - p)\delta_0 + p\delta_1$$

is the Bernoulli distribution with parameter $p$. We have,

$$\mathbb{E}X = p, \quad \text{Var}(X) = p(1 - p).$$

Notation: $X \sim \text{Ber}(p)$.

3) The binomial distribution. For an integer $n \geq 1$ and $p \in [0, 1]$, let $X$ be a random variable taking values $\{0, 1, \ldots, n\}$ with probabilities

$$\mathbb{P}(X = k) = \binom{n}{k}p^k(1 - p)^{n-k}, \quad 0 \leq k \leq n.$$ 

Then

$$\mu_X = \sum_{k=0}^{n} \binom{n}{k}p^k(1 - p)^{n-k}\delta_k$$

is the Binomial distribution with parameters $n$ and $p$. It can be directly checked that $X$ has the same law as $X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables with parameter $p$, which gives a very convenient probabilistic representation of $X$. In other words,

$$X$$

is the number of successes in $n$ independent Bernoulli trials.
We have,
\[ EX = E(X_1 + \ldots + X_n) = nEX_1 = np \]
and
\[ \text{Var}(X) = n \text{Var}(X_1) = np(1-p). \]
Notation: \( X \sim \text{Bin}(n,p) \).

4) **The Poisson distribution.** For \( \lambda > 0 \), let \( X \) be a random variable taking values \( \{0,1,2,\ldots\} \) with probabilities
\[ P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0. \]

Then
\[ \mu_X = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \delta_k \]
is the Poisson distribution with parameter \( \lambda \). We will see later that this distribution arises as an appropriate limit of the Binomial distribution with parameters \( n \) and \( \lambda/n \) as \( n \to \infty \). In other words, \( X \) is “the number of successes in \( n \) independent Bernoulli trials each with probability of success \( \frac{\lambda}{n} \)” as \( n \to \infty \), so that the *rate* of success is \( \lambda \). This distribution models well the number of events occurring in a fixed interval of time if these events occur with a constant mean rate \( \lambda \), independently of the time since the last event, say the number of calls in a busy call centre.

We have,
\[ E X = \lambda, \quad \text{Var}(X) = \lambda. \]
Notation: \( X \sim \text{Poiss}(\lambda) \).

5) **The geometric distribution.** For \( p \in [0,1] \), let \( X \) be a random variable taking values \( \{1,2,\ldots\} \) with probabilities
\[ P(X = k) = (1-p)^{k-1}p, \quad k \geq 1. \]

Then
\[ \mu_X = \sum_{k=1}^{\infty} (1-p)^{k-1}p \delta_k \]
is the Geometric distribution with parameter \( p \). It can be directly checked that
\( X \) has the same law as \( \inf\{n \geq 1, X_n = 1\} \),
where \( X_1, X_2, \ldots \) are i.i.d. Bernoulli random variables with parameter \( p \). In other words,
\( X \) is the number of trials in independent Bernoulli trials until first success.

We have,
\[ E X = \frac{1}{p}, \quad \text{Var}(X) = \frac{1-p}{p^2}. \]
Notation: \( X \sim \text{Geom}(p) \).
6) **The uniform distribution.** For a Borel set \( K \) in \( \mathbb{R}^n \) of positive finite Lebesgue measure (volume) \(|K|\), let \( X \) be a random variable with density function

\[
f(x) = \frac{1}{|K|} \mathbf{1}_K(x), \quad x \in \mathbb{R}^n.
\]

Then

\[
\mu_X(A) = \int_A f(x) \, dx = \frac{|A \cap K|}{|K|}, \quad A \in \mathcal{B}(\mathbb{R}^n).
\]

We say that \( \mu_X \) is the uniform measure on \( K \). We have,

\[
\mathbb{E}X = \frac{1}{|K|} \int_K x dx \quad \text{(the barycentre of } K)\.
\]

Notation: \( X \sim \text{Unif}(K) \).

In particular, if \( K = [0, 1] \) in \( \mathbb{R} \), \( X \) is uniform on the unit interval \([0, 1]\) and we have

\[
\mathbb{E}X = \frac{1}{2}, \quad \text{Var}(X) = \frac{1}{12}.
\]

7) **The exponential distribution.** For \( \lambda > 0 \), let \( X \) be a random variable with density function

\[
f(x) = \lambda e^{-\lambda x} \mathbf{1}_{(0, \infty)}(x), \quad x \in \mathbb{R}.
\]

We say that \( \mu_X \) (or \( X \)) has the exponential distribution with parameter \( \lambda \). This is a continuous analogue of the geometric distribution. It has the so-called memory-less property: for every \( s, t > 0 \),

\[
\mathbb{P}(X > s + t | X > s) = \mathbb{P}(X > t)
\]

which characterises it uniquely among continuous distributions (see exercises). We have,

\[
\mathbb{E}X = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.
\]

Notation: \( X \sim \text{Exp}(\lambda) \).

8) **The gamma distribution.** For \( \beta, \lambda > 0 \), let \( X \) be a random variable with density function

\[
f(x) = \frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x} \mathbf{1}_{(0, \infty)}(x), \quad x \in \mathbb{R},
\]

where

\[
\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} \, dt,
\]

is the Gamma function. We say that \( \mu_X \) (or \( X \)) has the Gamma distribution with parameters \( \beta \) and \( \lambda \). When \( \beta = n \) is a positive integer, we have a nice probabilistic representation,

\[
X \text{ has the same law as } X_1 + \cdots + X_n.
\]
where $X_1, \ldots, X_n$ are i.i.d. exponential random variables with parameter $\lambda$. We have,

$$
\mathbb{E}X = \frac{\beta}{\lambda}, \quad \text{Var}(X) = \frac{\beta}{\lambda^2}.
$$

Notation: $X \sim \text{Gamma}(\beta, \lambda)$.

9) The beta distribution. For $\alpha, \beta > 0$, let $X$ be a random variable with density function

$$
f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}1_{(0,1)}(x), \quad x \in \mathbb{R},
$$

where

$$
B(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
$$

is the Beta function. We say that $\mu_X$ (or $X$) has the Beta distribution with parameters $\alpha, \beta$. This distribution appears naturally as a marginal of a random vector uniform on the centred unit Euclidean ball. We have,

$$
\mathbb{E}X = \frac{\alpha}{\alpha + \beta}, \quad \text{Var}(X) = \frac{\alpha \beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}.
$$

Notation: $X \sim \text{Beta}(\alpha, \beta)$.

10) The Cauchy distribution. Let $X$ be a random variable with density function

$$
f(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.
$$

We say that $\mu_X$ (or $X$) has the standard Cauchy distribution. It has the following stability property: for every $a_1, \ldots, a_n \in \mathbb{R},$

$$
a_1X_1 + \cdots + a_nX_n
$$

has the same law as $(\sum |a_i|)X$,

where $X_1, \ldots, X_n$ are i.i.d. copies of $X$. Cauchy random variables are not integrable.

Notation: $X \sim \text{Cauchy}(1)$.

11) The Gaussian distribution. Let $X$ be a random variable with density function

$$
f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \in \mathbb{R}.
$$

We say that $\mu_X$ (or $X$) has the standard Gaussian (or normal) distribution. We have,

$$
\mathbb{E}X = 0, \quad \text{Var}(X) = 1.
$$

Notation: $X \sim N(0,1)$.

For $\mu \in \mathbb{R}$ and $\sigma > 0$ consider

$$
Y = \mu + \sigma X.
$$

This a Gaussian random variable with parameters $\mu$ and $\sigma$. It has density

$$
g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}.
$$
We have, 
\[ EY = \mu, \quad \text{Var}(Y) = \sigma^2. \]

Notation: \( Y \sim N(\mu, \sigma^2) \).

The key property of the Gaussian distribution is that \textit{sums of independent Gaussians are Gaussian}. Formally, let \( Y_1 \sim N(\mu_1, \sigma_1^2), Y_2 \sim N(\mu_2, \sigma_2^2) \) be two independent Gaussian random variables. Then,

\[ Y_1 + Y_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2). \quad (5.1) \]

We prove this later. Because of the central limit theorem, Gaussian random variables are ubiquitous.

5.2 Gaussian vectors

Let \( X_1, \ldots, X_n \) be i.i.d. standard Gaussian random variables. The vector 
\[ X = (X_1, \ldots, X_n) \]
is called a standard Gaussian random vector in \( \mathbb{R}^n \). It has density 
\[ f(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} = (2\pi)^{-n/2} e^{-|x|^2/2}, \quad x \in \mathbb{R}^n \]
(here \( |x| = \sqrt{\sum x_i^2} \) is the Euclidean norm of \( x \)). Note that \( X \) enjoys at the same time two important features: 1) \( X \) has independent components (its law is a product measure), 2) because the density of \( X \) is rotationally invariant, so is \( X \), that is for every orthonormal matrix \( U \in O(n) \),
\[ UX \text{ has the same law as } X. \]

We have, 
\[ \mathbb{E}X = 0 \quad (\in \mathbb{R}^n), \quad \text{Cov}(X) = \text{Id}_{n \times n}. \]

Notation: \( X \sim N(0, I_{n \times n}) \).

We say that a random vector \( Y \) in \( \mathbb{R}^m \) is \textbf{Gaussian}, if
\[ Y \text{ has the same law as } AX + b, \]
for some \( m \times n \) matrix \( A \) and vector \( b \in \mathbb{R}^m \), where \( X \sim N(0, I_{n \times n}) \). In other words, Gaussian vectors are defined as affine images of standard Gaussian vectors. We have,
\[ \mathbb{E}Y = b, \quad Q = \text{Cov}(Y) = AA^\top. \]

Notation: \( Y \sim N(b, Q) \).
In particular, if $m = n$ and $A$ is nonsingular, then $Y$ has density

$$g(x) = \frac{1}{\sqrt{2\pi}^{\frac{n}{2}}} e^{-\frac{1}{2}(Q^{-1}(x-b),(x-b))}, \quad x \in \mathbb{R}^n$$

where

$$(x,y) = \sum_{i=1}^{n} x_i y_i$$

is the standard scalar product on $\mathbb{R}^n$.

All of the claims made here are standard but very important computations and we leave the details as exercise.

5.3 Sums of independent random variables

Recall that the convolution of two integrable functions $f, g: \mathbb{R} \to \mathbb{R}$ is defined as a function

$$x \mapsto (f \ast g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$$

which by Fubini’s theorem is well-defined because $w(x,y) = f(x)g(y)$ is integrable on $\mathbb{R}^2$, so $w(x-y,y)$ is also integrable on $\mathbb{R}^2$.

Convolutions appear naturally when we take sums of independent random variables.

5.1 Theorem. Let $X$ and $Y$ be independent random variables. Then the law of $X + Y$ is given by

$$\mu_{X+Y}(A) = \int_A \mu_Y(A-x) d\mu_X(x) = \int_A \mu_X(A-y) d\mu_Y(y), \quad A \in \mathcal{B}(\mathbb{R}).$$

In particular, if $X$ has density $f$, then $X + Y$ has density

$$h(x) = \int_{\mathbb{R}} f(x-y) d\mu_Y(y).$$

If both $X, Y$ have densities, say $f, g$ respectively, then

$X + Y$ has density $f \ast g$.

Proof. By independence, $\mu_{(X,Y)} = \mu_X \otimes \mu_Y$, thus

$$\mu_{X+Y}(A) = \mu_{(X,Y)} \{ (x,y) \in \mathbb{R}^2, \ x + y \in A \} = \int_{(x,y) \in \mathbb{R}^2, \ x + y \in A} d\mu_X(x) d\mu_Y(y)$$

$$= \int_x \left[ \int_{y \in A-x} d\mu_Y(y) \right] d\mu_X(x),$$

where the last equality follows by Fubini’s theorem. Since $\int_{y \in A-x} d\mu_Y(y) = \mu_Y(A-x)$, the first identity follows. Note that swapping the roles of $X$ and $Y$ above gives also the identity

$$\mu_{X+Y}(A) = \int_{\mathbb{R}} \mu_X(A-y) d\mu(y).$$
If \( X \) has density \( f \), we have \( \mu_X(A - y) = \int_{A-y} f(x)dx \), so by a change of variables \( x = z - y \) and Fubini’s theorem, we get

\[
\mu_{X+Y}(A) = \int_{\mathbb{R}} \int_{A-y} f(x)d\mu_Y(y)
\]

\[
= \int_{\mathbb{R}} \int_{A} f(z-y)dzd\mu_Y(y)
\]

\[
= \int_{A} \left[ \int_{\mathbb{R}} f(z-y)d\mu_Y(y) \right] dz,
\]

so \( h(z) = \int_{\mathbb{R}} f(z-y)d\mu_Y(y) = \mathbb{E}f(z - Y) \) is the density of \( X + Y \). Finally, if \( Y \) also has density, say \( g \), then this becomes \( h(z) = \int_{\mathbb{R}} f(z-y)g(y)dy \), that is \( h = f * g \).

Sometimes we use the notation \( \mu_X * \mu_Y \) to denote \( \mu_{X+Y} \). To illustrate this theorem, we consider the example of sums of independent Gaussians.

**5.2 Example.** Let \( X \sim N(0,1) \), \( Y \sim N(0,\sigma^2) \) be independent. The densities of \( X \) and \( Y \) are respectively

\[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},
\]

\[
g(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2}}.
\]

Thus the density of \( X + Y \) is given by

\[
h(z) = \int_{\mathbb{R}} f(z-x)g(x)dx = \frac{1}{2\pi\sigma} \int_{\mathbb{R}} e^{-\frac{1}{2}(z-x)^2 - \frac{1}{2\sigma^2}x^2} dx
\]

\[
= \frac{1}{2\pi\sigma} \int_{\mathbb{R}} e^{-\frac{1}{2}(\frac{1}{1+\sigma^2} u^2 + \frac{1}{\sigma^2})} e^{-\frac{1}{2\sigma^2}u^2} \cdot \sqrt{\frac{\sigma^2}{1+\sigma^2}} du
\]

\[
= \frac{1}{\sqrt{2\pi\sqrt{1+\sigma^2}}} e^{-\frac{1}{2} \frac{1}{1+\sigma^2} z^2},
\]

that is

\[
X + Y \sim N(0, 1 + \sigma^2).
\]

Using this, linearity and the fact that for \( Y \sim N(\mu,\sigma^2) \) we can write \( Y = \mu + \sigma X \) for a standard Gaussian \( X \), we can easily deduce (5.1).

**5.4 Density**

Recall that a random variable \( X \) has density \( f \) if for every \( t \in \mathbb{R} \),

\[
F_X(t) = \int_{-\infty}^{t} f(x)dx.
\]

How to find out whether \( X \) has density and if that is the case, determine it using its distribution function \( F_X \)?

**5.3 Lemma.** Let \( F : \mathbb{R} \to \mathbb{R} \) be a nondecreasing, right-continuous function such that \( F' \) exists a.e. Then for every \( a < b \), we have

\[
\int_{a}^{b} F' \leq F(b) - F(a).
\]

47
Proof. By Fatou’s lemma,
\[
\int_a^b F'(t)\,dt = \int_a^b \liminf_{\delta \to 0^+} \frac{F(t + \delta) - F(t)}{\delta} \,dt 
\leq \liminf_{\delta \to 0^+} \int_a^b \frac{F(t + \delta) - F(t)}{\delta} \,dt
\]
\[
= \liminf_{\delta \to 0^+} \frac{1}{\delta} \left( \int_b^{b+\delta} F(t)\,dt - \int_a^{a+\delta} F(t)\,dt \right)
\]
and the right hand side equals \( F(b) - F(a) \) by the right-continuity of \( F \).

5.4 Corollary. Under the assumptions of Lemma 5.3, if additionally \( \lim_{t \to -\infty} F(t) = 0 \) and \( \lim_{t \to +\infty} F(t) = 1 \), then for every \( x \in \mathbb{R} \), we have
\[
\int_{-\infty}^x F' \leq F(x) \quad \text{and} \quad \int_{x}^{\infty} F' \leq 1 - F(x).
\]

5.5 Theorem. If \( X \) is a random variable such that \( F'_X \) exists a.e. and \( \int_{-\infty}^{\infty} F'_X = 1 \), then \( X \) is continuous with density
\[
f(x) = \begin{cases} F'_X(x), & \text{if } F'_X(x) \text{ exists}, \\ 0, & \text{otherwise}. \end{cases}
\]

Proof. By Corollary 5.4, it remains to show that for every \( x \in \mathbb{R} \), we have \( \int_x^{\infty} F'_X \geq F(x) \).

This follows from
\[
\int_{-\infty}^x F'_X + \int_{x}^{\infty} F'_X = \int_{-\infty}^{\infty} F'_X = 1
\]
and \( \int_{x}^{\infty} F'_X \leq 1 - F_X(x) \). \( \square \)
5.5 Exercises

1. There are \( n \) different coupons and each time you obtain a coupon it is equally likely to be any of the \( n \) types. Let \( Y_i \) be the additional number of coupons collected, after obtaining \( i \) distinct types, before a new type is collected (including the new one). Show that \( Y_i \) has the geometric distribution with parameter \( \frac{n-i}{n} \) and find the expected number of coupons collected before you have a complete set.

2. The double exponential distribution with parameter \( \lambda > 0 \) has density \( f(x) = \frac{\lambda}{2} e^{-\lambda|x|} \). Find its distribution function, sketch its plot, find the mean and variance. Let \( X \) and \( Y \) be i.i.d. exponential random variables with parameter 1. Find the distribution of \( X - Y \).

3. Let \( X \) and \( Y \) be independent Poisson random variables with parameters \( \mu \) and \( \lambda \). Show that \( X + Y \) is a Poisson random variable with parameter \( \mu + \lambda \).

4. Let \( X \) be a uniform random variable on \((0, 1)\). Find the distribution function and density of \( Y = -\ln X \). What is the distribution of \( Y \) called?

5. Let \( X \) be a Poisson random variable with parameter \( \lambda \). Show that \( P(X \geq k) = P(Y \leq \lambda) \), for \( k = 1, 2, \ldots \), where \( Y \) is a random variable with the Gamma distribution with parameter \( k \).

6. Let \( X \) and \( Y \) be independent exponential random variables with parameters \( \lambda \) and \( \mu \). Show that \( \min\{X, Y\} \) has the exponential distribution with parameter \( \lambda + \mu \).

7. Let \( X_1, X_2, \ldots \) be independent exponential random variables with parameter 1. Show that for every \( n \), the distribution of \( X_1 + \cdots + X_n \) is Gamma(\( n \)). Generalise this to sums of independent random variables with Gamma distributions: if \( X_1, \ldots, X_n \) are independent with \( X_i \sim \Gamma(\beta_i) \), then \( \sum_{i=1}^{n} X_i \sim \Gamma(\sum_{i=1}^{n} \beta_i) \).

8. Let \((X, Y)\) be a random vector in \( \mathbb{R}^2 \) with density \( f(x, y) = cxy \mathbf{1}_{0 < x < y < 1} \). Find \( c \) and \( P(X + Y < 1) \). Are \( X \) and \( Y \) independent? Find the density of \((X/Y, Y)\). Are \( X/Y \) and \( Y \) independent?

9. Let \( X \) and \( Y \) be independent standard Gaussian random variables. Show that \( X/Y \) has the Cauchy distribution.

10. Let \( X = (X_1, \ldots, X_n) \) be a random vector in \( \mathbb{R}^n \) uniformly distributed on the simplex \( \{x \in \mathbb{R}^n, x_1 + \cdots + x_n \leq 1, x_1, \ldots, x_n \geq 0\} \). Find \( E X_1, E X_1^2, E X_1 X_2 \), the covariance matrix of \( X \) and its determinant.

11. Let \( U_1, \ldots, U_n \) be a sequence of i.i.d. random variables, each uniform on \([0, 1]\). Let \( U_1^*, \ldots, U_n^* \) be its nondecreasing rearrangement, that is \( U_1^* \leq \cdots \leq U_n^* \). In particular, \( U_1^* = \min\{U_1, \ldots, U_n\} \) and \( U_n^* = \max\{U_1, \ldots, U_n\} \). Show that the vector
$(U_1^*, \ldots, U_n^*)$ is uniform on the simplex \( \{ x \in \mathbb{R}^n, \ 0 \leq x_1 \leq \ldots \leq x_n \leq 1 \} \). Find \( \mathbb{E}U_k^* \) for \( 1 \leq k \leq n \).

12. Show the lack of memory property characterises the exponential distribution. Specifically, let \( X \) be a random variable such that for every positive \( s \) and \( t \), \( \mathbb{P}(X > s) > 0 \) and \( \mathbb{P}(X > s + t | X > s) = \mathbb{P}(X > t) \). Show that \( X \) has the exponential distribution.

13. Let \( X \) be a random variable such that there is a function \( g: \mathbb{R} \rightarrow \mathbb{R} \) such that \( F_X(t) = \int_{-\infty}^{t} g(x)dx \) for every \( t \in \mathbb{R} \). Then \( X \) is continuous and \( g \) is the density of \( X \).

14. Let \( U_1, U_2, U_3 \) be independent uniform random variables on \([-1, 1]\). Find the density of \( U_1 + U_2 \) and \( U_1 + U_2 + U_3 \).

15. Let \( X \) and \( Y \) be independent random variables with densities \( f \) and \( g \) respectively. Show that \( Z = X/Y \) has density \( h(z) = \int_{-\infty}^{\infty} \frac{g(y)}{|y|} f(yz)dy, \ z \in \mathbb{R} \).

16. Let \( X \) be a standard Gaussian random variable and \( Y \) be an exponential random variable with parameter 1, independent of \( X \). Show that \( \sqrt{2Y}X \) has the symmetric (two-sided) exponential distribution with parameter 1.

17. Let \( X_1, X_2, X_3 \) be i.i.d. standard Gaussian random variables. Find the mean and variance of \( Y = 3X_1 - X_2 + 2X_3 \). Find its density.

18. Show that a continuous Gaussian random vector in \( \mathbb{R}^n \) has independent components if and only if they are uncorrelated.

19. Give an example of a random vector \( (X,Y) \) such that \( X \) and \( Y \) are uncorrelated Gaussian random variables but \( X \) and \( Y \) are not independent.

20. Let \( (X,Y) \) be a standard Gaussian random vector in \( \mathbb{R}^2 \). Let \( \rho \in (-1, 1) \) and define
\[
[U \ V] = \begin{bmatrix}
\frac{\sqrt{1+\rho^2}}{\sqrt{1-\rho^2}} X + \frac{\sqrt{1-\rho^2}}{\sqrt{1+\rho^2}} Y \\
\frac{\sqrt{1-\rho^2}}{\sqrt{1+\rho^2}} Y + \frac{\sqrt{1+\rho^2}}{\sqrt{1-\rho^2}} X
\end{bmatrix}.
\]
Find the density of \( (U,V) \). Is this a Gaussian random vector? What is its covariance matrix? What is the distribution of \( U \) and \( V \)? Determine the values of \( \rho \) for which \( U \) and \( V \) are independent.

21. Let \( \rho \in (-1, 1) \) and let \( (U,V) \) be a random vector in \( \mathbb{R}^2 \) with density
\[
f(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2)\right\}, \quad (u,v) \in \mathbb{R}^2.
\]
Is it a Gaussian random vector? Find the covariance matrix of \( (U,V) \). Find the distributions of the marginals \( U \) and \( V \). Determine the values of \( \rho \) for which \( U \) and \( V \) are independent.
22. Suppose \((X, Y)\) is a centred (i.e., \(EX = EY = 0\)) Gaussian random vector in \(\mathbb{R}^2\) with 
\[
\text{Cov}([X, Y]) = [\begin{array}{cc}2 & 1 \\ 1 & 1 \end{array}].
\]
Find, a) the density of \((X, Y)\), b) the density of \(X + 3Y\), c) all \(\alpha \in \mathbb{R}\) for which \(X + Y\) and \(X + \alpha Y\) are independent.

23. Let \(G\) be a standard Gaussian vector in \(\mathbb{R}^n\) and let \(U\) be an \(n \times n\) orthogonal matrix. Find the density of \(UG\). Are the components of this vector independent?

24. Let \(g\) be a standard Gaussian random variable. Show that 
\[
\mathbb{E}g^{2m} = 1 \cdot 3 \cdot \ldots \cdot (2m - 1),
\]
\(m = 1, 2, \ldots\).

25. Using Fubini’s theorem and the fact that the standard Gaussian density integrates to 1, find the volume of a Euclidean ball in \(\mathbb{R}^n\) of radius 1. What is the radius of a Euclidean ball of volume 1? What is its asymptotics for large \(n\)?
6 Important inequalities and notions of convergence

6.1 Basic probabilistic inequalities

One of the simplest and very useful probabilistic inequalities is a tail bound by expectation: the so-called Chebyshev’s inequality.

6.1 Theorem (Chebyshev’s inequality). If \( X \) is a nonnegative random variable, then for every \( t > 0 \),

\[
P( X \geq t ) \leq \frac{1}{t} \mathbb{E} X.
\]

Proof. Since \( X \geq X \mathbf{1}_{\{X \geq t\}} \geq t \mathbf{1}_{\{X \geq t\}} \), taking the expectation yields

\[
\mathbb{E} X \geq \mathbb{E} t \mathbf{1}_{\{X \geq t\}} = t \mathbb{P}(X \geq t).
\]

There are several variants, easily deduced from Chebyshev’s inequality by monotonicity of certain functions. For a nonnegative random variable \( X \) and \( t > 0 \), using the power function \( x^p, p > 0 \), we get

\[
P( X \geq t ) = \mathbb{P}(X^p \geq t^p) \leq \frac{1}{t^p} \mathbb{E} X^p.
\] (6.1)

For a real-valued random variable \( X \), every \( t \in \mathbb{R} \) and \( \lambda > 0 \), using the exponential function \( e^{\lambda x} \), we have

\[
P( X \geq t ) = \mathbb{P}(\lambda X \geq \lambda t) \leq \frac{1}{e^{\lambda t}} \mathbb{E} e^{\lambda X}.
\] (6.2)

For a real-valued random variable \( X \), every \( t \in \mathbb{R} \), using the square function \( x^2 \) and variance, we have

\[
P( |X - \mathbb{E} X| \geq t ) \leq \frac{1}{t^2} \mathbb{E} |X - \mathbb{E} X|^2 = \frac{1}{t^2} \text{Var}(X).
\] (6.3)

Another general and helpful inequality is about convex functions. Recall that a function \( f: \mathbb{R} \to \mathbb{R} \) is convex if \( f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \) for every \( \lambda \in [0, 1] \) and \( x, y \in \mathbb{R} \). By induction, this can be extended to

\[
f \left( \sum_{i=1}^{n} \lambda_i x_i \right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)
\]

for every \( \lambda_1, \ldots, \lambda_n \geq 0 \) such that \( \sum_{i=1}^{n} \lambda_i = 1 \) and every \( x_1, \ldots, x_n \in \mathbb{R} \). The weights \( \lambda_i \) can of course be interpreted in probabilistic terms: if \( X \) is a random variable taking the value \( x_i \) with probability \( \lambda_i \), then \( \sum \lambda_i x_i = \mathbb{E} X \), whereas \( \sum \lambda_i f(x_i) = \mathbb{E} f(X) \), so we have

\[
f(\mathbb{E} X) \leq \mathbb{E} f(X).
\]

This generalises to arbitrary random variables and is called Jensen’s inequality.

52
6.2 Theorem (Jensen’s inequality). If $f: \mathbb{R} \to \mathbb{R}$ is a convex function and $X$ is a random variable such that both $\mathbb{E}X$ and $\mathbb{E}f(X)$ exist, then

$$f(\mathbb{E}X) \leq \mathbb{E}f(X).$$

We shall present two proofs.

Proof 1. Suppose $f$ is differentiable. Then by convexity, a tangent line at $x_0$ is below the graph, so

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

(which holds for every $x_0$ and $x$). We set $x = X, x_0 = \mathbb{E}X$ and take the expectation of both sides to get

$$\mathbb{E}f(X) \geq \mathbb{E}[f(\mathbb{E}X) + f'(\mathbb{E}X)(X - \mathbb{E}X)] = f(\mathbb{E}X) + f'(\mathbb{E}X)\mathbb{E}(X - \mathbb{E}X) = f(\mathbb{E}X).$$

If $f$ is not differentiable, this argument can be rescued by using the fact that convex functions have left and right derivatives defined everywhere (because the divided differences of convex functions are monotone).

Proof 2. Recall that a function is convex if and only if its epigraph is a convex set. By a separation type argument, this gives that the convex function is a pointwise supremum over a countable collection of linear functions. Specifically, let $f: \mathbb{R} \to \mathbb{R}$ be a convex function and consider the family of linear functions with rational coefficients which are below $f$,

$$\mathcal{A} = \{ \ell: \mathbb{R} \to \mathbb{R}, \ell(x) = ax + b, a, b \in \mathbb{Q}, \ell \leq f \}.$$

Then

$$f(x) = \sup_{\ell \in \mathcal{A}} \ell(x), \quad x \in \mathbb{R}.$$ 

Jensen’s inequality follows: for every $\ell \in \mathcal{A}$, by linearity, $\mathbb{E}\ell(X) = \ell(\mathbb{E}X)$, thus

$$\mathbb{E}f(X) = \mathbb{E}\sup_{\ell \in \mathcal{A}} \ell(X) \geq \sup_{\ell \in \mathcal{A}} \mathbb{E}\ell(X) = \sup_{\ell \in \mathcal{A}} \ell(\mathbb{E}X) = f(\mathbb{E}X).$$

The so-called Hölder’s inequality is a very effective tool used to factor out the expectation of a product.

6.3 Theorem (Hölder’s inequality). Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For random variables $X$ and $Y$, we have

$$\mathbb{E}|XY| \leq \left(\mathbb{E}|X|^p\right)^{1/p} \left(\mathbb{E}|Y|^q\right)^{1/q}.$$

In particular, when $p = q = 2$, this gives the Cauchy-Schwarz inequality

$$\mathbb{E}|XY| \leq \sqrt{\mathbb{E}|X|^2 \mathbb{E}|Y|^2}.$$
Proof 1. We can assume without loss of generality that \( E|X|^p \) and \( E|Y|^q \) are finite (otherwise the right hand side is +\( \infty \) and there is nothing to prove). The key ingredient is an elementary inequality for numbers.

Claim. For \( p, q \geq 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( x, y \geq 0 \), we have

\[
x y \leq \frac{x^p}{p} + \frac{y^q}{q}.
\]

Proof. By the concavity of the log function, we have

\[
\log \left( \frac{x^p}{p} + \frac{y^q}{q} \right) \geq \frac{1}{p} \log x^p + \frac{1}{q} \log y^q = \log xy.
\]

Setting \( x = \frac{|X|^p}{(E|X|^p)^{1/p}} \), \( y = \frac{|Y|^q}{(E|Y|^q)^{1/q}} \), taking the expectation and simplifying yields the desired inequality.

Proof 2. By homogeneity we can assume that \( E|X|^p = 1 \) and \( E|Y|^q = 1 \). We can also assume that \( |Y| > 0 \) a.e. (otherwise we consider \( \max\{|Y|, \frac{1}{n}\} \) and pass to the limit by Lebesgue’s monotone convergence theorem). Define a new probability measure \( \tilde{P}(A) = E|Y|^q 1_A, A \in \mathcal{F} \). In other words, \( \tilde{E}Z = E|Y|^q Z \) for every (\( \tilde{P} \)-integrable) random variable \( Z \). Then, by the convexity of \( x \mapsto x^p \) and Jensen’s inequality,

\[
(E|X||Y|)^p = (E|X|^q|Y|^{1-q})^p \leq E|X|^p|Y|^{(1-q)p} = E|X|^p|Y|^{(1-q)p+q} = E|X|^p = 1.
\]

6.2 \( L_p \)-spaces

Given a probability space \( (\Omega, \mathcal{F}, P) \) and \( p \in (0, \infty) \), we define

\[ L_p = L_p(\Omega, \mathcal{F}, P) = \{ X: \Omega \to \mathbb{R}, X \text{ is a random variable with } E|X|^p < \infty \} \]

which is called the \( L_p \) space (on \( \Omega \)). Technically, \( L_p \) is defined as the set of the abstract classes of random variables which are equal a.e., but we tacitly assume that and skip such details. We set

\[
||X||_p = (E|X|^p)^{1/p}, \quad X \in L_p.
\]

We also extend this to \( p = \infty \) by setting

\[ L_\infty = \{ X: \Omega \to \mathbb{R}, X \text{ is a random variable with } |X| \leq M \text{ a.s., for some } M > 0 \} \]

and

\[
||X||_\infty = \text{ess sup} X = \inf\{M \geq 0, |X| \leq M \text{ a.s.}\}
\]
(the essential supremum of \(X\)) with the usual convention that \(\inf\emptyset = +\infty\). Equivalently,
\[
\|X\|_\infty = \inf\{t \in \mathbb{R}, \ F_X(t) = 1\}
\]
(exercise). We also have
\[
\|X\|_p \underset{p \to \infty}{\longrightarrow} \|X\|_\infty
\]
(another exercise). The quantity \(\|X\|_p\) is called the \(p\)-th moment of \(X\). It is monotone in \(p\), which is an easy consequence of Jensen’s inequality.

**6.4 Example.** Let \(0 < p < q\). Take \(r = \frac{q}{p}\) and \(f(x) = |x|^r\) which is convex. Thus for a random variable \(X\) which is in \(L_q\), using Jensen’s inequality, we have
\[
\mathbb{E}|X|^q = \mathbb{E}f(|X|^p) \geq f(\mathbb{E}|X|^p) = (\mathbb{E}|X|^p)^{q/p},
\]
equivalently,
\[
\|X\|_q \geq \|X\|_p.
\]
In other words, the function \(p \mapsto \|X\|_p\) of moments of the random variable \(X\) is nondecreasing.

**6.5 Example.** Hölder’s inequality can be restated as: for random variables \(X\) and \(Y\) and \(p, q \in [1, \infty]\) with \(\frac{1}{p} + \frac{1}{q} = 1\), we have
\[
\mathbb{E}|XY| \leq \|X\|_p \|Y\|_q.
\]
The case \(p = 1, q = \infty\) follows by taking the limit in Hölder’s inequality.

Hölder’s inequality gives the following helpful variational formula for \(p\)th moments, \(p \in [1, \infty]\).

**6.6 Theorem.** Let \(p \in [1, \infty]\). For \(X \in L_p\), we have
\[
\|X\|_p = \sup\{\mathbb{E}XY, \ Y \text{ is a random variable with } \mathbb{E}|Y|^q \leq 1\},
\]
where \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof.** To see that the supremum does not exceed the \(p\)th moment, simply apply Theorem 6.3. To see the opposite inequality, consider \(Y = \text{sgn}(X)|X|^{p-1}\|X\|^{-p/q}\). Then \(\mathbb{E}XY = \|X\|_p\), so in fact we can write “max” instead of “sup” in (6.5). Using this linearisation, we can effortlessly establish the triangle inequality for the \(p\)th moment, the so-called Minkowski’s inequality.

**6.7 Theorem** (Minkowski’s inequality). Let \(p \in [1, \infty]\). Let \(X\) and \(Y\) be random variables. Then
\[
\|X + Y\|_p \leq \|X\|_p + \|Y\|_p.
\]
Proof. Invoking (6.5),

\[ \|X + Y\|_p = \sup \{E(X + Y)Z, \ E|Z|^p \leq 1 \}. \]

By linearity, \( E(X + Y)Z = EXZ + EYZ \). Using that \( \sup \{ f + g \} \leq \sup f + \sup g \) and applying again (6.5) finishes the proof. \( \square \)

6.8 Remark. For every \( 0 < p < 1 \) Minkowski’s inequality fails (for instance, take \( X \) and \( Y \) to be i.i.d. Ber(\( \alpha \))). Let us derive its analogue. Observe that for \( 0 < p < 1 \) and every real numbers \( x, y \), we have

\[ |x + y|^p \leq |x|^p + |y|^p. \]  \hspace{1cm} (6.6)

If \( x + y = 0 \), the inequality is trivial. Otherwise, note that \( |t|^p \geq |t| \) for \( |t| \leq 1 \), so using this and the triangle inequality yields

\[ \left( \frac{|x|}{|x + y|} \right)^p + \left( \frac{|y|}{|x + y|} \right)^p \geq \frac{|x|}{|x + y|} + \frac{|y|}{|x + y|} = \frac{|x| + |y|}{|x + y|} \geq \frac{|x + y|}{|x + y|} = 1. \]

Given two random variables, applying (6.6) for \( x = X(\omega), y = Y(\omega) \) and taking the expectation gives

\[ E|X + Y|^p \leq E|X|^p + E|Y|^p, \quad p \in (0, 1]. \]  \hspace{1cm} (6.7)

In other words,

\[ \|X + Y\|_p^p \leq \|X\|_p^p + \|Y\|_p^p, \quad p \in (0, 1]. \]  \hspace{1cm} (6.8)

The next two theorems justify that \( L_p \) are in fact Banach spaces (normed spaces which are complete, that is every Cauchy sequence converges).

6.9 Theorem. For every \( p \in [1, \infty] \), \( (L_p, \| \cdot \|_p) \) is a normed space.

Proof. To check that \( X \mapsto \|X\|_p \) is a norm on \( L_p \), it is to be verified that

1) \( \|X\|_p \geq 0 \) with equality if and only if \( X = 0 \) a.s.
2) \( \|\lambda X\|_p = |\lambda|\|X\|_p \), for every \( \lambda \in \mathbb{R} \)
3) \( \|X + Y\|_p \leq \|X\|_p + \|Y\|_p \).

1) and 2) follow easily form the properties of integral and essential supremum. 3) follows from Minkowski’s inequality. \( \square \)

6.10 Theorem. Let \( p \in [1, \infty] \). If \( (X_n)_{n \geq 1} \) is a Cauchy sequence in \( L_p \), that is for every \( \varepsilon > 0 \), there is a positive integer \( N \) such that for every \( n, m \geq N \), we have \( \|X_n - X_m\|_p \leq \varepsilon \), then there is a random variable \( X \) in \( L_p \) such that \( \|X_n - X\|_p \to 0 \). In other words, \( (L_p, \| \cdot \|_p) \) is complete, hence it is Banach space.
Proof. Assume first that $1 \leq p < \infty$. By the Cauchy condition, there is a subsequence $n_k$ such that

$$\|X_{n_{k+1}} - X_{n_k}\|_p \leq 2^{-k}, \quad k = 1, 2, \ldots$$

Let

$$Y_k = \sum_{j=1}^{k} |X_{n_{j+1}} - X_{n_j}|.$$ 

The sequence $(Y_k)$ is nondecreasing, hence it pointwise converges, say to $Y$, $\lim_{k \to \infty} Y_k = Y$. Since $\|Y_k\|_p \leq 1$, by Fatou’s lemma,

$$EY^p = E \liminf_k Y_k^p \leq \liminf_k EY_k^p \leq 1,$$

that is $Y \in L^p$. In particular, $Y < \infty$ a.s. Consequently, the sequence

$$X_{n_k} = X_{n_1} + \sum_{j<k} (X_{n_{j+1}} - X_{n_j})$$

converges a.s. say to $X$. It remains to show that $\|X_n - X\|_p \to 0$. For a fixed $m$, by Fatou’s lemma, we get

$$E|X_m - X|^p = E \liminf_k |X_m - X_{n_k}|^p \leq \liminf_k E|X_m - X_{n_k}|^p,$$

thus by the Cauchy condition, for every $\varepsilon > 0$, there is $N$ such that for every $m > N$,

$$E|X_m - X|^p \leq \varepsilon.$$

This finishes the argument.

For $p = \infty$, we consider the sets

$$A_k = \{|X_k| > \|X_k\|_{\infty}\}$$

$$B_{n,m} = \{|X_m - X_n| > \|X_m - X_n\|_{\infty}\}.$$ 

Their union $E$ is of measure zero, whereas on $E^c$, the variables $X_n$ converge uniformly to a bounded random variable $X$ (because $\mathbb{R}$ is complete).

The case $p = 2$ is the most important because $L_2$ is Hilbert space. The scalar product $\langle \cdot, \cdot \rangle : L_2 \times L_2 \to \mathbb{R}$ is defined by

$$\langle X, Y \rangle = EXY, \quad X, Y \in L_2.$$

Then

$$\|X\|_2 = \sqrt{\langle X, X \rangle}.$$ 

Crucially, we have the parallelogram identity: for $X, Y \in L_2$, we have

$$\|X + Y\|_2^2 + \|X - Y\|_2^2 = 2(\|X\|_2^2 + \|Y\|_2^2).$$ (6.9)

A consequence of this is that balls in $L_2$ are round and orthogonal projection is well defined.
6.11 Theorem. Let \( H \) be a complete linear subspace of \( L_2 \). Then for every random variable \( X \) in \( L_2 \), there is a unique random variable \( Y \in H \) such that the following two conditions hold:

(i) \( Y \) is closest to \( X \) in \( H \), that is

\[
\|X - Y\|_2 = \inf\{\|X - Z\|_2, Z \in H\},
\]

(ii) for every \( Z \in H \), \( \langle X - Y, Z \rangle = 0 \), that is \( X - Y \) is orthogonal to \( H \).

The uniqueness is understood as follows: if \( \tilde{Y} \in H \) satisfies either (i) or (ii), then \( \|Y - \tilde{Y}\|_2 = 0 \), that is \( Y = \tilde{Y} \) a.s.

Proof. Let \( d \) denote the infimum in (i). Then, there are \( Y_n \in H \) such that

\[
\|X - Y_n\|_2 \to d.
\]

By the parallelogram law,

\[
\|X - Y_n\|^2_2 + \|X - Y_m\|^2_2 = 2 \left( \left\| \frac{X - Y_n + Y_m}{2} \right\|_2^2 + \left\| \frac{Y_n - Y_m}{2} \right\|_2^2 \right) \\
\geq 2d + \|Y_n - Y_m\|^2_2.
\]

Since the left hand side converges to \( 2d \) as \( m, n \to \infty \), we conclude that \( (Y_n) \) is a Cauchy sequence in \( H \). Since \( H \) is assumed to be complete, \( \|Y_n - Y\|_2 \to 0 \) for some \( Y \in H \). Thus, \( \|X - Y\|_2 = d \), which establishes (i).

To get (ii), fix \( Z \in H \) and note that for every \( t \in \mathbb{R} \), by (i), we have

\[
\|X - (Y + tZ)\|_2 \geq \|X - Y\|_2,
\]

which after squaring and rearranging gives

\[
t^2\|Z\|_2^2 - 2t\langle X - Y, Z \rangle \geq 0.
\]

Since this holds for all small \( t \) (both positive and negative), necessarily the linear term has to vanish, that is \( \langle X - Y, Z \rangle = 0 \).

For the uniqueness, suppose \( \tilde{Y} \) satisfies (i). Then, by the parallelogram law,

\[
2d = \|X - Y\|^2_2 + \|X - \tilde{Y}\|^2_2 = 2 \left( \left\| \frac{X - Y + \tilde{Y}}{2} \right\|_2^2 + \left\| \frac{Y - \tilde{Y}}{2} \right\|_2^2 \right) \\
\geq 2d + \|Y - \tilde{Y}\|^2_2,
\]

so \( \|Y - \tilde{Y}\|^2_2 \leq 0 \), hence \( \|Y - \tilde{Y}\|_2 = 0 \) and consequently, \( \tilde{Y} = Y \) a.s. If \( \tilde{Y} \) satisfies (ii), then since \( \tilde{Y} - Y \in H \), we get \( \langle X - \tilde{Y}, \tilde{Y} - Y \rangle = 0 \). Since also \( \langle X - Y, \tilde{Y} - Y \rangle = 0 \), we get \( \langle \tilde{Y} - Y, \tilde{Y} - Y \rangle = 0 \), so \( \tilde{Y} = Y \) a.s. \( \square \)
6.3 Notions of convergence

A sequence of random variables \((X_n)\) converges to a random variable \(X\)

a) **almost surely** if \(P\left(\{\omega \in \Omega, \lim_{n \to \infty} X_n(\omega) = X(\omega)\}\right) = 1\), denoted \(X_n \xrightarrow{\text{a.s.}} X\).

b) **in probability** if for every \(\varepsilon > 0\), \(P(|X_n - X| > \varepsilon) \xrightarrow{n \to \infty} 0\), denoted \(X_n \xrightarrow{\text{P}} 0\).

c) **in \(L^p\), \(p > 0\)** if \(\mathbb{E}|X_n - X|^p \xrightarrow{n \to \infty} 0\), denoted \(X_n \xrightarrow{\text{\(L^p\)}} X\).

For instance, let \(\Omega = \{1, 2\}\) and \(P(1) = P(2) = \frac{1}{2}\), \(X_n(1) = -\frac{1}{n}\), \(X_n(2) = \frac{1}{n}\).

We have

a) \(X_n \xrightarrow{\text{a.s.}} 0\) because \(X_n(\omega) \to 0\) for every \(\omega \in \Omega\),

b) \(X_n \xrightarrow{\text{P}} 0\) because \(P(|X_n| > \varepsilon) = P\left(\frac{1}{n} > \varepsilon\right) \to 0\),

c) \(X_n \xrightarrow{\text{\(L^p\)}} 0\) because \(\mathbb{E}|X_n|^p = \frac{2}{n^p} \to 0\).

We have two results, saying that the convergence in probability is the weakest among the three.

6.12 Theorem. If a sequence of random variables \((X_n)\) converges to \(X\) a.s. then it also converges in probability, but in general not conversely.

**Proof.** By the definition of the limit of a sequence,

\[
\{\lim_{n} X_n = X\} = \bigcap_{l \geq 1} \bigcup_{N \geq 1} \bigcap_{n \geq N} \left\{|X_n - X| < \frac{1}{l}\right\}.
\]

For any events \(A_l\), \(P\left(\bigcap_{l \geq 1} A_l\right) = 1\) if and only if \(P(A_l) = 1\) for all \(l \geq 1\). Therefore, \(X_n \xrightarrow{\text{a.s.}} 0\) is equivalent to: for every \(l \geq 1\),

\[
P\left(\bigcup_{N \geq 1} \bigcap_{n \geq N} \left\{|X_n - X| < \frac{1}{l}\right\}\right) = 1.
\]

By monotonicity with respect to \(N\),

\[
P\left(\bigcup_{N \geq 1} \bigcap_{n \geq N} \left\{|X_n - X| < \frac{1}{l}\right\}\right) \leq \lim_{N \to \infty} P\left(\bigcap_{n \geq N} \left\{|X_n - X| < \frac{1}{l}\right\}\right).
\]

Finally, observe that by the inclusion \(\bigcap_{n \geq N} \left\{|X_n - X| < \frac{1}{l}\right\} \subseteq \left\{|X_N - X| < \frac{1}{l}\right\}\), we have

\[
1 = \lim_{N \to \infty} P\left(\bigcap_{n \geq N} \left\{|X_n - X| < \frac{1}{l}\right\}\right) \leq \lim_{N \to \infty} P\left(\left\{|X_N - X| < \frac{1}{l}\right\}\right),
\]
so passing to the complements, for every \( l \geq 1, \)
\[
0 \leq \lim_{N \to \infty} \mathbb{P} \left( \left\{ |X_N - X| \geq \frac{1}{l} \right\} \right) \leq 0.
\]

Therefore, for every \( \varepsilon > 0, \)
\[
\lim_{N \to \infty} \mathbb{P} \left( \{ |X_N - X| \geq \varepsilon \} \right) = 0, \text{ that is } X_n \xrightarrow{p} 0. \]
The following example of a sequence convergent in probability but not a.s. finishes the proof.

6.13 Example. Let \( \Omega = [0, 1] \) and \( \mathbb{P} (\cdot) \) be the uniform probability measure. Let \( X_1 = 1, X_2 = 1_{[0,1/2]}, X_3 = 1_{[1/2,1]}, X_4 = 1_{[0,1/4]}, X_5 = 1_{[1/4,1/2]}, X_6 = 1_{[1/2,3/4]}, \) \( X_7 = 1_{[3/4,1]}, \) etc., \( X_{2^n}, X_{2^n+1}, \ldots, X_{2^{n+1}-1} \) are indicators of a wandering interval of length \( 2^{-n} \) shifting to right by \( 2^{-n} \) every increment of the index. We have
a) \( X_n \xrightarrow{p} 0 \) because for every \( \varepsilon > 0, \) \( \mathbb{P}(|X_n| > \varepsilon) \leq 2^{-k} \) when \( 2^k \leq n < 2^{k+1}, \) which goes to 0 as \( n \) goes to \( \infty. \)
b) \( X_n \xrightarrow{a.s} 0 \) because for every \( \omega \in (0,1), \) the sequence \( (X_n(\omega)) \) contains infinitely many 0 and 1, so it is not convergent; moreover, if \( X_n \xrightarrow{a.s} X \) for some random variable \( X \) other than 0, then by Theorem 6.12, \( X_n \xrightarrow{p} X \) and from the uniqueness of limits in probability (homework!), \( X = 0 \) a.s., contradiction.
c) \( X_n \xrightarrow{L_p} 0 \) because \( \mathbb{E}|X_n|^p = 2^{-kp} \) when \( 2^k \leq n < 2^{k+1}, \) which goes to 0 as \( n \) goes to \( \infty. \)

6.14 Theorem. If a sequence of random variables \( (X_n) \) converges to \( X \) in \( L_p \) for some \( p > 0, \) then it also converges in probability, but in general not conversely.

Proof. By Chebyshev’s inequality (6.1),
\[
\mathbb{P}(|X_n - X| > \varepsilon) \leq \frac{1}{\varepsilon^p} \mathbb{E}|X_n - X|^p \xrightarrow{n \to \infty} 0,
\]
so \( X_n \xrightarrow{p} X. \) The following example of a sequence convergent in probability but not in \( L_p \) finishes the proof.

6.15 Example. Let \( \Omega = [0, 1] \) and \( \mathbb{P} (\cdot) \) be the uniform probability measure. Let \( X_n = n^{1/p}1_{[0,1/n]} \). We have
a) \( X_n \xrightarrow{p} 0 \) because for every \( \varepsilon > 0, \) \( \mathbb{P}(|X_n| > \varepsilon) \leq \frac{1}{\varepsilon} \) which goes to 0 as \( n \) goes to \( \infty. \)
b) \( X_n \xrightarrow{L_p} 0 \) because \( \mathbb{E}|X_n|^p = n^{1/p} = 1; \) moreover, if \( X_n \xrightarrow{L_p} X \) for some random variable \( X \) other than 0, then by Theorem 6.14, \( X_n \xrightarrow{p} X \) and from the uniqueness of limits in probability (homework!), \( X = 0 \) a.s., contradiction.
c) $X_n \xrightarrow{a.s.}{n} 0$ because for every $\omega > 0$, the sequence $X_n(\omega)$ becomes eventually constant 0.

Theorems (6.12), (6.14) and Examples 6.13, 6.15 can be summarised in the following diagram.

\[
\begin{array}{c}
\text{a.s.} \quad \text{in } L_p \\
\text{in } P \\
\end{array}
\]

We record a few basic algebraic properties of the three notions of convergence.

1) If $X_n$ converges to $X$ a.s./in probability/in $L_p$ and $Y_n$ converges to $Y$ a.s./in probability/in $L_p$, then $X_n + Y_n$ converges to $X + Y$ a.s./in probability/in $L_p$.

2) If $X_n$ converges to $X$ a.s./in probability and $Y_n$ converges to $Y$ a.s./in probability, then $X_n \cdot Y_n$ converges to $X \cdot Y$ a.s./in probability.

3) If $0 < p < q$ and $X_n$ converges to $X$ in $L_q$, then $X_n$ converges to $X$ in $L_p$.

Immediately, 1) and 2) for the almost sure convergence follow from those statements for sequences of numbers since the intersection of two events of probability 1 is of probability 1.

Property 1) for $L_p$ convergence follows from Minkowski’s inequality (Theorem 6.7) and Property 3) follows from the monotonicity of moments (Example 6.4).

Establishing 1) and 2) directly from definition is cumbersome. Instead, we first prove a convenient equivalent condition for convergence in probability in terms of almost sure convergence.

\textbf{6.16 Theorem} (Riesz). If a sequence $(X_n)$ of random variables converges to a random variable $X$ in probability, then there is a subsequence $(X_{n_k})_k$ which converges to $X$ almost surely.

\textit{Proof.} Since for every $\varepsilon$, $P(\{|X_n - X| > \varepsilon\}) \to 0$, then we can find an index $n_1$ such that $P(\{|X_{n_1} - X| > 2^{-1}\}) < 2^{-1}$. By the same logic, we can find an index $n_2 > n_1$ such that $P(\{|X_{n_2} - X| > 2^{-2}\}) < 2^{-2}$, etc. We get a subsequence $(X_{n_k})_k$ such that $P(\{|X_{n_k} - X| > 2^{-k}\}) < 2^{-k}$ for every $k$. Since the series $\sum_{k=1}^{\infty} P(\{|X_{n_k} - X| > 2^{-k}\})$ converges, by the first Borel-Cantelli lemma (Lemma 3.14), with probability 1 only finitely many events $A_k = \{|X_{n_k} - X| > 2^{-k}\}$ occur. When this happens, $X_{n_k} \to X$, so $X_{n_k} \xrightarrow{k} X$. 

61
6.17 Theorem. A sequence \((X_n)\) of random variables converges to a random variable \(X\) in probability if and only if every subsequence \((X_{n_k})_k\) contains a further subsequence \((X_{n_{k_l}})_l\) which converges to \(X\) almost surely.

Proof. (\(\Rightarrow\)) It follows directly from Theorem 6.16.

(\(\Leftarrow\)) If \((X_n)\) does not converge to \(X\) in probability, then there is \(\varepsilon > 0\) such that 
\[
\mathbb{P}(|X_n - X| > \varepsilon) \not\to 0.
\]
Consequently, there is \(\varepsilon' > 0\) and a subsequence \((X_{n_k})_k\) for which 
\[
\mathbb{P}(|X_{n_k} - X| > \varepsilon) > \varepsilon'.
\]
By the assumption, there is a subsequence \((X_{n_{k_l}})_l\) convergent to \(X\) almost surely, in particular, in probability, so 
\[
\mathbb{P}(|X_{n_{k_l}} - X| > \varepsilon) \to 0.
\]
This contradiction finishes the proof. \(\square\)

Going back to the algebraic properties 1) and 2) for convergence in probability, we can easily justify them using that they hold for convergence almost surely. For 1), say 
\[
S_n = X_n + Y_n
\]
does not converge in probability to \(S = X + Y\). Then as in the proof of Theorem 6.17, \(\mathbb{P}(|S_{n_k} - S| > \varepsilon) > \varepsilon'\) for some \(\varepsilon, \varepsilon' > 0\) and a subsequence \((n_k)\). Using Theorem 6.17, there is a further subsequence \((n_{k_l})\) such that \((X_{n_{k_l}})_l\) converges to \(X\) a.s. and a further subsequence (for simplicity, denote it the same) such that \((Y_{n_{k_l}})_l\) converges to \(Y\) a.s.. Then 
\[
S_{n_{k_l}} \xrightarrow{a.s.} S,
\]
which contradicts \(\mathbb{P}(|S_{n_k} - S| > \varepsilon) > \varepsilon'.\)
6.4 Exercises

1. Show that the probability that in \( n \) throws of a fair die the number of sixes lies between \( \frac{1}{6}n - \sqrt{n} \) and \( \frac{1}{6}n + \sqrt{n} \) is at least \( \frac{31}{36} \).

2. Let \( X \) be a random variable with density \( \frac{1}{2}e^{-|x|} \) on \( \mathbb{R} \). Show that for every \( p \geq 1 \),
\[
c_1 p \leq \|X\|_p \leq c_2 p
\]
for some absolute constants \( c_1, c_2 > 0 \).

3. Let \( g \) be a standard Gaussian random variable. Show that for every \( p \geq 1 \),
\[
c_1 p \leq \|X\|_p \leq c_2 p
\]
for some universal constants \( c_1, c_2 > 0 \).

4. Show that for every random variable \( X \), we have
\[
\|X\|_{\infty} = \inf \{ t \in \mathbb{R}, F_X(t) = 1 \}.
\]

5. Show that for every random variable \( X \), we have
\[
\|X\|_p \xrightarrow{p \to \infty} \|X\|_{\infty}.
\]

6. If \( E|X|^{p_0} < \infty \) for some \( p_0 > 0 \), then \( E\log_+ |X| < \infty \) and
\[
(E|X|^{p_0})^{1/p_0} \xrightarrow{p_0 \to \infty} e^{E\log |X|}.
\]

7. Let \( X \) be a random variable with values in an interval \([0,a]\). Show that for every \( t \) in this interval, we have
\[
P(X \geq t) \geq \frac{EX - t}{a - t}.
\]

8. Prove the Payley-Zygmund inequality: for a nonnegative random variable \( X \) and every \( \theta \in [0,1] \), we have
\[
P(\theta X > X) \geq (1 - \theta)^2 \frac{(EX)^2}{EX^2}.
\]

9. Prove that for nonnegative random variables \( X \) and \( Y \), we have
\[
E\frac{X}{Y} \geq \frac{(EX)^2}{EY}.
\]

10. Let \( p \in (0,1) \) and \( q < 0 \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then for every random variables \( X \) and \( Y \), we have
\[
E|XY| \geq (E|X|^p)^{1/p}(E|Y|^q)^{1/q}.
\]

11. Let \( X_1, X_2, \ldots \) be i.i.d. positive random variables with \( EX_1^3 < \infty \). Let
\[
a_n = E\left( \frac{X_1 + \ldots + X_n}{n} \right)^3.
\]
Prove that \( a_n^2 \leq a_{n-1} a_{n+1}, \ n \geq 2 \).
12. Let \( X, X_1, X_2, \ldots \) be identically distributed random variables such that \( \Pr(X > t) > 0 \) for every \( t > 0 \). Suppose that for every \( \eta > 1 \), we have \( \lim_{t \to \infty} \frac{\Pr(X > \eta t)}{\Pr(X > t)} = 0 \). For \( n \geq 1 \), let \( a_n \) be the smallest number \( a \) such that \( n \Pr(X > a) \leq 1 \). Show that for every \( \varepsilon > 0 \), we have \( \max_{i \leq n} X_i \leq (1 + \varepsilon)a_n \) with high probability as \( n \to \infty \), i.e. \( \Pr(\max_{i \leq n} X_i \leq (1 + \varepsilon)a_n) \to 1 \) as \( n \to \infty \).

13. Let \( X \) be a random variable such that \( \mathbb{E}e^{\delta |X|} < \infty \) for some \( \delta > 0 \). Show that \( \mathbb{E}|X|^p < \infty \) for every \( p > 0 \).

14. Let \( X \) be a random variable such that \( \mathbb{E}e^{tX} < \infty \) for every \( t \in \mathbb{R} \). Show that the function \( t \mapsto \log \mathbb{E}e^{tX} \) is convex on \( \mathbb{R} \).

15. Let \( X \) be a random variable such that \( \mathbb{E}|X|^p < \infty \) for every \( p > 0 \). Show that the function \( p \mapsto \log \|X\|_{1/p} \) is convex on \( (0, \infty) \).

16. Let \( \varepsilon_1, \ldots, \varepsilon_n \) be independent random signs, that is \( \Pr(\varepsilon_i = -1) = \frac{1}{2} = \Pr(\varepsilon_i = 1) \), \( i \leq n \). Prove that there is a positive constant \( c \) such that for every \( n \geq 1 \) and real numbers \( a_1, \ldots, a_n \), we have
\[
\Pr\left( \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| > \frac{1}{2} \sqrt{\sum_{i=1}^{n} a_i^2} \right) \geq c.
\]

17. Let \( \varepsilon_1, \varepsilon_2, \ldots \) be i.i.d. symmetric random signs. Show that there is a constant \( c > 0 \) such that for every \( n \geq 1 \) and reals \( a_1, \ldots, a_n \), we have
\[
\Pr\left( \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \right) \geq c.
\]

18. Let \( \varepsilon_1, \varepsilon_2, \ldots \) be i.i.d. symmetric random signs. Show that there is a constant \( c > 0 \) such that for every \( n \geq 1 \) and reals \( a_1, \ldots, a_n \), we have
\[
\Pr\left( \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| \geq \sqrt{\sum_{i=1}^{n} a_i^2} \right) \geq c.
\]

19. The goal is to prove Bernstein's inequality: for every \( n \), every real numbers \( a_1, \ldots, a_n \) and \( t > 0 \), we have
\[
\Pr\left( \left| \sum_{i=1}^{n} a_i \varepsilon_i \right| > t \right) \leq 2 \exp\left\{ -\frac{t^2}{2 \sum_{i=1}^{n} a_i^2} \right\},
\]
where \( \varepsilon_1, \ldots, \varepsilon_n \) are i.i.d. symmetric random signs.

a) Show that \( \cosh(t) \leq e^{t^2/2}, t \in \mathbb{R} \).

b) Find \( \mathbb{E}e^{a_i \varepsilon_i} \).

c) Let \( S = \sum a_i \varepsilon_i \). Show that for every \( t, \lambda > 0 \), we have \( \Pr(S > t) \leq e^{-\lambda x} \mathbb{E}e^{\lambda x} \).
d) Optimising over $\lambda$ conclude that $\mathbb{P}(S > t) \leq e^{-t^2/(2 \sum a_i^2)}$.

e) Using symmetry, conclude that $\mathbb{P}(|S| > t) \leq 2e^{-t^2/(2 \sum a_i^2)}$.

20. **Hoeffding’s lemma**: for a random variable $X$ such that $a \leq X \leq b$ a.s. for some $a < b$, we have $\mathbb{E}e^{u(X-\mathbb{E}X)} \leq \exp\left\{\frac{u^2(b-a)^2}{8}\right\}$, $u \in \mathbb{R}$.

21. **Hoeffding’s inequality**: for independent random variables $X_1, \ldots, X_n$ such that $a_i \leq X_i \leq b_i$ a.s. for some $a_i < b_i$, $i \leq n$, we have

$$
\mathbb{P}\left(|\sum_{i=1}^n X_i - \mathbb{E}\sum_{i=1}^n X_i| > t\right) \leq 2 \exp\left\{-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right\}, \quad t \geq 0.
$$

22. **Khinchin’s inequality**: for every $p > 0$, there are positive constants $A_p, B_p$ which depend only on $p$ such that for every $n$ and every real numbers $a_1, \ldots, a_n$, we have

$$
A_p \left(\sum_{i=1}^n a_i^2\right)^{1/2} \leq \left(\mathbb{E}\left|\sum_{i=1}^n a_i \varepsilon_i\right|^p\right)^{1/p} \leq B_p \left(\sum_{i=1}^n a_i^2\right)^{1/2},
$$

where $\varepsilon_1, \ldots, \varepsilon_n$ are i.i.d. symmetric random signs.

23. Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs. Show that

$$
\mathbb{P}\left(\limsup_{n \to \infty} \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{2n \log n}} \leq 1\right) = 1.
$$

24. Let $X$ be an integrable random variable and define

$$
X_n = \begin{cases} 
-n, & X < -n \\
X, & |X| \leq n \\
n, & X > n.
\end{cases}
$$

Does the sequence $X_n$ converge a.s., in $L_1$, in probability?

25. Show that if $X_n \xrightarrow{p} X$ and $X_n \xrightarrow{p} Y$, then $\mathbb{P}(X = Y) = 1$ (in other words, the limit in probability is unique).

26. Let $X_1, X_2, \ldots$ be i.i.d. integrable random variables. Prove that $\frac{1}{n} \max_{k \leq n} |X_k|$ converges to 0 in probability.

27. Show that if $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$, then $X_n Y_n \xrightarrow{p} XY$.

28. Prove that a sequence of random variables $X_n$ converges a.s. if and only if for every $\varepsilon > 0$, $\lim_{N \to \infty} \mathbb{P}\left(\bigcap_{n,m \geq N} |X_n - X_m| < \varepsilon\right) = 1$ (the Cauchy condition).

29. Prove that a sequence of random variables $X_n$ converges in probability if and only if for every $\varepsilon > 0$, $\lim_{n,m \to \infty} \mathbb{P}\left(|X_n - X_m| > \varepsilon\right) = 1$ (the Cauchy condition).

30. Does a sequence of independent random signs $\varepsilon_1, \varepsilon_2, \ldots$ converge a.s.?
31. Let $X_1, X_2, \ldots$ be independent random variables, $X_n \sim \text{Poiss}(1/n)$. Does the sequence $X_n$ converge a.s., in $L_1$, in $L_2$, in probability?

32. Show that if for every $\delta > 0$ we have $\sum_{n=1}^{\infty} P(|X_n - X| > \delta) < \infty$, then $X_n \xrightarrow{a.s.} X$.

33. Show that if there is a sequence of positive numbers $\delta_n$ convergent to 0 such that $\sum_{n=1}^{\infty} P(|X_n - X| > \delta_n) < \infty$, then $X_n \xrightarrow{a.s.} X$.

34. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $P(|X| < 1) = 1$. Show that $X_1 X_2 \cdots X_n$ converges to 0 a.s. and in $L_1$.

35. Let $V$ be the linear space of all random variables on a probability space $(\Omega, \mathcal{F}, P)$ (two random variables are considered equal if they are equal a.s.). Define $\rho: V \times V \to \mathbb{R}$,

$$\rho(X, Y) = \mathbb{E} \left| \frac{X - Y}{1 + |X - Y|} \right|.$$ 

Show that this a metric on $V$, $(V, \rho)$ is complete and $X_n \xrightarrow{P} X$ if and only if $\rho(X_n, X) \to 0$.

36. Let $(\Omega, \mathcal{F}, P)$ be a discrete probability space. Show that for every sequence of random variables $(X_n)$ on this space, $X_n \xrightarrow{P} X$ if and only if $X_n \xrightarrow{a.s.} X$.

37. Show that in general almost sure convergence is not metrisable.

38. Weierstrass theorem. Let $f: [0, 1] \to \mathbb{R}$ be a continuous function. For $x \in [0, 1]$ and an integer $n \geq 1$, let $S_{n,x}$ be a binomial random variable with parameters $n$ and $x$. Let

$$Q_n(x) = \mathbb{E} f \left( \frac{S_{n,x}}{n} \right).$$

(a) Show that $Q$ is a polynomial of degree $n$ (in $x$) (Bernstein’s polynomial of $f$).

(b) Using that $f$ is bounded and uniformly continuous, combined with Chebyshev’s inequality, show that for every $\varepsilon > 0$, there is $n_0$ such that for all $n \geq n_0$ and $x \in [0, 1]$, we have

$$\mathbb{E} \left| f \left( \frac{S_{n,x}}{n} \right) - f(x) \right| < \varepsilon.$$ 

(c) Conclude that for every $\varepsilon > 0$, there is a polynomial $Q$ such that $\sup_{x \in [0, 1]} |f(x) - Q(x)| < \varepsilon$. 

66
7 Laws of large numbers

Suppose we roll a die \( n \) times and the outcomes are \( X_1, X_2, \ldots, X_n \). We expect that the average \( \frac{X_1 + \cdots + X_n}{n} \) should be approximately 3.5 (the expectation of \( X_1 \)) as \( n \) becomes large. Laws of large numbers establish that rigorously, in a fairly general situation.

Formally, we say that a sequence of random variables \( X_1, X_2, \ldots \) satisfies the weak law of large numbers if \( \frac{X_1 + \cdots + X_n}{n} - \mathbb{E}[X_1 + \cdots + X_n] \) converges to 0 in probability and the sequence satisfies the strong law of large numbers if the convergence is almost sure. In particular, for a sequence of identically distributed random variables, we ask whether \( \frac{X_1 + \cdots + X_n}{n} \xrightarrow{n \to \infty} \mathbb{E}[X_1] \). Consider two examples when no reasonable law of large numbers holds and the opposite.

7.1 Example. Let \( X_1, X_2, \ldots \) be i.i.d. standard Cauchy random variables. Then it can be checked that \( \bar{S}_n = \frac{X_1 + \cdots + X_n}{n} \) has the same distribution as \( X_1 \), so \( \bar{S}_n \) is a “well spread out” random variable which in no reasonable sense should be close to its expectation (which in fact does not exist!), or any other constant.

7.2 Example. Let \( \varepsilon_1, \varepsilon_2, \ldots \) be i.i.d. symmetric random signs, that is \( \mathbb{P}(\varepsilon_i = \pm 1) = \frac{1}{2} \). Let \( \bar{S}_n = \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n} \). By Bernstein’s inequality (Exercise 6.19), \( \mathbb{P}(|\bar{S}_n| > t) \leq 2e^{-nt^2/2} \), so the series \( \sum_{n=1}^{\infty} \mathbb{P}(|\bar{S}_n| > t) \) converges, so \( \bar{S}_n \xrightarrow{a.s. n \to \infty} 0 = \mathbb{E}[\varepsilon_1] \) (check!). In other words, the sequence \( (\varepsilon_n) \) satisfies the strong law of large numbers.

7.1 Weak law of large numbers

Using the second moment, we can easily get a very simple version of the weak law of large numbers for uncorrelated random variables with uniformly bounded variance.

7.3 Theorem (The \( L_2 \) law of large numbers). Let \( X_1, X_2, \ldots \) be random variables such that \( \mathbb{E}|X_i|^2 < \infty \) for every \( i \). If

\[
\frac{1}{n^2} \text{Var}(X_1 + \ldots + X_n) \xrightarrow{n \to \infty} 0,
\]

then denoting \( S_n = X_1 + \ldots + X_n \),

\[
\frac{S_n}{n} - \mathbb{E}\frac{S_n}{n} \xrightarrow{L_2 n \to \infty} 0.
\]

In particular, this holds when the \( X_i \) are uncorrelated with bounded variance, that is \( \text{Var}(X_i) \leq M \) for every \( i \) for some \( M \).

Proof. We have

\[
\mathbb{E}\left| \frac{S_n}{n} - \mathbb{E}\frac{S_n}{n} \right|^2 = \frac{1}{n^2} \mathbb{E}|S_n - \mathbb{E}S_n|^2 = \frac{1}{n^2} \text{Var}(X_1 + \ldots + X_n) \xrightarrow{n \to \infty} 0.
\]

Since

\[
\text{Var}(X_1 + \ldots + X_n) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j),
\]

\[67\]
when the $X_i$ are uncorrelated with bounded variance, we have
\[
\frac{1}{n^2} \text{Var}(X_1 + \ldots + X_n) \leq \frac{M}{n^2} = \frac{M}{n}
\]
which goes to 0 as $n \to \infty$.

Since convergence in $L_2$ implies convergence in probability, the above is in fact stronger than a weak law of large numbers.

7.4 Example. Let $X$ be a random vector in $\mathbb{R}^n$ uniformly distributed on the cube $[-1,1]^n$, that is $X = (X_1, \ldots, X_n)$ with the $X_i$ being i.i.d. uniform on $[-1,1]$. The assumptions of the above $L_2$ law of large numbers are satisfied for $X_1^2, X_2^2, \ldots$, so in particular
\[
\frac{X_1^2 + \ldots + X_n^2}{n} - \mathbb{E}X_1^2 \xrightarrow{p \ n \to \infty} 0.
\]
Note that $\mathbb{E}X_1^2 = \frac{1}{3}$. By definition, this convergence in probability means that for every $\varepsilon > 0$,
\[
\mathbb{P}\left(\left|\frac{X_1^2 + \ldots + X_n^2}{n} - \frac{1}{3}\right| > \varepsilon\right) \xrightarrow{\ n \to \infty\ } 0,
\]
or equivalently,
\[
\mathbb{P}\left(\sqrt{n(1/3 - \varepsilon)} < \sqrt{X_1^2 + \ldots + X_n^2} < \sqrt{n(1/3 + \varepsilon)}\right) \xrightarrow{\ n \to \infty\ } 1.
\]
In words, a random point in a high dimensional cube is typically near the boundary of the Euclidean ball centered at 0 of radius $\sqrt{n/3}$.

7.5 Example. Let $X_1, X_2, \ldots$ be i.i.d. random variables uniform on $\{1, \ldots, n\}$. For $k \geq 1$, let
\[
\tau_k = \inf\{m \geq 1, |\{X_1, \ldots, X_m\}| = k\}.
\]
This random variable can be though of as the first index (time) when we have collected $k$ coupons if the $X_i$ are though of as coupons given to us one by one and selected uniformly at random (with replacement) among $n$ different coupons. We are interested in the behaviour of $\tau_n$ as $n \to \infty$ (the time needed to collect the entire set of $n$ coupons).

For convenience we set $\tau_0 = 0$ and of course $\tau_1 = 1$. Let
\[
T_k = \tau_k - \tau_{k-1}, \quad k \geq 1,
\]
which is time we wait to get a coupon of a next type after we have collected $k-1$ different coupons. We have,
\[
\mathbb{P}(T_k = l) = \left(\frac{k-1}{n}\right)^{l-1} \left(1 - \frac{k-1}{n}\right), \quad l = 1, 2, \ldots,
\]
that is
\[
T_k \sim \text{Geom}\left(1 - \frac{k-1}{n}\right)
\]
68
and $T_1, \ldots, T_n$ are independent. Plainly,
\[ \tau_n = T_1 + \ldots + T_n. \]

Thus
\[ \mathbb{E} \tau_n = \sum_{k=1}^{n} \mathbb{E} \tau_k = \sum_{k=1}^{n} \left( 1 - \frac{k-1}{n} \right)^{-1} = n \sum_{k=1}^{n} \frac{1}{n-k+1} = n \sum_{k=1}^{n} \frac{1}{k}, \]
so for large $n$, we have $\mathbb{E} \tau_n \sim n \log n$. Moreover, thanks to independence,
\[ \text{Var}(\tau_n) = \sum_{k=1}^{n} \text{Var}(T_k) \leq n \sum_{k=1}^{n} \left( 1 - \frac{k-1}{n} \right)^{-2} = n^2 \sum_{k=1}^{n} \frac{1}{k^2} < 2n^2. \]

If we let
\[ t_n = \frac{\tau_n}{n \log n}, \]
we obtain
\[ \mathbb{E}(t_n - 1)^2 = \frac{1}{n^2 \log^2 n} \mathbb{E}(\tau_n - n \log n)^2 \leq \frac{2}{n^2 \log^2 n} \left( \mathbb{E}(\tau_n - \mathbb{E} \tau_n)^2 + (\mathbb{E} \tau_n - n \log n)^2 \right) \leq \frac{4}{\log^2 n} + \frac{2}{\log^2 n}. \]

This gives that $t_n \to 1$ in $L_2$ and in probability.

Our goal is to prove the weak law of large numbers for i.i.d. sequences under optimal assumptions on integrability.

7.6 Theorem (The weak law of large numbers). If $X_1, X_2, \ldots$ are i.i.d. random variables such that
\[ t \mathbb{P}(|X_1| > t) \xrightarrow{t \to \infty} 0, \quad (7.1) \]
then
\[ \frac{X_1 + \ldots + X_n}{n} - \mu_n \xrightarrow{p \to \infty} 0, \quad (7.2) \]
where $\mu_n = \mathbb{E} X_1 1_{\{|X_1| \leq n\}}$.

7.7 Remark. The assumption is optimal in the following sense: condition (7.1) is necessary for existence of a sequence $a_n$ such that
\[ \frac{X_1 + \ldots + X_n}{n} - a_n \xrightarrow{p \to \infty} 0 \]
(see exercises).

To prove the theorem, we first establish a fairly general lemma.

7.8 Lemma. Let $\{X_{n,k}\}_{n \geq 1, 1 \leq k \leq n}$ be a triangular array of random variables such that for every $n$, $X_{n,1}, \ldots, X_{n,n}$ are independent (i.e. they are independent within each row). Let $(b_n)$ be a sequence of positive numbers such that $b_n \to \infty$. Let
\[ \tilde{X}_{n,k} = X_{n,k} 1_{\{|X_{n,k}| \leq b_n\}} \]
and
\[ \tilde{S}_n = \sum_{k=1}^{n} \tilde{X}_{n,k}. \]

If the following two conditions are satisfied
\[
(i) \sum_{k=1}^{n} P(\{|X_{n,k}| > b_n\}) \xrightarrow{n \to \infty} 0, \\
(ii) b_n^{-2} \sum_{k=1}^{n} \mathbb{E} \tilde{X}^2_{n,k} \xrightarrow{n \to \infty} 0,
\]
then
\[ \frac{S_n - \mathbb{E} \tilde{S}_n}{b_n} \xrightarrow{n \to \infty} 0, \]
where \( S_n = \sum_{k=1}^{n} X_{n,k}. \)

**Proof.** Fix \( \varepsilon > 0. \) First note that
\[
P \left( \frac{|S_n - \mathbb{E} \tilde{S}_n|}{b_n} > \varepsilon \right) = P \left( \frac{|S_n - \mathbb{E} \tilde{S}_n|}{b_n} > \varepsilon, S_n \neq \tilde{S}_n \right) + P \left( \frac{|S_n - \mathbb{E} \tilde{S}_n|}{b_n} > \varepsilon, S_n = \tilde{S}_n \right) \leq P \left( S_n \neq \tilde{S}_n \right) + P \left( \frac{|S_n - \mathbb{E} \tilde{S}_n|}{b_n} > \varepsilon \right).
\]

We show that each of the two terms on the right hand side goes to 0 as \( n \) goes to \( \infty. \)

For the first term, since \( S_n \neq \tilde{S}_n \) implies that for some \( k, X_{n,k} \neq \tilde{X}_{n,k} \) which in turn implies that \(|X_{n,k}| > b_n\), by the union bound, we have,
\[
P \left( S_n \neq \tilde{S}_n \right) \leq \sum_{k=1}^{n} P \left( X_{n,k} \neq \tilde{X}_{n,k} \right) \leq \sum_{k=1}^{n} P \left( |X_{n,k}| > b_n \right) \xrightarrow{n \to \infty} 0,
\]
by (i). It remains to handle the second term. By Chebyshev’s inequality, the independence of the \( \tilde{X}_{n,k} \) and a simple bound \( \text{Var}(Y) \leq \mathbb{E}Y^2 \), we get
\[
P \left( \frac{|S_n - \mathbb{E} \tilde{S}_n|}{b_n} > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E} \left( \frac{|S_n - \mathbb{E} \tilde{S}_n|}{b_n} \right)^2 = \frac{1}{\varepsilon^2 b_n^2} \text{Var}(S_n) = \frac{1}{\varepsilon^2 b_n^2} \sum_{k=1}^{n} \text{Var}(\tilde{X}_{n,k}) \leq \frac{1}{\varepsilon^2 b_n^2} \sum_{k=1}^{n} \mathbb{E} \tilde{X}^2_{n,k}.
\]

The right hand side goes to 0 as \( n \) goes to \( \infty \) thanks to (ii). This finishes the proof. \( \square \)

**Proof of Theorem 7.6.** We use Lemma 7.8 with \( X_{n,k} = X_k \) and \( b_n = n \). Then
\[ \tilde{X}_{n,k} = X_k 1_{|X_k| \leq n} \]
which has the same law as \( X_1 1_{|X_1| \leq n} \). It suffices to check (i) and (ii) of Lemma 7.8 because its assertion is exactly (7.2), due to the fact that
\[ \mathbb{E} \tilde{S}_n = \sum_{k=1}^{n} \mathbb{E} X_k 1_{|X_k| \leq n} = n \mu_n. \]
For (i), we simply have
\[ \sum_{k=1}^{n} P(|X_{n,k}| > b_n) = \sum_{k=1}^{n} P(|X_k| > n) = nP(|X_1| > n) \longrightarrow 0, \]
by (7.1). For (ii), we have
\[ b_n^{−2} \sum_{k=1}^{n} \mathbb{E}X_{n,k}^2 = \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{E}X_k^2 1_{|X_k| \leq n} = \frac{1}{n} \mathbb{E}X_1^2 1_{|X_1| \leq n}. \]

We compute
\[ \mathbb{E}X_1^2 1_{|X_1| \leq n} = \mathbb{E} \int_{0}^{\infty} 2t 1_{(t \leq |X_1|, |X_1| \leq n)} dt \leq \int_{0}^{n} 2tP(|X_1| > t) dt. \]

Let \( f(t) = tP(|X_1| > t) \). It thus remains to show that
\[ \frac{1}{n} \int_{0}^{n} f(t) dt \longrightarrow 0. \]
This is a consequence of two properties of \( f \): \( f(t) \leq t \) and \( f(t) \longrightarrow 0 \) (by (7.1)). These in turn imply that \( M = \sup f < \infty \). Thus the following standard Cesàro-type lemma finishes the proof.

7.9 Lemma. Let \( f : [0, +\infty) \rightarrow [0, +\infty) \) be a bounded function with \( f(t) \longrightarrow 0 \). Then
\[ \frac{1}{n} \int_{0}^{n} f(t) dt \longrightarrow 0. \]

Proof. Let \( M = \sup f \). We fix \( \varepsilon > 0 \) and choose \( L \) such that \( f(t) < \varepsilon \) for all \( t > L \). Then,
\[ \frac{1}{n} \int_{0}^{n} f(t) dt = \frac{1}{n} \left( \int_{0}^{L} f + \int_{L}^{n} f \right) \leq \frac{1}{n} \left( LM + (n - L) \varepsilon \right) < \frac{LM}{n} + \varepsilon < 2\varepsilon, \]
provided that \( n > \frac{LM}{\varepsilon} \). This finishes the proof.

7.10 Remark. The same argument gives a similar result for sequences: if for a sequence \((a_n)\) of real numbers we have \( a_n \rightarrow a \), then
\[ \frac{a_1 + \cdots + a_n}{n} \rightarrow a. \]

7.11 Remark. Our weak law of large numbers – Theorem 7.6 – in particular gives the following: if \( X_1, X_2, \ldots \) are i.i.d. random variables with \( \mathbb{E}|X_1| < \infty \), then
\[ \frac{S_n}{n} - \mathbb{E}X_1 \overset{p}{\longrightarrow} 0 \]
(which will be strengthen to a.s. convergence in the next section, which is the content of the strong law of large numbers). Indeed,
\[ tP(|X_1| > t) = \mathbb{E}(t1_{|X_1| > t}) \leq \mathbb{E}(|X_1|1_{|X_1| > t}) \longrightarrow 0 \]
\( \overset{t \to \infty}{\longrightarrow} 0 \).
by Lebesgue’s dominated convergence theorem: pointwise $|X_1|1_{X_1 > t} \xrightarrow{t \to \infty} 0$ and $|X_1|$ is an integrable majorant (see also Exercise 4.4). Moreover, similarly,

$$\mu_n = E X_1 1_{X_1 \leq n} \xrightarrow{n \to \infty} E X_1,$$

which combined with the conclusion of Theorem 7.6,

$$\frac{S_n}{n} - \mu_n \xrightarrow{p} 0,$$

gives

$$\frac{S_n}{n} - E X_1 \xrightarrow{p} 0.$$

7.12 Remark. Recall Exercise 4.5: if $\lim_{t \to \infty} t^p \mathbb{P} (|X_1| > t) = 0$, then for every $0 < \delta < 1$, we have $\mathbb{E} |X_1|^{1-\delta} < \infty$. This shows that assumption (7.1) of Theorem 7.6 “almost” implies that $X_1$ is integrable. An example of a random variable with $\mathbb{P} (|X_1| > t) = \frac{1}{t \log t}$, $t > e$, shows that the weak law still holds ((7.1) is fulfilled), even though the expectation does not exist. Note that if $X_1$ is a standard Cauchy random variable, then for $t > 1$, we have

$$t^p \mathbb{P} (|X_1| > t) = 2t \int_t^{\infty} \frac{dx}{\pi(1 + x^2)} \geq 2t \int_t^{\infty} \frac{dx}{2\pi x^2} = \frac{1}{\pi},$$

so (7.1) does not hold. As discussed in Example 7.1, in this case no reasonable law-of-large-numbers-type convergence should hold.

7.13 Example. We shall describe the so-called St. Petersburg paradox. Let $X_1, X_2, \ldots$ be i.i.d. random variables with the following discrete distribution

$$\mathbb{P} (X_1 = 2^k) = 2^{-k}, \quad k = 1, 2, \ldots$$

Such distribution models this simple casino game: you sit down and they toss a coin until the first head shows up, which finishes the game and your pay is $2^k$ dollars, where $k$ is the number of tosses. The expected value of your payout is thus

$$\mathbb{E} X_1 = \frac{1}{2} \cdot 2 + \frac{1}{2^2} \cdot 2^2 + \cdots = +\infty.$$

How much should the casino charge for this game? Is there any fair charge? (Obviously, they cannot charge “$+\infty$”.) Suppose you want to play $n$ games, $n$ is large. Your payout after $n$ games is

$$S_n = X_1 + \cdots + X_n.$$

A fair charge per game should be a “typical value” of $S_n/n$. we cannot apply Theorem 7.6. Instead, we can estimate it using Lemma 7.8 which gives us extra flexibility in the choice of the normalising constant $b_n$. We want to choose $b_n$ as small as possible with conditions (i) and (ii) of the lemma being satisfied. Since

$$\mathbb{P} (X_1 \geq 2^m) = \sum_{j=0}^{\infty} 2^{-j} = 2^{-m+1},$$

72
the quantity in (i) for \( b_n = 2^{m_n} \) equals

\[ n \mathbb{P}(X_1 \geq b_n) = n \cdot 2^{-m_n + 1}. \]

If we thus choose \( m_n = \log_2 n + \omega_n \) with \( 0 < \omega_n \to \infty \) slowly and such that \( m_n \) is an integer, we get that \( n \mathbb{P}(X_1 > b_n) \to 0 \). Moreover, for the quantity in (ii), since \( b_n = n \omega_n \sum_{j \leq m_n} 2^j \cdot 2^{-j} < n 2^{m_n + 1} = \frac{2n}{b_n} = \frac{2}{2^{\omega_n}} \to 0. \)

Thus, by Lemma 7.8,

\[ \frac{S_n - \mathbb{E} \tilde{S}_n}{b_n} \xrightarrow{p} 0. \]

It remains to find \( \mathbb{E} \tilde{S}_n \). We have,

\[ \mathbb{E} \tilde{S}_n = n \mathbb{E} X_1 1_{X_1 \leq b_n} = n \sum_{j \leq m_n} 2^j \cdot 2^{-j} = nm_n. \]

Consequently, choosing \( \omega_n \) to be asymptotically \( \frac{\omega_n}{\log_2 \log_2 n} \), that is such that \( \frac{\omega_n}{\log_2 \log_2 n} \to 1 \), we get

\[ \frac{S_n - \mathbb{E} \tilde{S}_n}{b_n} = \frac{S_n - n \lfloor \log_2 n \rfloor}{n \cdot 2^{\omega_n}} = \frac{S_n}{n \cdot 2^{\omega_n}} - \lfloor \log_2 n \rfloor, \]

thus,

\[ \frac{S_n}{n \log_2 n} \xrightarrow{p} 1. \]

As a result, \( S_n / n \) is typically like \( \log_2 n \), so a fair charge for playing \( n \) games should be \( \log_2 n \) per game.

### 7.2 Strong law of large numbers

Strong laws of large numbers concern a.s. convergence. The following simple lemma turns out to be quite useful in such situations (see Exercises 6.31 and 6.32).

#### 7.14 Lemma

Let \( X_1, X_2, \ldots \) be a sequence of random variables such that for every \( \varepsilon \), we have

\[ \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty. \]  

(7.3)

Then,

\[ X_n \xrightarrow{a.s.} 0. \]

This also holds if for some sequence \( \varepsilon_1, \varepsilon_2, \ldots \) of positive numbers convergent to 0, we have

\[ \sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon_n) < \infty. \]

(7.4)

73
Proof. From (7.3), by the first Borel-Cantelli lemma, we get
\[ \forall \varepsilon > 0 \quad \mathbb{P}(|X_n| > \varepsilon \text{ for infinitely many } n) = 0, \]
or, equivalently,
\[ \forall \varepsilon > 0 \quad \mathbb{P}(|X_n| \leq \varepsilon \text{ eventually}) = 1. \]
Since the intersection of countably many certain events is a certain event, we get
\[ \mathbb{P}\left(\forall k = 1, 2, \ldots |X_n| \leq \frac{1}{k} \text{ eventually}\right) = 1. \]
By the definition of a limit, this implies that \( \mathbb{P}(X_n \to 0) = 1 \). Similarly, by (7.4), we get
\[ \mathbb{P}(|X_n| \leq \varepsilon_n \text{ eventually}) = 1. \]
Since \( e_n \to 0 \), by the sandwich theorem, \( X_n \to 0 \) with probability 1.

As a warm-up we show the strong law of large numbers under the generous assumption of a finite 4th moment. Even though we shall not need this result to prove it under optimal assumptions, the technique employed here of analysing high enough moments is quite useful and important.

7.15 Theorem. Let \( X_1, X_2, \ldots \) be independent random variables such that for all \( i \), \( \mathbb{E}|X_i|^4 \leq C \) and \( \mathbb{E}X_i = \mu \) for some constants \( C > 0 \) and \( \mu \in \mathbb{R} \). Then
\[ \frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s. \ n \to \infty} \mu. \]

Proof. Without loss of generality, we can assume that \( \mu = 0 \) (otherwise, we consider \( X_i - \mu \); for the assumption, we can use the triangle inequality, \( \|X_i\|_4 \leq \|X_i\|_4 + \mu \)).
We have,
\[ \mathbb{E}(X_1 + \cdots + X_n)^4 = \sum_{i,j,k,l=1}^{n} \mathbb{E}X_iX_jX_jX_k. \]
There are 5 types of terms: \( \mathbb{E}X_i^4, \mathbb{E}X_i^2X_j^2, \mathbb{E}X_iX_jX_k^2, \mathbb{E}X_iX_jX_kX_l \) with \( i, j, k, l \) distinct here. By independence and \( \mu = 0 \), the last 3 types vanish, thus
\[ \mathbb{E}(X_1 + \cdots + X_n)^4 = \sum_{i=1}^{n} \mathbb{E}X_i^4 + 3 \sum_{i \neq j} \mathbb{E}X_i^2\mathbb{E}X_j^2 \leq nC + 3n^2C \leq 4Cn^2, \]
where we use in the estimate that \( \mathbb{E}X_i^2 = \|X_i\|_2^2 \leq \|X_i\|_4^2 \leq C^{1/2} \) (by the monotonicity of moments – Example 6.4). Consequently, for \( \varepsilon > 0 \), by Chebyshev’s inequality,
\[ \mathbb{P}\left(\frac{|X_1 + \cdots + X_n|}{n} > \varepsilon \right) \leq \frac{1}{\varepsilon^4} \frac{\mathbb{E}(X_1 + \cdots + X_n)^4}{n^4} \leq \frac{1}{\varepsilon^4} \frac{4C}{n^2}. \]
Lemma 7.14 finishes the proof.

Our major goal here is to show Etemadi’s strong law of large numbers which assumes only pairwise independence.
7.16 Theorem (Etemadi). Let $X_1, X_2, \ldots$ be pairwise independent identically distributed random variables such that $\mathbb{E}|X_1| < \infty$. Let $\mu = \mathbb{E}X_1$. Then,

$$
\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s.} \mu.
$$

Proof. Since $X_n = X_n^+ - X_n^-$ and both the positive part $X_n^+$ and the negative part $X_n^-$ of $X_n$ satisfy the same assumptions as $X_n$, we can assume that $X_n \geq 0$ for every $n$.

As we have seen in Theorem 7.15, it is useful to be able to control higher moments to prove a.s. convergence. Since here we only assume existence of the first moment, we employ the technique of truncating (as in the weak law – Theorem 7.6). We divide the whole proof into several steps.

**Step I (truncation).** Let

$$
\tilde{X}_k = X_k 1_{|X_k| \leq k}, \quad k \geq 1
$$

and

$$
\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n, \quad \tilde{\tilde{S}}_n = \tilde{X}_1 + \cdots + \tilde{X}_n.
$$

We claim that for $\frac{\tilde{S}_n}{n} \xrightarrow{a.s.} \mu$, it suffices to show that

$$
\frac{\tilde{\tilde{S}}_n}{n} \xrightarrow{a.s.} \mu. \quad (7.5)
$$

We have,

$$
\sum_{k=1}^{\infty} \mathbb{P}(X_k \neq \tilde{X}_k) = \sum_{k=1}^{\infty} \mathbb{P}(|X_k| \geq k) = \sum_{k=1}^{\infty} \mathbb{P}(|X_1| \geq k) \leq \int_0^{\infty} \mathbb{P}(|X_1| \geq t) \, dt = \mathbb{E}|X_1| < \infty.
$$

Thus, by the first Borel-Cantelli lemma,

$$
\mathbb{P}(X_n = \tilde{X}_n \text{ eventually}) = 1.
$$

Since on the event “$X_n = \tilde{X}_n$ eventually”, we have

$$
\left| \frac{X_1 + \cdots + X_n}{n} - \frac{\tilde{X}_1 + \cdots + \tilde{X}_n}{n} \right| \leq \frac{R}{n}
$$

for some (random) $R$, our claim about (7.5) follows.

**Step II (variance bounds).** Here we show that

$$
\sum_{k=1}^{\infty} \frac{1}{k^2} \text{Var}(\tilde{X}_k) < \infty. \quad (7.6)
$$
First note that
\[ \text{Var}(\tilde{X}_k) \leq \mathbb{E} \tilde{X}_k^2 = \int_0^\infty 2t \mathbb{P}(|\tilde{X}_k| \geq t) \, dt \]
\[ = \int_0^k 2t \mathbb{P}(|\tilde{X}_k| \geq t) \, dt + \int_k^\infty 2t \mathbb{P}(|\tilde{X}_k| \geq t) \, dt \]
\[ \leq \int_0^k 2t \mathbb{P}(|X_k| \geq t) \, dt \]
\[ = \int_0^k 2t \mathbb{P}(|X_1| \geq t) \, dt. \]

Thus,
\[ \sum_{k=1}^\infty \frac{1}{k^2} \text{Var}(\tilde{X}_k) \leq \sum_{k=1}^\infty \frac{1}{k^2} \int_0^k 2t \mathbb{P}(|X_1| \geq t) \, dt \]
\[ = \sum_{k=1}^\infty \int_0^\infty \frac{1}{k^2} 1_{t<k} \cdot 2t \mathbb{P}(|X_1| \geq t) \, dt \]
\[ = \int_0^\infty \left( \sum_{k>t} \frac{1}{k^2} \right) \cdot 2t \mathbb{P}(|X_1| \geq t) \, dt. \]

It is an elementary exercise to show that
\[ \sum_{k>t} \frac{1}{k^2} \leq \frac{2}{t}. \]
Then the right hand side get upper-bounded by
\[ 4 \int_0^\infty \mathbb{P}(|X_1| \geq t) \, dt = 4 \mathbb{E}|X_1|, \]
so (7.6) follows.

**Step III (convergence on a subsequence).** Fix \( \alpha > 1 \). Let \( k_n = \lfloor \alpha^n \rfloor \). Our goal in this step is to show that
\[ \frac{\tilde{S}_{k_n}}{k_n} \xrightarrow{\text{a.s.}} \mu, \]
that is (7.5) holds on the subsequence \( k_n \). In the next step, thanks to the monotonicity of \( S_n \) (recall we assume that the \( X_n \) are nonnegative), we will extend this from the subsequence \( k_n \) to the convergence on all the terms.

Fix \( \epsilon > 0 \). We have,
\[ \sum_{n=1}^\infty \mathbb{P} \left( \left| \frac{\tilde{S}_{k_n}}{k_n} - \mathbb{E} \tilde{S}_{k_n} \right| > \epsilon \right) \leq \epsilon^{-2} \sum_{n=1}^\infty \text{Var}(\tilde{S}_{k_n}) = \epsilon^{-2} \sum_{n=1}^\infty \frac{1}{k_n^2} \sum_{j=1}^{k_n} \text{Var}(\tilde{X}_j), \]
where in the last equality we use pairwise independence. Changing the order of summation gives,
\[ \epsilon^{-2} \sum_{n=1}^\infty \frac{1}{k_n^2} \sum_{j=1}^{k_n} \text{Var}(\tilde{X}_j) = \epsilon^{-2} \sum_{j=1}^\infty \text{Var}(\tilde{X}_j) \sum_{n:k_n \geq j} \frac{1}{k_n^2}. \]
By a simple estimate \( \alpha^n \geq k_n \geq \frac{\alpha^n}{2} \),
\[ \sum_{n:k_n \geq j} \frac{1}{k_n^2} \leq \sum_{n:2^n \geq j} \frac{4}{\alpha^{2n}} \leq \frac{4}{j^2} \frac{1}{1 - \alpha^{-2}}, \]
and, as a result,
\[ \sum_{n=1}^\infty \mathbb{P} \left( \left| \frac{\tilde{S}_{k_n}}{k_n} - \mathbb{E} \tilde{S}_{k_n} \right| > \epsilon \right) \leq \frac{4}{\epsilon^2(1-\alpha^{-2})} \sum_{j=1}^\infty \frac{1}{j^2} \text{Var}(\tilde{X}_j) < \infty., \]
by (7.6). Therefore, by virtue of Lemma 7.14,

$$\frac{\hat{S}_{k_n} - E\hat{S}_{k_n}}{k_n} \xrightarrow{n \to \infty} 0.$$ 

Moreover,

$$\frac{E\hat{S}_{k_n}}{k_n} = \frac{E\hat{X}_1 + \cdots + E\hat{X}_{k_n}}{k_n} = \frac{EX_1 + \cdots + EX_{k_n} - EX_11_{X_1 > k_n}}{k_n} \xrightarrow{n \to \infty} \mu$$

because $EX_11_{X_1 > n} \to 0$ as $n \to 0$ (recall the Cesàro-type lemma – Remark 7.10).

**Step IV (convergence on all terms).** For every positive integer $m$, there is $n$ such that $k_n \leq m < k_{n+1}$ and then

$$\frac{\hat{S}_{k_n}}{k_{n+1}} \leq \frac{\hat{S}_m}{m} \leq \frac{\hat{S}_{k_{n+1}}}{k_n}$$

(because $\hat{X}_j \geq 0$ for every $j$). Thus, for every $\alpha > 1$, with probability 1,

$$\mu^{-1} \leq \liminf_{n \to \infty} \frac{\hat{S}_{k_n}}{k_{n+1}} \leq \liminf_{m \to \infty} \frac{\hat{S}_m}{m} \leq \limsup_{m \to \infty} \frac{\hat{S}_m}{m} \leq \limsup_{n \to \infty} \frac{\hat{S}_{k_{n+1}}}{k_n} = \mu \alpha.$$ 

Taking, say $\alpha = 1 + \frac{1}{l}$ and letting $l \to \infty$, we get that with probability 1,

$$\mu \leq \liminf_{m \to \infty} \frac{\hat{S}_m}{m} \leq \limsup_{m \to \infty} \frac{\hat{S}_m}{m} \leq \mu.$$ 

This shows (7.5) and finishes the proof.

**7.17 Remark.** The assumption of integrability in the strong law of large numbers, Theorem 7.16, is necessary: if $X_1, X_2, \ldots$ are i.i.d. random variables such that there is a constant $c \in \mathbb{R}$ for which $P (\lim_{n \to \infty} X_1 + \cdots + X_n = c) > 0$, then $E|X_1| < \infty$ and $c = EX_1$. We leave the proof as an exercise.

**7.18 Remark.** If we know that the expectation is infinite, the strong law in some sense still holds, but the limit is also infinite: if $X_1, X_2, \ldots$ are i.i.d. random variables such that one of the expectations $EX_1^+, EX_1^-$ is $+\infty$ and the other one is finite, then $\limsup_{n \to \infty} \frac{|X_1 + \cdots + X_n|}{n} = +\infty$ a.s. We leave the proof as an exercise.

**7.19 Remark.** Let $(\Omega_0, F_0, P_0)$ be the infinite product of a probability space $(\Omega, F, P)$. Fix an event $A \in F$. By the strong law of large numbers, for $P_0$-a.e. point $\omega = (\omega_1, \omega_2, \ldots) \in \Omega_0$, we have

$$\frac{1_A(\omega_1) + \cdots + 1_A(\omega_n)}{n} \xrightarrow{n \to \infty} E1_A = P (A).$$

In other words, the probability of $A$ is the limit of its frequency as the number of trials goes to $\infty$. This justifies the so-called frequentist definition of probability.

We refer to the exercises for additional extensions and applications of the strong law of large numbers.
7.3 Exercises

1. Let $X$ be a random variable and let $X'$ be its independent copy. Show that for every $t \geq 0$,
   \[
P (|X - X'| > 2t) \leq 2P (|X| > t).
   \]
   Moreover, if $a \geq 0$ is chosen such that $P (X \leq a) \geq u$ and $P (X \geq -a) \geq u$ for some $u \in [0, 1]$, then
   \[
P (|X - X'| > t) \geq uP (|X| > t + a).
   \]
   In particular, if $m$ is a median of $X$, then
   \[
P (|X - X'| > t) \geq \frac{1}{2}P (|X - m| > t).
   \]

2. Let $X_1, \ldots, X_n$ be independent symmetric random variables, that is $X_i$ has the same distribution as $-X_i$. Then for every $t \geq 0$,
   \[
P (|X_1 + \ldots + X_n| > t) \geq \frac{1}{2}P \left( \max_{j \leq n} |X_j| > t \right).
   \]
   In particular, if the $X_i$ are identically distributed, then
   \[
P (|X_1 + \ldots + X_n| > t) \geq \frac{1}{2} \left[ 1 - \exp \left\{ -n \mathbb{P} (|X_1| > t) \right\} \right].
   \]

3. Using the symmetrisation from Exercise 7.1 and inequalities from Exercise 7.2, justify Remark 7.7.

4. Let $X_1, X_2, \ldots$ be independent random variables with
   \[
P (X_n = n + 1) = P (X_n = -(n + 1)) = \frac{1}{2(n + 1) \log(n + 1)},
   \]
   \[
P (X_n = 0) = 1 - \frac{1}{(n + 1) \log(n + 1)}, \quad n \geq 1.
   \]
   Show that $(X_n)_{n=1}^\infty$ satisfies the weak law of large numbers, that is $\frac{X_1 + \ldots + X_n}{n}$ converges in probability. Show that $\sum_{n=1}^\infty P (|X_n| > n) = \infty$ and conclude that $(X_n)_{n=1}^\infty$ does not satisfy the strong law of large numbers.

5. Let $X_1, X_2, \ldots$ be i.i.d. integrable random variables with distribution function $F$. Define the sequence of empirical distribution functions by
   \[
   F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_i \leq x\}}
   \]
   (which are random). Show that for every $x \in \mathbb{R}$, we have $P (F_n(x) \to F(x)) = 1$.

6. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $P (|X_i| < 1) = 1$. Show that $X_1 X_2 \cdots X_n$ converges to $0$ a.s. and in $L_1$.

7. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter $\lambda$. Show that
a) \( Y_n = \frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2} \)

b) \( Z_n = \frac{X_1X_2 + X_2X_3 + \cdots + X_nX_{n+1}}{n} \)

converge a.s. and find their limits.

8. Let \( X_1, X_2, \ldots \) be i.i.d. random variables with density \( g \) which is positive. Show that for every continuous function \( f \) such that \( \int_{\mathbb{R}} |f| < \infty \), we have

\[
\frac{1}{n} \sum_{i=1}^{n} f(X_i) \xrightarrow{a.s.} \int_{\mathbb{R}} f.
\]

(This provides a method of numerical integration.)

9. Let \( f \) be a continuous function on \([0, 1]\) taking values in \([0, 1]\). Let \( X_1, Y_1, X_2, Y_2, \ldots \) be independent random variables uniformly distributed on \([0, 1]\). Let \( Z_i = 1_{\{f(X_i) > Y_i\}} \).

Show that 

\[
\frac{1}{n} \sum_{i=1}^{n} Z_i \xrightarrow{a.s.} \int_{0}^{1} f.
\]

10. Let \( X_1, X_2, \ldots \) be i.i.d. random variables such that \( P(X_i = 1) = p = 1 - P(X_i = -1) \) with \( \frac{1}{2} < p < 1 \). Let \( S_n = X_1 + \cdots + X_n \) (a random walk with a drift to the right).

Show that 

\[
S_n \xrightarrow{a.s.} \infty.
\]

11. Find

\[
\lim_{n \to \infty} \int_{0}^{1} \cdots \int_{0}^{1} \frac{\sqrt{x_1^2 + \cdots + x_n^2}}{x_1 + \cdots + x_n} \, dx_1 \ldots dx_n
\]

(or show the limit does not exist).

12. Find

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \int_{0}^{1} \cdots \int_{0}^{1} \sqrt{x_1^2 + \cdots + x_n^2} \, dx_1 \ldots dx_n
\]

(or show the limit does not exist).

13. Suppose that \( f \) is a continuous function on \([0, 1]\). Find

\[
\lim_{n \to \infty} \int_{0}^{1} \cdots \int_{0}^{1} f(\sqrt{x_1 + \cdots + x_n}) \, dx_1 \ldots dx_n
\]

(or show the limit does not exist).

14. Let \( X_1, X_2, \ldots \) be i.i.d. random variables uniform on \([-1, 1]\). Does the sequence

\[
\frac{X_1 + X_2^2 + \cdots + X_n^2}{n}, \quad n = 1, 2, \ldots,
\]

converge a.s.?

15. We say that a number \( x \in [0, 1] \) is **simply normal** in an integer base \( b \geq 2 \), if its sequence of digits is **uniform** in the sense that each of the digits \( \{0, 1, \ldots, b-1\} \) occurs with the same density \( 1/b \), that is, formally, for every \( d \in \{0, 1, \ldots, b-1\} \), we have

\[
\lim_{n \to \infty} \frac{1}{n} |\{j \leq n, \ d_j(x) = d\}| = \frac{1}{b},
\]

where \( x = \sum_{k=1}^{\infty} \frac{d_k(x)}{b^k} \) with \( d_1(x), d_2(x), \ldots \in \{0, 1, \ldots, b-1\} \). Show Borel’s theorem: **almost every number** \( x \in [0, 1] \) is **simply normal in every base**. On the other hand,
it is not known whether \( e - 2 \), \( \pi - 3 \), or \( \sqrt{2} - 1 \) are simply normal in any given base (although, it is of course widely believed so). It is not even known whether \( \sqrt{2} \) has infinitely many 5’s in its decimal expansion!

16. We say that a random variable \( X \) is \textit{singular} if its cumulative distribution function is continuous and there is a subset \( A \) of \( \mathbb{R} \) of Lebesgue measure 0 with \( \mathbb{P}(X \in A) = 1 \). Fix \( p \in (0, 1), p \neq \frac{1}{2} \). Let \( X_1, X_2, \ldots \) be i.i.d. random variables with \( \mathbb{P}(X_i = 1) = p, \mathbb{P}(X_i = 0) = 1 - p, i = 1, 2, \ldots \). Show that \( Y = \sum_{n=1}^{\infty} \frac{X_n}{2^n} \) is singular. What is the distribution of \( Y \) when \( p = \frac{1}{2} \)?

17. Let \( b \) be an integer, \( b \geq 3 \). Let \( X_1, X_2, \ldots \) be i.i.d. random variables with \( \mathbb{P}(X_i = 1) = \mathbb{P}(X_i = 0) = \frac{1}{2}, i = 1, 2, \ldots \). Show that \( Y = \sum_{n=1}^{\infty} \frac{X_n}{b^n} \) is singular (in particular, for \( b = 3 \) we get the distribution from 2.22).


19. Justify Remark 7.18

20. Give an example of a sequence of i.i.d. random variables \( X_1, X_2, \ldots \) for which \( \sum_{n=1}^{\infty} \frac{X_n}{n} \) converges in probability but not a.s.

21. Let \( f: [0, +\infty) \to \mathbb{R} \) be a continuous bounded function. Define its Laplace transform

\[
L(t) = \int_0^\infty e^{-tx} f(x) dx, \quad t > 0.
\]

Show that \( L \) is \( C^\infty \) and \( L^{(n)}(t) = \int_0^\infty (-x)^n e^{-tx} f(x) dx \). Let \( S_n \) be the sum of \( n \) i.i.d. exponential random variables with parameter \( t \). Show that

\[
\mathbb{E}f(S_n) = (-1)^{n-1} \frac{t^n L^{(n-1)}(t)}{(n-1)!}
\]

and deduce the following inversion formula for the Laplace transform

\[
f(y) = \lim_{n \to \infty} (-1)^{n-1} \frac{(n/y)^n L^{(n-1)}(n/y)}{(n-1)!}, \quad y > 0.
\]
8 Weak convergence

8.1 Definition and equivalences

We start with a general definition. We say that a sequence \((\mu_n)\) of probability measures on a metric space \((E, d)\) converges \textit{weakly} to a Borel probability measure \(\mu\) on \((E, d)\) if for every bounded continuous function \(f: E \to \mathbb{R}\), we have

\[
\int_E f d\mu_n \xrightarrow{n \to \infty} \int_E f d\mu.
\]

The following equivalences explain weak convergence on the level of sets. It is sometimes referred to as the Portmanteau theorem.

8.1 Theorem. Let \(\mu, \mu_1, \mu_2, \ldots\) be Borel probability measures on a metric space \((E, d)\). The following are equivalent

(i) \(\mu_n \to \mu\) weakly,
(ii) \(\limsup \mu_n(F) \leq \mu(F)\) for all closed sets \(F\) in \(E\),
(iii) \(\liminf \mu_n(G) \geq \mu(G)\) for all open sets \(G\) in \(E\),
(iv) \(\mu_n(A) \to \mu(A)\) for all Borel sets \(A\) in \(E\) with \(\mu(\partial A) = 0\).

Proof. (i) \(\Rightarrow\) (ii): Fix \(\varepsilon > 0\) and let

\[
g_\varepsilon(x) = \left(1 - \frac{1}{\varepsilon} d(x, F)\right)_+, \quad x \in E,
\]

where as usual \(d(x, F) = \inf \{d(x, y), y \in F\}\) is the distance from \(x\) to the set \(F\). We also define

\[
F_\varepsilon = \{x \in E, \ d(x, F) \leq \varepsilon\},
\]

the \(\varepsilon\)-enlargement of \(F\). Note that for every \(x \in E\),

\[
1_F(x) \leq g_\varepsilon(x) \leq 1_{F_\varepsilon}(x).
\]

Moreover, the function \(g_\varepsilon\) is bounded and continuous (it is \(1/\varepsilon\)-Lipschitz). Thus,

\[
\limsup \mu_n(F) = \limsup \int_E 1_F d\mu_n
\leq \limsup \int_E g_\varepsilon d\mu_n
= \int_E g_\varepsilon d\mu
\leq \int_E 1_{F_\varepsilon} d\mu
= \mu(F_\varepsilon),
\]

81
where in the second equality we used (i) and the inequalities follow from the pointwise bounds $1_F \leq g_\varepsilon \leq 1_{F_\varepsilon}$. Letting $\varepsilon \to 0$ in a decreasing way, we get $\mu(F_\varepsilon) \to \mu(F)$, by continuity of probability measures. This shows (ii).

(ii) $\Leftrightarrow$ (iii): We use complements.

(ii) & (iii) $\Rightarrow$ (iv): Let $A \subset E$ be a Borel set such that $\mu(\partial A) = 0$. Since $\text{cl}(A) = A \cup \partial A$ and $\text{int}(A) = A \setminus \partial A$, we get

$$\limsup_{\varepsilon \to 0} \mu_n(A) \leq \limsup_{\varepsilon \to 0} \mu_n(\text{cl}(A)) = \mu(\text{cl}(A)) = \mu(A),$$

$$\liminf_{\varepsilon \to 0} \mu_n(A) \geq \liminf_{\varepsilon \to 0} \mu_n(\text{int}(A)) = \mu(\text{int}(A)) = \mu(A).$$

These show that $\lim \mu_n(A) = \mu(A)$.

(iv) $\Rightarrow$ (ii): Let $F \subset E$ be a closed set and for $\varepsilon > 0$. We set as before $F_\varepsilon = \{x \in E, d(x, F) \leq \varepsilon\}$. Let $S_\varepsilon = \{x \in E, d(x, F) = \varepsilon\}$. Note that $\partial F_\varepsilon \subset S_\varepsilon$. Since the sets $S_\varepsilon$ are disjoint for different $\varepsilon$, for only countably many $\varepsilon$ we can have $\mu(S_\varepsilon) > 0$. Thus we can find a decreasing sequence $\varepsilon_k \to 0$ such that $\mu(S_{\varepsilon_k}) = 0$ for every $k$. Consequently, $\mu(\partial F_{\varepsilon_k}) = 0$ for every $k$. We get by (iv) that

$$\limsup_{\varepsilon \to 0} \mu_n(F) \leq \limsup_{\varepsilon \to 0} \mu_n(F_{\varepsilon_k}) = \mu(F_{\varepsilon_k})$$

and $\mu(F_{\varepsilon_k}) \to \mu(F)$ by continuity. These show (ii).

(iii) $\Rightarrow$ (i): Let $f : E \to \mathbb{R}$ be a bounded continuous function. Suppose $f \geq 0$ (otherwise, we consider $f - \inf f$). Using Fatou’s lemma and (iii) (sets $\{x \in E, f(x) > t\}$ are open), we get

$$\liminf_{\varepsilon \to 0} \int_E f d\mu_n = \liminf_{\varepsilon \to 0} \int_0^\infty \mu_n(\{x \in E, f(x) > t\}) dt$$

$$\geq \int_0^\infty \liminf_{\varepsilon \to 0} \mu_n(\{x \in E, f(x) > t\}) dt$$

$$\geq \int_0^\infty \mu(\{x \in E, f(x) > t\}) dt$$

$$= \int_E f d\mu.$$

Since this inequality holds for an arbitrary function, applying it to $-f$ and combining the two gives $\int_E f d\mu_n \to \int_E f d\mu$, as required.

The following equivalences show that even the smaller set of all Lipschitz functions captures the same.

8.2 Theorem. Let $\mu, \mu_1, \mu_2, \ldots$ be Borel probability measures on a metric space $(E,d)$. The following are equivalent
(i) \( \mu_n \to \mu \) weakly,

(ii) for every bounded uniformly continuous function \( f: E \to \mathbb{R} \), we have

\[
\int_E f \, d\mu_n \to \int_E f \, d\mu,
\]

(iii) for every bounded Lipschitz function \( f: E \to \mathbb{R} \), we have

\[
\int_E f \, d\mu_n \to \int_E f \, d\mu.
\]

Proof. The implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) are clear. To show (iii) \( \Rightarrow \) (i), we use condition (ii) of Theorem 8.1: let \( F \) be a closed set and for \( \varepsilon > 0 \), let

\[
g_\varepsilon(x) = \left(1 - \frac{1}{\varepsilon} \, d(x, F)\right)_+, \quad x \in E,
\]

which is a bounded Lipschitz function. Therefore using (iii), we can repeat verbatim the argument “(i) \( \Rightarrow \) (ii)” of the proof of Theorem 8.1 to show that \( \limsup \mu_n(F) \leq \mu(F) \).

We immediately get the following corollaries.

8.3 Corollary. Let \( \mu \) and \( \nu \) be Borel probability measures on a metric space \( (E, d) \). If

\[
\int_E f \, d\mu = \int_E f \, d\nu
\]

for every function \( f: E \to \mathbb{R} \) which is bounded and Lipschitz, then \( \mu = \nu \).

Proof. Letting \( \mu_1 = \mu_2 = \ldots = \mu \), we get by the assumption and Theorem 8.2 (iii) that \( \mu_n \to \nu \) weakly. Therefore, for every closed set \( F \), we have \( \mu(F) = \limsup \mu_n(F) \leq \nu(F) \). By symmetry, we get the reverse inequality as well, thus \( \mu(F) = \nu(F) \). Since the closed sets generate the Borel \( \sigma \)-algebra, by Dynkin’s theorem on \( \pi \)-\( \lambda \) systems, we get \( \mu = \nu \).

8.4 Corollary. Weak limits are uniquely determined, that is if \( \mu_n \to \mu \) and \( \mu_n \to \nu \) weakly, then \( \mu = \nu \).

Proof. It follows from Theorem 8.2 (iii) and the previous corollary.

Convergence in distribution of random variables

We say that a sequence of random variables \( (X_n) \) converges to a random variable \( X \) in distribution (or in law), denoted \( X_n \xrightarrow{d} X \), if \( \mu_{X_n} \to \mu_X \) weakly, that is for every continuous bounded function \( f: \mathbb{R} \to \mathbb{R} \), we have

\[
\mathbb{E} f(X_n) \to \mathbb{E} f(X).
\]
Of course, this definition easily extends to random variables taking values in a metric space. Note that this notion of convergence only depends on the law of random variables involved and not on their particular realisations as functions on a probability space (in fact, they can be defined on different probability spaces). Particularly, if \(X_n \xrightarrow{d} X\) and say \(X'\) is another random variable with the same distribution as \(X\) (i.e. \(\mu_{X'} = \mu_X\)), then we can also write \(X_n \xrightarrow{d} X'\).

In the case of real-valued random variables, there is an intuitive equivalent formulation in terms of distribution functions.

**8.5 Theorem.** A sequence \((X_n)\) of random variables converges in distribution to a random variable \(X\) if and only if

\[
F_{X_n}(t) \xrightarrow{n \to \infty} F_X(t)
\]

for every point of continuity of \(F_X\). (8.1)

**Proof.** \((\Rightarrow)\) For parameters \(t \in \mathbb{R}\) and \(\varepsilon > 0\) define the continuous bounded functions

\[
g_{t,\varepsilon}(x) = \begin{cases} 
1, & x \leq t, \\
1 - \frac{x-t}{\varepsilon}, & t < x \leq t + \varepsilon, \\
0, & x > t + \varepsilon.
\end{cases}
\]

The idea is that these functions are continuous approximations of indicator functions. We have,

\[
\limsup P(X_n \leq t) = \limsup E1_{\{X_n \leq t\}} \leq \limsup E g_{t,\varepsilon}(X_n) = E g_{t,\varepsilon}(X) \leq E1_{\{X \leq t + \varepsilon\}} = P(X \leq t + \varepsilon).
\]

Letting \(\varepsilon \to 0\) gives

\[
\limsup F_{X_n}(t) \leq F_X(t).
\]

On the other hand, since

\[
\liminf P(X_n \leq t) = \liminf E1_{\{X_n \leq t\}} \geq \liminf E g_{t-\varepsilon,\varepsilon}(X_n) = E g_{t-\varepsilon,\varepsilon}(X) \geq E1_{\{X \leq t - \varepsilon\}} = P(X \leq t - \varepsilon)
\]

after taking \(\varepsilon \to 0\), we get

\[
\liminf F_{X_n}(t) \geq F_X(t-).
\]

If \(t\) is a point of continuity of \(F_X\), \(F_X(t-) = F_X(t)\) and we obtain \(\lim F_{X_n}(t) = F_X(t)\), which means \(X_n \xrightarrow{d} X\).

\((\Leftarrow)\) We first show a lemma which allows us to relate condition (8.1) to a.s. convergence.

**8.6 Lemma.** If random variables \(X, X_1, X_2, \ldots\) satisfy (8.1), then there are random variables \(Y, Y_1, Y_2, \ldots\) such that \(Y_n\) has the same distribution as \(X_n\), \(Y\) has the same distribution as \(X\) and \(Y_n \to Y\) a.s.
Proof. Let $F_n = F_{X_n}$ be the distribution function of $X_n$ and let $F = F_X$ be the distribution function of $X$. Let $\Omega = (0, 1)$, $\mathcal{F}$ be the Borel subsets of $(0, 1)$ and $\mathbb{P}(\cdot)$ be uniform. For every $x \in (0, 1)$ define the “inverse” distribution functions

$$Y_n(x) = \sup\{y \in \mathbb{R}, \, F_n(y) < x\}$$

and similarly

$$Y(x) = \sup\{y \in \mathbb{R}, \, F(y) < x\}.$$

By the construction, $F_{Y_n} = F_n$ and $F_Y = F$. Note that $Y_n$ and $Y$ are nondecreasing right-continuous functions whose only discontinuities are jumps which happen at at most countably many points. If we let $\Omega_0$ be the set of points where $Y$ is continuous, then $\mathbb{P}(\Omega_0) = 1$. Fix $x \in \Omega_0$. We claim that $Y_n(x) \to Y(x)$, which then gives $Y_n \to Y$ a.s.

We have

1. $\lim \inf Y_n(x) \geq Y(x)$, for suppose $y < Y(x)$ is a continuity point of $F$; then $F(y) < x$ (since $x \in \Omega_0$), so for large $n$, $F_n(y) < x$ and by the definition of the supremum, $y \leq Y_n(x)$. Taking $\lim \inf$, we get $\lim \inf Y_n(x) \geq y$ for every $y < Y(x)$, so $\lim \inf Y_n(x) \geq Y(x)$.

2. $Y(x) \geq \lim \sup Y_n(x)$, for suppose $y > Y(x)$ is a continuity point of $F$; then $F(y) > x$, so for large $n$, $F_n(y) > x$ which gives $y \geq Y_n(x)$. Taking $\lim \sup$ finishes the argument.

Let $Y_n$ and $Y$ be as in Lemma 8.6, $Y_n \xrightarrow{a.s.} Y$. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Since we also have $f(Y_n) \xrightarrow{a.s.} f(Y)$, so by Lebesgue’s dominated convergence theorem ($f$ is bounded),

$$\mathbb{E}g(X_n) = \mathbb{E}g(Y_n) \to \mathbb{E}g(Y) = \mathbb{E}g(X).$$

\[ \square \]

8.7 Example. Let $\varepsilon$ be a symmetric random sign. Consider the sequence $(X_n)_{n=1}^\infty = (\varepsilon, -\varepsilon, \varepsilon, -\varepsilon, \ldots)$. Since $-\varepsilon$ has the same distribution as $\varepsilon$, we have $F_{X_n} = F_{\varepsilon}$ for every $n$, so $X_n \xrightarrow{d} \varepsilon$. On the other hand, the sequence $(X_n)$ does not converge in probability, for suppose $X_n \xrightarrow{p} X$ for some random variable $X$. Then for $n, m$ large enough $\mathbb{P}(|X_n - X_m| > 1) \leq \mathbb{P}(|X_n - X| > 1/2) + \mathbb{P}(|X - X_m| > 1/2) \leq 1/4$. Taking $n$ and $m$ of different parity, we get $\mathbb{P}(|X_n - X_m| > 1) = \mathbb{P}(|2\varepsilon| > 1) = 1$, a contradiction.

8.8 Example. Let $X$ be a random variable and consider the sequence $X_n = X + \frac{1}{n}$. For any reasonable definition of “convergence in distribution” we should have $X_n \to X$.

Note that for a fixed $t \in \mathbb{R}$, we have

$$\lim F_{X_n}(t) = \lim \mathbb{P}(X_n \leq t) = \lim \mathbb{P}\left(X \leq t - \frac{1}{n}\right) = F(t-),$$

85
which is \( F(t) \) if and only if \( t \) is a continuity point of \( F \). This explains why in the definition we make this exclusion.

Convergence in distribution is metrisable. For two distribution functions \( F, G : \mathbb{R} \to [0, 1] \), we let

\[
\rho(F, G) = \inf \{ \varepsilon > 0, \forall x \in \mathbb{R} \ G(x - \varepsilon) - \varepsilon \leq F(x) \leq G(x + \varepsilon) + \varepsilon \},
\]

which is called the Lévy metric. We naturally extend this definition for random variables, setting \( \rho(X, Y) = \rho(F_X, F_Y) \) for random variables \( X, Y \) defined on the same probability space.

**8.9 Theorem.** Let \( L_0 \) be the set of all random variables on a given probability space. Then \( \rho \) is a metric on \( L_0 \) and \( (L_0, \rho) \) is separable and complete. Moreover, \( X_n \to X \) in distribution if and only if \( \rho(X_n, X) \to 0 \).

We leave the proofs as exercises.

### 8.2 Relations to other notions of convergence and basic algebraic properties

Lemma 8.6 explains the relation between convergence in distribution and a.s. convergence. If the random variables are defined on the same probability space, then convergence in distribution is the weakest of all types of convergences we have seen.

**8.10 Theorem.** Let \( X, X_1, X_2, \ldots \) be random variables such that \( X_n \xrightarrow{d} X \). Then, we also have \( X_n \xrightarrow{p} X \).

**Proof.** Suppose the assertion does not hold. Then, there is a bounded continuous function \( f : \mathbb{R} \to \mathbb{R} \) and \( \varepsilon > 0 \) such that \( |\mathbb{E}f(X_n) - \mathbb{E}f(X)| > \varepsilon \) for infinitely many \( n \), say for \( n_1 < n_2 < \ldots \). By Theorem 6.16, there is a subsequence \( n_{k_i} \) such that \( X_{n_{k_i}} \) converges to \( X \) a.s., but then, by Lebesgue’s dominated convergence theorem, we get a contradiction with \( |\mathbb{E}f(X_{n_{k_i}}) - \mathbb{E}f(X)| > \varepsilon \). \( \square \)

We record basic algebraic properties and defer their proofs to exercises.

**8.11 Theorem.** Let \( (X_n), (Y_n) \) be sequences of random variables such that \( X_n \xrightarrow{d} X \) and \( Y_n \xrightarrow{d} c \) for some random variable \( X \) and a constant \( c \in \mathbb{R} \). Then

\[
X_n + Y_n \xrightarrow{n \to \infty} X + c
\]

and

\[
X_n Y_n \xrightarrow{n \to \infty} cX.
\]

86
8.12 Remark. There are examples showing that if the sequence \((Y_n)\) converges to a (non-constant) random variable \(Y\), then it is not true that \(X_n + Y_n \xrightarrow{n\to\infty} X + Y\), \(X_nY_n \xrightarrow{n\to\infty} XY\).

We finish with a simple fact concerning images of convergent sequences under “essentially” continuous maps.

8.13 Theorem. Let \(h: E \to E'\) be a Borel function between two metric spaces and let \(D_h\) be the set of its discontinuity points. If \(X, X_1, X_2, \ldots\) are \(E\)-valued random variables such that \(X_n \xrightarrow{n\to\infty} X\) and \(P(X \in D_h) = 0\), then \(h(X_n) \xrightarrow{n\to\infty} h(X)\).

Proof. We shall use condition (ii) of Theorem 8.1. Let \(F\) be a closed set in \(E'\). We have,

\[
\limsup P(h(X_n) \in F) = \limsup P(X_n \in h^{-1}(F)) \leq \limsup P(X \in \text{cl}(h^{-1}(F))) \\
\leq P(X \in \text{cl}(h^{-1}(F))),
\]

where the last inequality follows because \(X_n \xrightarrow{n\to\infty} X\). Since \(\text{cl}(h^{-1}(F)) \subset h^{-1}(F) \cup D_h\) and \(P(X \in D_h) = 0\), the right hand side equals \(P(X \in h^{-1}(F)) = P(h(X) \in F)\), showing that \(h(X_n) \xrightarrow{n\to\infty} h(X)\).

8.3 Compactness

Being able to extract convergent subsequences often helps. For real-valued random variables, we can work with their distribution functions to establish weak convergence. Since distribution functions are bounded and monotone, extracting convergent subsequences is always possible, as stated in the next theorem.

8.14 Theorem (Helly’s selection theorem). If \((F_n)\) is a sequence of distribution functions, then there is a subsequence \((F_{n_k})\) and a right-continuous nondecreasing function \(F: \mathbb{R} \to [0, 1]\) such that \(F_{n_k}(t) \xrightarrow{k\to\infty} F(t)\) for every point \(t\) of continuity of \(F\).

8.15 Remark. In general, \(F\) may not be a distribution function – it may happen that \(F(\infty) < 1\) or \(F(-\infty) > 0\).

Proof. To construct the desired subsequence we use a standard diagonal argument. Let \(q_1, q_2, \ldots\) be a sequence of all rationals. Since the sequence \(F_n(q_1)\) is bounded, it has a convergent subsequence, say \(F_{n_k^{(1)}}(q_1)\) converges to \(G(q_1)\). Then we look at the sequence \(F_{n_k^{(1)}}(q_2)\) which is bounded, so it has a convergent subsequence, say \(F_{n_k^{(2)}}(q_2)\) converges to \(G(q_2)\), etc. We obtain subsequences \((n_k^{(i)})\) such that \((n_k^{(i+1)})\) is a subsequence of \((n_k^{(i)})\) and \(F_{n_k^{(i)}}(q_l)\) converges to \(G(q_l)\). Choose the diagonal subsequence \(n_k = n_k^{(k)}\). Then \(F_{n_k^{(k)}}(q_l)\) converges to \(G(q_l)\) for every \(l\). The function \(G: \mathbb{Q} \to [0, 1]\) obtained as the limit is nondecreasing. We extend it to the nondecreasing function \(F: \mathbb{R} \to [0, 1]\) by \(F(x) = \inf\{G(q), q \in \mathbb{Q}, q > x\}, \quad x \notin \mathbb{Q}\).
The function $F$, as monotone, satisfies $F(x-) \leq F(x) \leq F(x+)$ for every $x$. At the points $x$, where $F$ is not right-continuous, we modify it and set $F(x) = F(x+)$ (there are at most countably many such points).

It remains to check that $F_{n_k}$ converges to $F$ at its points of continuity. Let $x$ be such a point and let $q,r$ be rationals such that $q < x < r$. Then

$$F(q) = G(q) = \liminf_k F_{n_k}(q) \leq \liminf_k F_{n_k}(x) \leq \limsup_k F_{n_k}(x) \leq \limsup_k F_{n_k}(r) = G(r) = F(r).$$

Letting $q,r \to x$, we get $F(q), F(r) \to F(x)$, so $\liminf_k F_{n_k}(x) = \limsup_k F_{n_k}(x) = F(x)$.

To capture when the limiting function is a distribution function of a random variable, we need the notion of tightness. A sequence $(X_n)$ of random variables is tight if for every $\varepsilon > 0$, there is $M > 0$ such that $\mathbb{P}(|X_n| \leq M) > 1 - \varepsilon$ for every $n$.

8.16 Remark. If there is $\delta > 0$ such that $C = \sup_n \mathbb{E}|X_n|^{\delta} < \infty$, then the sequence $(X_n)$ is tight. Indeed, by Chebyshev’s inequality,

$$\mathbb{P}(|X_n| > M) \leq M^{-\delta} \mathbb{E}|X_n|^{\delta} \leq \frac{C}{M^{\delta}}$$

which is less than $\varepsilon$ for $M$ large enough.

The main result of this section is the following compactness type result. It gives a necessary and sufficient condition for existence of convergent subsequences in distribution in terms of tightness.

8.17 Theorem. A sequence of random variables $(X_n)$ is tight if and only every subsequence $(X_{n_k})_k$ has a subsequence $(X_{n_{k_l}})_l$ which converges in distribution to some random variable.

Proof. Let $F_n$ be the distribution function of $X_n$.

$(\Rightarrow)$ By Helly’s theorem applied to $(F_{n_k})_k$, there is a subsequence $(F_{n_{k_l}})_l$ which converges to a right-continuous nondecreasing function $F : \mathbb{R} \to [0, 1]$ pointwise at the points of continuity of $F$. It remains to check that $F$ is a distribution function, that is $F(-\infty) = 0$ and $F(+\infty) = 1$. By tightness, there is $M > 0$ such that $F_n(M) - F_n(-M) > 1 - \varepsilon$, for every $n$ and we can further arrange that $-M$ and $M$ are points of continuity of $F$. Taking $n = n_{k_l}$ and letting $l \to \infty$, we get $F(M) - F(-M) \geq 1 - \varepsilon$. Since $\varepsilon$ is arbitrary and $F$ is monotone, this yields $F(-\infty) = 0$ and $F(+\infty) = 1$.

$(\Leftarrow)$ If $(X_n)$ is not tight, there is $\varepsilon > 0$ and an increasing sequence of indices $n_k$ such that $\mathbb{P}(|X_{n_k}| \leq k) \leq 1 - \varepsilon$ for every $k$. By the assumption, $X_{n_{k_l}} \xrightarrow{d} X$. Let $x < 0 < y$ be points of continuity of $F_X$. Then

$$F_X(y) - F_X(x) = \lim_l (F_{n_{k_l}}(y) - F_{n_{k_l}}(x)) \leq \limsup_l (F_{n_{k_l}}(k_l) - F_{n_{k_l}}(-k_l)) \leq 1 - \varepsilon.$$
Taking $x \to -\infty$ and $y \to \infty$ gives $1 \leq 1 - \varepsilon$, a contradiction.

8.18 Remark. Theorem 8.17 can be greatly generalised to the setting of arbitrary separable complete metric spaces (the so-called Polish spaces), which is called Prokhorov’s theorem. As we have seen, in the real-valued case, we take huge advantage of CDFs. In general, the proof is much more complicated. It can be also quite easily deduced from the Banach-Alaoglu’s concerning compactness of weak$^*$ convergence (which can be identified with the notion of weak convergence).
8.4 Exercises

1. Give an example of Borel probability measures $\mu, \mu_1, \mu_2, \ldots$ on $\mathbb{R}$ and a Borel set $B$ in $\mathbb{R}$ such that $\mu_n \to \mu$ weakly and $\mu_n(B) \not\to \mu(B)$.

2. Let $X_1, X_2, \ldots$ be random variables such that $P(X_n = \frac{k}{n}) = \frac{1}{n}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. Does the sequence $(X_n)$ converge in distribution? If yes, find the limiting distribution.

3. Let $U_1, U_2, \ldots$ be i.i.d. random variables uniformly distributed on $[0, 1]$. Let $X_n = \min\{U_1, U_2, \ldots\}$. Show that $nX_n$ converges in distribution to an exponential random variable with parameter one.

4. Suppose that $X, X_1, X_2, \ldots$ are nonnegative integer-valued random variables. Show that $X_n \xrightarrow{d} X$, if and only if $P(X_n = k) \to P(X = k)$, for every $k = 0, 1, 2, \ldots$.

5. For $p \in [0, 1]$, let $X_p$ be a Geometric random variable with parameter $p$. Show that the sequence $(\frac{1}{n}X_{1/n})$ converges in distribution to an exponential random variable with parameter 1.

6. Suppose that a sequence of random variables converges in distribution to a constant. Then it also converges in probability.


8. Prove Scheffé’s lemma: If $X_1, X_2, \ldots$ is a sequence of continuous random variables with densities $f_1, f_2, \ldots$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$ for some probability density $f$, then $\int_{\mathbb{R}} |f - f_n| \xrightarrow{n \to \infty} 0$. Conclude that then $X_n \xrightarrow{d} X$ for a random variable $X$ with density $f$ (in other words, pointwise convergence of densities implies convergence in distribution). Considering $f_n(x) = (1 + \cos(2\pi nx))1_{[0, 1]}(x)$, show that the converse statement does not hold.

9. Let $X_1, X_2, \ldots$ be i.i.d. random variables uniform on $\{1, 2, \ldots, n\}$. Let $N_n = \min\{l \geq 2, X_k = X_l \text{ for some } k < l\}$.

Show that $P(N_n > k) = \prod_{j=1}^{k-1} \left(1 - \frac{j}{n}\right)$, for every integer $k \geq 1$ (the birthday problem). For every $t \geq 0$ show that $\lim_{n \to \infty} P\left(\frac{N_n}{\sqrt{n}} > t\right) = e^{-t^2/2}$ and show that the sequence $(\frac{N_n}{\sqrt{n}})$ converges in distribution to a random variable with density $xe^{-x^2/2}1_{x \geq 0}$.

10. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter 1. Let $M_n = \max\{X_1, \ldots, X_n\}$. Show that $M_n - \log n$ converges in distribution to a random variable with the distribution function $e^{-e^{-x}}$, $x \in \mathbb{R}$. 

90
11. Let $U_1, \ldots, U_{2n+1}$ be i.i.d. random variables uniform on $[0, 1]$. Order them in a nondecreasing way and call the $n + 1$ term (the middle one) $M_n$. Show that $M_n$ has density $(2n + 1)\binom{2n}{n} x^n (1 - x)^n \mathbf{1}_{[0,1]}(x)$. Find $\mathbb{E} M_n$ and $\text{Var}(M_n)$. Show that $\sqrt{8n} (M_n - \frac{1}{2})$ converges in distribution to a standard Gaussian random variable.

12. Show that for positive $t$, $\int_t^{\infty} e^{-x^2/2} \, dx \leq \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}$ and $\int_t^{\infty} e^{-x^2/2} \, dx \geq \frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}$.

Conclude that for a standard Gaussian random variable $Z$ and positive $t$,

$$\lim_{t \to \infty} \frac{\mathbb{P}(Z > t)}{\frac{1}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}} = 1.$$ 

13. Let $X_1, X_2, \ldots$ be i.i.d. standard Gaussian random variables. For $n = 2, 3, \ldots$ let $b_n$ be such that $\mathbb{P}(X_1 > b_n) = \frac{1}{n}$. Show that $\lim_{n \to \infty} \frac{b_n}{\sqrt{2 \log n}} = 1$. Let $M_n = \max\{X_1, \ldots, X_n\}$. Show that $b_n(M_n - b_n)$ converges in distribution to a random variable with the distribution function $e^{-e^{-x}}$, $x \in \mathbb{R}$.

*Hint: Using Exercise 8.12, first show that for every $a \in \mathbb{R}$, $\lim_{t \to \infty} \frac{\mathbb{P}(Z > t + \frac{x}{2})}{\mathbb{P}(X_1 > t)} = e^{-a}$. 

14. Assume that $X_1, X_2, \ldots$ are i.i.d. standard Gaussian random variables. Define $M_n = \max\{X_1, \ldots, X_n\}$. Show that $\frac{M_n}{\sqrt{2 \log n}} \xrightarrow{p} 1$.

15. Let $(X_n)$ and $(Y_n)$ be two sequences of random variables such that $X_n \xrightarrow{d} 0$ and $X_n Y_n \xrightarrow{d} Z$ for some random variable $Z$. Prove that for a function $f$ differentiable at 0, we have $(f(X_n) - f(0))Y_n \xrightarrow{d} f'(0)Z$.

16. Prove that if $X_n \xrightarrow{d} X$, then $\mathbb{E}|X| \leq \liminf \mathbb{E}|X_n|$.

17. Prove that if $X_n \xrightarrow{d} X$ and $\sup_n \mathbb{E}|X_n|^{p+\varepsilon} < \infty$ for some $p, \varepsilon > 0$, then $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|X_n|^p \to \mathbb{E}|X|^p$ and $\mathbb{E}X_n^p \to \mathbb{E}X^p$.

18. Suppose that $X_1, X_2, \ldots$ are nonnegative random variables such that for some $0 < \alpha < \beta$, we have $\mathbb{E}X_\alpha^n \to 1$ and $\mathbb{E}X_\beta^n \to 1$ as $n \to \infty$. Then $X_n \xrightarrow{1}$ in probability.


20. Let $X = (X_1, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$ uniformly distributed on the sphere \(\{x \in \mathbb{R}^n, x_1^2 + \ldots + x_n^2 = n\}\). Show that $X_i$ converges in distribution to a standard Gaussian random variable.
9 Characteristic functions

9.1 Definition and basic properties

The characteristic function of a random variable $X$ is the function $\phi_X : \mathbb{R} \to \mathbb{C}$ defined as

$$\phi_X(t) = \mathbb{E}e^{itX}, \quad t \in \mathbb{R}.$$  

(For complex valued random variables, say $Z = X + iY$, we of course define $\mathbb{E}Z = \mathbb{E}X + i\mathbb{E}Y$ provided that $\mathbb{E}|Z| < \infty$.) Since $e^{ix} = \cos x + i\sin x, x \in \mathbb{R}$ is a complex number of modulus 1, $e^{itX}$ is a bounded random variable hence its expectation exists, so $\phi_X$ is well-defined on $\mathbb{R}$. We also use the notation $\phi_\mu(t)$ to denote the characteristic function of a Borel probability measure $\mu$ on $\mathbb{R}$ (i.e., of a random variable with law $\mu$),

$$\phi_\mu(t) = \int_{\mathbb{R}} e^{itx}d\mu(x), \quad t \in \mathbb{R}.$$  

9.1 Example. For a symmetric random sign $\varepsilon$,

$$\phi_\varepsilon(t) = \mathbb{E}e^{it\varepsilon} = e^{it} + e^{-it} = \cos t.$$  

9.2 Example. For an exponential random variable $X$ with parameter $\lambda$,

$$\phi_X(t) = \mathbb{E}e^{itX} = \int_{-\infty}^{\infty} e^{itx}f_X(x)dx = \int_{0}^{\infty} \lambda e^{(it-\lambda)x}dx = \lambda e^{\frac{-\lambda x}{it-\lambda}}\bigg|_0^\infty = \lambda \frac{\lambda}{it-\lambda}$$  

(when taking the limit $x \to \infty$, we use that $e^{itx}$ is bounded).

We gather several basic properties in the following theorem.

9.3 Theorem. Let $X$ be a random variable with characteristic function $\phi_X$. Then

(i) $|\phi_X(t)| \leq 1, t \in \mathbb{R},$

(ii) $\phi_X(0) = 1,$

(iii) $\phi_{aX+b}(t) = e^{itb}\phi_X(at),$

(iv) $\phi_X$ is uniformly continuous,

(v) if $\mathbb{E}|X|^n < \infty$ for some positive integer $n$, then the $n$th derivative $\phi_X^{(n)}(t)$ exists, equals $\phi_X^{(n)}(t) = i^n\mathbb{E}X^n e^{itX}$ and is uniformly continuous.

Proof. (i), (ii), (iii): These are easy to directly verify, $|\phi_X(t)| = |\mathbb{E}e^{itX}| \leq \mathbb{E}|e^{itX}| = 1$ and $\phi_X(0) = \mathbb{E}e^{i0X} = 1$ and $\phi_{aX+b}(t) = \mathbb{E}e^{it(aX+b)} = e^{itb}\phi_X(at)$.

(iv): For every $t, h \in \mathbb{R},$

$$|\phi_X(t+h) - \phi_X(t)| = |\mathbb{E}e^{itX}(e^{ihX} - 1)| \leq \mathbb{E}|e^{ihX} - 1| \xrightarrow{h \to 0} 0$$
where the limit is justified by Lebesgue’s dominated convergence theorem ($|e^{ihX} - 1| \to 0$ pointwise and the sequence is bounded by 2). This implies the continuity of $\phi_X$ at $t$. The continuity is uniform because the bound does not depend on $t$.

(v): Fix $n$ such that $\mathbb{E}|X|^n < \infty$. First, we inductively show that for $0 \leq k \leq n$,

$$
\phi_X^{(k)}(t) = \mathbb{E}(iX)^k e^{itX},
$$

This is clear for $k = 0$ and for $k < n$, inductively, we have

$$
\phi_X^{(k+1)}(t) = \lim_{h \to 0} \frac{\phi_X^{(k)}(t+h) - \phi_X^{(k)}(t)}{h} = \lim_{h \to 0} \mathbb{E} \left[ (iX)^k e^{itX} \frac{e^{ihX} - 1}{h} \right] = \mathbb{E} \left[ (iX)^k e^{itX} \lim_{h \to 0} \frac{e^{ihX} - 1}{h} \right].
$$

The last equality is justified by Lebesgue’s dominated convergence theorem because

$$
\left| (iX)^k e^{itX} \frac{e^{ihX} - 1}{h} \right| \leq |X|^k |X| = |X|^{k+1}
$$

and by the assumption $\mathbb{E}|X|^{k+1} < \infty$; we also used that for $t \in \mathbb{R}$, $|e^{it} - 1| \leq |t|$ which can be justified as follows

$$
|e^{it} - 1| = \left| \frac{1}{i} \int_0^t e^{ix} \, dx \right| \leq \int_0^t |e^{ix}| \, dx = t
$$

when $t \geq 0$ and similarly for $t < 0$. Finally, $\lim_{h \to 0} \frac{e^{ihX} - 1}{h} = iX$ which finishes the inductive argument. Having the formula, uniform continuity follows as in (iii). \qed

9.4 Example. Let $X$ be a standard Gaussian random variable. We have,

$$
\phi_X(t) = \int_{\mathbb{R}} e^{itx} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = e^{-t^2/2} \int_{\mathbb{R}} e^{-(x-it)^2/2} \frac{dx}{\sqrt{2\pi}} = e^{-t^2/2},
$$

where the last step would need proper justification (e.g., integrating along an appropriate contour and using $\int_{\mathbb{R}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$). Instead, we use Theorem 9.3 (iv),

$$
\phi_X(t) = i\mathbb{E}X e^{itX} = -\mathbb{E}X \sin(tX) + i\mathbb{E}X \cos(tX).
$$

Since $X$ is symmetric and cos is even, $\mathbb{E}X \cos(tX) = 0$ and integrating by parts,

$$
\phi_X(t) = -\mathbb{E}X \sin(tX) = -\int x \sin(tx) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = \int \sin(tx) (e^{-x^2/2})' \frac{dx}{\sqrt{2\pi}} = -t \int \cos(tx) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}},
$$

which is $-t\mathbb{E}\cos(tx) = -t\mathbb{E}e^{itX} = -t\phi_X(t)$ (by the symmetry of $X$, again, $\mathbb{E}\sin(tX) = 0$), so $\phi_X(t) = -t\phi_X(t)$. That is, $\phi_X(t) = -t\phi_X(t)$, equivalently, $(e^{it/2}\phi_X(t))' = 0$ which finally gives $e^{t^2/2}\phi_X(t) = \phi_X(0) = 1$.

If $Y \sim N(\mu, \sigma^2)$, then $Y = \mu + \sigma X$ and we thus get

$$
\phi_Y(t) = \mathbb{E}e^{it(Y+\sigma X)} = e^{it\mu} \mathbb{E}e^{it\sigma X} = e^{it\mu - \sigma^2 t^2/2}.
$$

(9.1)
9.5 Theorem. If $X$ and $Y$ are independent random variables, then
\[ \phi_{X+Y} = \phi_X \cdot \phi_Y. \]

Proof. Clearly, $E e^{it(X+Y)} = E e^{itX}e^{itY} = E e^{itX}E e^{itY}$. \hfill \Box

9.2 Inversion formulae

One of the crucial properties of characteristic functions is that they determine the distribution uniquely. En route to proving that, we establish an inversion formula, quite standard in Fourier analysis. We first need a lemma.

9.6 Lemma. For two independent random variables $X$ and $Y$ and every $t \in \mathbb{R}$, we have
\[ E e^{-itY} \phi_X(Y) = E \phi_Y(X-t). \]

Proof. Changing the order of taking expectation, we have
\[ E Y e^{-itY} \phi_X(Y) = E Y e^{-itY} E X e^{iYX} = E X, Y e^{iY(X-t)} = E X E Y e^{iY(X-t)} = E X \phi_Y(X-t). \]

9.7 Theorem (Inversion formula). For a random variable $X$, at every point $x$ of continuity of its distribution function $F_X$, we have
\[ F_X(x) = \lim_{a \to \infty} \int_{-\infty}^{x} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \phi_X(s)e^{-s^2/2\sigma^2} ds \right) dt. \]

Proof. Let $G$ be a standard Gaussian random variable, independent of $X$. For $a > 0$, consider $X_a = X + a^{-1}G$. Since $X_a$ converges pointwise to $X$ as $a \to \infty$, by Lebesgue’s dominated convergence theorem $E g(X_a) \to E g(X)$ for every bounded continuous function $g$, thus $X_a \overset{d}{\to} 0$ as $a \to \infty$ (Theorem 8.5). Consequently, for every continuity point $x$ of $F_X$, we have
\[ F_X(x) = \lim_{a \to \infty} F_{X_a}(x). \]

Let us find the distribution function of $X_a$. We have,
\[ F_{X_a}(x) = \mathbb{P}(X + a^{-1}G \leq x) = E X, G 1_{\{X + a^{-1}G \leq x\}} = E X E G 1_{\{X + a^{-1}G \leq x\}} = E X \mathbb{P}(X + a^{-1}G \leq x). \]

For any $y \in \mathbb{R}$, the density of $y + a^{-1}G$ at $t$ is $\frac{a}{\sqrt{2\pi}} e^{-a^2(t-y)^2/2}$, thus
\[ F_{X_a}(x) = E X \int_{-\infty}^{x} \frac{a}{\sqrt{2\pi}} e^{-a^2(t-x)^2/2} dt = \int_{-\infty}^{x} E X \frac{a}{\sqrt{2\pi}} e^{-a^2(t-x)^2/2} dt. \]

Note a simple but very powerful observation involving independence.
Note that $e^{-a^2s^2/2}$ is the characteristic function of $aG$ at $s$ (Example 9.4), so by Lemma 9.6,

$$
E_X \frac{a}{\sqrt{2\pi}} e^{-a^2(t-X)^2/2} = \frac{a}{\sqrt{2\pi}} E_X \phi_{aG}(X - t) = \frac{a}{\sqrt{2\pi}} e^{-itaG} \phi_X(aG).
$$

Writing this explicitly using the density of $aG$ yields

$$
a \sqrt{2\pi} E_X e^{-itaG} \phi_X(aG) = a \sqrt{2\pi} E_X \phi_X(aG)^t = a \sqrt{2\pi} \int_{-\infty}^{\infty} e^{-ist} \phi_X(s)e^{-s^2/2a^2} ds.
$$

Plugging this back,

$$
F_{X_a}(x) = \int_{-\infty}^{x} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \phi_X(s)e^{-s^2/2a^2} ds \right) dt,
$$

which combined with $F_X(x) = \lim_{a \to \infty} F_{X_a}(x)$ remarked earlier finishes the proof.  

Now we can prove that characteristic functions determine distribution.

**9.8 Theorem.** Random variables $X$ and $Y$ have the same distribution (that is, $F_X = F_Y$) if and only if they have the same characteristic functions $\phi_X = \phi_Y$.

**Proof.** By Theorem 9.7, $F_X(x) = F_Y(x)$ for every $x \in \mathbb{R} \setminus B$, where $B$ is the union of the discontinuity points of $F_X$ and the discontinuity points of $F_Y$. For $x \in B$, take $x_n > x$ such that $x_n \in \mathbb{R} \setminus B$ and $x_n \to x$ (it is possible since $B$ is at most countable). Then $F_X(x_n) = F_Y(x_n)$ and by right-continuity, $F_X(x) = F_Y(x)$.

The inversion formula from Theorem 9.7 gives us several other interesting corollaries. Since the characteristic function determines distribution, it should be possible to reconstruct densities from characteristic functions.

**9.9 Theorem.** If $X$ is a random variable such that $\int_{\mathbb{R}} |\phi_X| < \infty$, then $X$ has density $f$ given by

$$
f(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-ist} \phi_X(s)ds
$$

which is bounded and uniformly continuous.

**9.10 Remark.** If $X$ is a continuous random variable with density $f$, then clearly

$$
\phi_X(t) = \int_{-\infty}^{\infty} e^{ist} f(s)ds.
$$

The two formulae have the same form!

**Proof.** For two continuity points $x < y$ of $F_X$, we have from Theorem 9.7,

$$
F_X(y) - F_X(x) = \lim_{a \to \infty} \int_{x}^{y} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist} \phi_X(s)e^{-s^2/2a^2} ds \right) dt.
$$
Since $|e^{-ist}\phi_X(s)e^{-\frac{s^2}{2\pi}}| \leq |\phi_X(s)|$, that is the integrand is dominated by $|\phi_X|$ which is integrable on $[x, y] \times \mathbb{R}$, by Lebesgue’s dominated convergence theorem,

$$F_X(y) - F_X(x) = \int_x^y \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ist}\phi_X(s)ds \right) dt,$$

which gives that $X$ has density given by the promised formula. The rest follows as for characteristic functions (recall the proof of Theorem 9.3 (iii)).

9.11 Corollary. If $X$ is a continuous random variable with density $f_X$ and characteristic function $\phi_X$ which is nonnegative, then $\int_{\mathbb{R}} \phi_X < \infty$ if and only if $f$ is bounded.

Proof. If $\int_{\mathbb{R}} \phi_X < \infty$, then by Theorem 9.9, $f$ is bounded. Conversely, let as in the proof of Theorem 9.9, $G$ be a standard Gaussian random variable independent of $X$. Then the density of $X + a^{-1}G$ at $x$ equals

$$\int_{\mathbb{R}} f_X(x - y)fa^{-1}G(y)dy.$$

On the other hand, it equals $\frac{d}{dx}F_X(x)$ and from the last identity in the proof of Theorem 9.9, this becomes

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx}\phi_X(s)e^{-\frac{s^2}{2\pi}} ds.$$

For $x = 0$ we thus get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(s)e^{-\frac{s^2}{2\pi}} ds = \int_{\mathbb{R}} f_X(-y)fa^{-1}G(y)dy.$$

If $f_X$ is bounded by, say $M$, we obtain that the right hand side is bounded by $M$, so

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(s)e^{-\frac{s^2}{2\pi}} ds \leq M.$$

As $a \to \infty$, by Lebesgue’s monotone convergence theorem, the left hand side converges to $\frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(s)ds$, which proves that $\int_{\mathbb{R}} \phi_X \leq 2\pi M$.

9.12 Example. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables uniform on $[-1, 1]$. Then $X_1 + \ldots + X_n$ for $n \geq 2$ has density

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(tx) \left( \frac{\sin t}{t} \right)^n dt.$$

Indeed, note that $\phi_{X_1}(t) = \frac{\sin t}{t}$, so $\phi_{X_1+\ldots+X_n}(t) = (\frac{\sin t}{t})^n$ which is integrable for $n \geq 2$ and the formula follows from Theorem 9.9.

We finish with two Fourier analytic identities.

9.13 Theorem (Parseval’s identities). If $X$ and $Y$ are continuous random variables with densities $f_X$ and $f_Y$, then

$$\int_{\mathbb{R}} \phi_X(s)\phi_Y(s)ds = \int_{\mathbb{R}} f_X(x)f_Y(x)dx.$$
\( (i) \) \( \int_{\mathbb{R}} |\phi_X|^2 < \infty \) if and only if \( \int_{\mathbb{R}} f_X^2 < \infty \) and then
\[
\int_{\mathbb{R}} f_X^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\phi_X|^2,
\]

\( (ii) \) if \( \int_{\mathbb{R}} f_X^2 < \infty \) and \( \int_{\mathbb{R}} f_Y^2 < \infty \), then
\[
\int_{\mathbb{R}} f_X f_Y = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_X \phi_Y.
\]

**Proof.** \( (i) \) Let \( X' \) be an independent copy of \( X \). Consider \( \tilde{X} = X - X' \). We have,
\[
\phi_{\tilde{X}}(t) = \phi_X(t) \phi_{-X'}(t) = \phi_X(t) \phi_X(t) \bar{\phi}_{X}(t) = |\phi_X(t)|^2.
\]
On the other hand, \( \tilde{X} \) is continuous with density given by convolution,
\[
f_{\tilde{X}}(y) = (f_X \ast f_{-X})(y) = \int_{\mathbb{R}} f_X(x) f_{-X}(y-x) \, dx.
\]
It can be seen from here that if \( \int f_X^2 < \infty \), then by the Cauchy-Schwarz inequality, \( f_{\tilde{X}} \) is bounded. Then by Corollary 9.11, \( \phi_{\tilde{X}} = |\phi_X|^2 \) is integrable. Conversely, if \( |\phi_X|^2 \) is integrable, then from Theorem 9.9 applied to \( \tilde{X} \), we get
\[
f_{\tilde{X}}(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \phi_{\tilde{X}} = \frac{1}{2\pi} \int_{\mathbb{R}} |\phi_X|^2.
\]
Since
\[
f_{\tilde{X}}(0) = (f_X \ast f_{-X})(0) = \int_{\mathbb{R}} f_X(x) f_{-X}(0-x) \, dx = \int_{\mathbb{R}} f_X(x) f_X(x) \, dx = \int_{\mathbb{R}} f_X^2,
\]
we get that \( \int f_X^2 = \frac{1}{2\pi} \int |\phi_X|^2 \). In particular, \( f_X^2 \) is integrable.

\( (ii) \) Apply \( (i) \) to the density \( \frac{f_X + f_Y}{2} \).

\[\square\]

### 9.3 Relations to convergence in distribution

The second crucial property of characteristic functions is that their pointwise convergence captures convergence in distribution. To establish that, we will need to use compactness-type arguments. We start with a lemma that will help us get tightness.

**9.14 Lemma.** For a random variable \( X \) and \( \delta > 0 \),
\[
\mathbb{P} \left( |X| > \frac{2}{\delta} \right) \leq \frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_X(t)| \, dt.
\]

**Proof.** Note that
\[
\int_{-\delta}^{\delta} |1 - \phi_X(t)| \, dt = \int_{-\delta}^{\delta} \left| 1 - \mathbb{E} e^{itX} \right| \, dt = 2\delta - \mathbb{E} \int_{-\delta}^{\delta} e^{itX} \, dt = 2\delta - \mathbb{E} \frac{e^{itX} - e^{-itX}}{iX} = 2\delta - 2\mathbb{E} \frac{\sin(\delta X)}{X}
\]

97
(this incidentally shows that the a priori complex number \( \int_{-\delta}^{\delta} |1 - \phi_X(t)| dt \) is real). Thus
\[
\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_X(t)| dt = 2 \mathbb{E} \left[ 1 - \frac{\sin(\delta X)}{\delta X} \right].
\]

Using \(|\sin x| \leq |x|\), we have \(1 - \frac{\sin x}{x} \geq 0\), so
\[
\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_X(t)| dt \geq 2 \mathbb{E} \left[ \left( 1 - \frac{\sin(\delta X)}{\delta}|X| \right) 1_{|\delta X| > 2} \right],
\]
where is the last equality we used that \(\sin x\) is even. Crudely, \(-\sin(\delta X) \geq -1\), hence
\[
\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_X(t)| dt \geq 2 \mathbb{E} \left[ \left( 1 - \frac{1}{\delta X} \right) 1_{|\delta X| > 2} \right] \geq 2 \mathbb{E} \left[ \frac{1}{2} 1_{|\delta X| > 2} \right] = P(\{|\delta X| > 2\}).
\]

\[\square\]

The main result about convergence in distribution is the following so-called (Lévy’s) continuity theorem.

**9.15 Theorem** (Lévy’s continuity theorem). Let \((X_n)\) be a sequence of random variables such that for every \(t \in \mathbb{R}\), \(\phi_{X_n}(t) \xrightarrow{\text{d}} \phi(t)\) for some function \(\phi : \mathbb{R} \to \mathbb{C}\) which is continuous at \(t = 0\). Then there is a random variable \(X\) such that \(\phi = \phi_X\) and \(X_n \xrightarrow{\text{d}} X\).

**9.16 Remark.** The converse to Lévy’s Theorem also holds: if \(X_n \xrightarrow{\text{d}} X\), then \(\phi_{X_n}(t) \to \phi_X(t)\) for every \(t \in \mathbb{R}\). Indeed, since \(\phi\) is continuous and bounded, \(\mathbb{E}\sin(tX_n) \to \mathbb{E}\sin(tX)\) and the same for the cosine function, so \(\phi_{X_n}(t) = \mathbb{E}\cos(tX_n) + i\mathbb{E}\sin(tX_n) \to \phi_X(t)\).

**Proof of Theorem 9.15.** Since \(|\phi_{X_n}(t)| \leq 1\) for every \(t\), by taking the limit, we have \(|\phi(t)| \leq 1\) for every \(t\).

**Step 1 (tightness).** Since \(\phi\) is continuous at 0 and \(\phi(0) = \lim_n \phi_{X_n}(0) = 1\), for every \(\varepsilon > 0\), there is \(\delta > 0\) such that \(|1 - \phi(t)| < \varepsilon\) for \(|t| < \delta\), so
\[
\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi(t)| dt \leq 2\varepsilon.
\]

By Lebesgue’s dominated convergence theorem,
\[
\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_{X_n}(t)| dt \xrightarrow{n \to \infty} \frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi(t)| dt,
\]
so for large \(n\),
\[
\frac{1}{\delta} \int_{-\delta}^{\delta} |1 - \phi_{X_n}(t)| dt < 3\varepsilon.
\]

98
By Lemma 9.14, we obtain
\[ \mathbb{P} \left( |X_n| > \frac{2}{\delta} \right) < \varepsilon. \]
This shows that the sequence \((X_n)\) is tight. By Theorem 8.17, there is a subsequence \((X_{n_k})\) which converges in distribution to a random variable, say \(X\). This is our candidate for the limit of \((X_n)\).

**Step 2 \((\phi = \phi_X)\).** Since \(X_{n_k} \overset{d}{\to} X\), we get \(\phi_{X_{n_k}} \to \phi_X \) at every point (Remark 9.16, but also \(\phi_{X_{n_k}} \to \phi \) at every point, so \(\phi = \phi_X\), which proves that \(\phi\) is a characteristic function.

**Step 3 \((X_n \overset{d}{\to} X)\).** If this is not the case, then, by the definition, there is a bounded continuous function \(g\) such that \(\mathbb{E}g(X_{n_k}) \to \mathbb{E}g(X)\). Therefore, there is \(\varepsilon > 0\) and a sequence \(m_k\) such that \(|\mathbb{E}g(X_{m_k}) - \mathbb{E}g(X)| > \varepsilon\). Since \((X_n)\) is tight, using Theorem 8.17 again, there is a convergent subsequence \(X_{m_{k_l}}\) to some random variable, say \(X'\). As in Step 2, \(\phi_{X'} = \phi = \phi_X\), so \(X'\) has the same distribution as \(X\) (Theorem 9.8) and \(|\mathbb{E}g(X_{m_{k_l}}) - \mathbb{E}g(X')| = |\mathbb{E}g(X_{m_{k_l}}) - \mathbb{E}g(X)| > \varepsilon\) contradicts that \(X_{m_{k_l}} \overset{d}{\to} X'\). \(\square\)

**9.17 Example.** In Levy’s theorem the continuity assumption is necessary. Let \(G\) be a standard Gaussian random variable and consider the sequence \(X_n = nG\). We have \(\phi_{X_n}(t) = \phi_{nG}(t) = \phi_G(nt) = e^{-n^2t^2/2}\), so
\[
\phi_{X_n}(t) \to \begin{cases} 0, & t \neq 0, \\ 1, & t = 0. \end{cases}
\]
The limiting function is discontinuous at 0. The sequence \(X_n\) does not converge in distribution because \(F_{X_n}(t) = \mathbb{P}(G \leq t/n) \to \mathbb{P}(G \leq 0) = 1/2\), but the limit is not a distribution function (an alternative argument: by Remark 9.16, if \(X_n \overset{d}{\to} X\), then \(\phi_{X_n}\) would converge to a characteristic function which is continuous).
9.4 Exercises

1. In the proof of Theorem 9.5 we implicitly used: if $X,Y$ are complex-valued integrable random variables which are independent, then $EXY = EXEY$. Fill out this gap.

Table 1: Characteristic functions of common discrete distributions (see Section 5.1).

<table>
<thead>
<tr>
<th>Distribution $\mu$</th>
<th>Characteristic function $\phi_\mu(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dirac delta $\delta_a$</td>
<td>$e^{ita}$</td>
</tr>
<tr>
<td>Ber($p$)</td>
<td>$1 - p + pe^{it}$</td>
</tr>
<tr>
<td>Bin($n,p$)</td>
<td>$(1 - p + pe^{it})^n$</td>
</tr>
<tr>
<td>Poiss($\lambda$)</td>
<td>$\exp{\lambda(e^{it} - 1)}$</td>
</tr>
<tr>
<td>Geom($p$)</td>
<td>$pe^{it} \over 1-(1-p)e^{it}$</td>
</tr>
</tbody>
</table>

Table 2: Characteristic functions of common continuous distributions (see Section 5.1).

<table>
<thead>
<tr>
<th>Distribution $\mu$</th>
<th>Density function $f_\mu(x)$</th>
<th>Characteristic function $\phi_\mu(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unif([0, $a$])</td>
<td>$\frac{1}{a} 1_{[0,a]}(x)$</td>
<td>$\frac{e^{iat} - 1}{iat}$</td>
</tr>
<tr>
<td>Exp($\lambda$)</td>
<td>$\lambda e^{-\lambda x} 1_{(0,\infty)}(x)$</td>
<td>$\lambda \over \lambda - e^{it}$</td>
</tr>
<tr>
<td>Sym-Exp</td>
<td>$\frac{1}{2}e^{-</td>
<td>x</td>
</tr>
<tr>
<td>Cauchy</td>
<td>$\frac{1}{\pi(1+x^2)} e^{-\lambda x} 1_{(0,\infty)}(x)$</td>
<td>$(1 - \frac{it}{\lambda})^{-\beta}$</td>
</tr>
<tr>
<td>Gamma($\beta, \lambda$)</td>
<td>$\frac{\lambda^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\lambda x} 1_{(0,\infty)}(x)$</td>
<td>$e^{iat-\sigma^2t^2/2}$</td>
</tr>
<tr>
<td>$N(a,\sigma^2)$</td>
<td>$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-a)^2}{2\sigma^2}}$</td>
<td>$\frac{1}{cosh(\pi t/2)}$</td>
</tr>
<tr>
<td>Hyperbolic cosine</td>
<td>$\frac{1}{\pi \cosh x}$</td>
<td>$\frac{1}{cosh(\pi t/2)}$</td>
</tr>
</tbody>
</table>

2. Justify Table 1.

3. Justify Table 2.

4. Decide whether the following functions are the characteristic functions of some distributions. If yes, describe the corresponding distribution.

   a) $\cos t$,
   b) $\cos^2 t$,
   c) $\frac{1}{2}(1 + e^{it})^2$,
   d) $\frac{1 + \cos t}{2}$,
   e) $(2 - e^{it})^{-1}$.
5. Suppose $\phi_1, \ldots, \phi_n$ are characteristic functions of some distributions. Let nonnegative numbers $p_1, \ldots, p_n$ be such that $\sum_{j=1}^n p_j = 1$. Show that $\sum_{j=1}^n p_j \phi_j$ is also a characteristic function.

6. Let $X_1, X_2, \ldots$ be i.i.d. with characteristic function $\phi$. Let $N$ be a Poisson random variable with parameter $\lambda > 0$, independent of the $X_j$. Find the characteristic function of $Y = \sum_{j=1}^N X_j$ (we adopt the convention that $\sum_{j=1}^0 X_j = 0$).

7. Suppose $\phi$ is the characteristic function of some random variable $X$. Decide whether the following functions are always characteristic functions
   a) $\phi^2$,
   b) $\text{Re}\phi$,
   c) $|\phi|^2$,
   d) $|\phi|$.

8. Show that if $\phi''(0)$ exists, then $\mathbb{E}X^2 < \infty$.

9. Let $X$ be a random variable with density $f(x) = \frac{C}{(1+x^2)\log(1+x^2)}$, $x \in \mathbb{R}$. Show that $\phi'_X(0)$ exits, but $\mathbb{E}|X| = +\infty$.

10. Let $X$ be a random variable such that $\phi_X(t) = 1 - ct^2 + o(t^2)$ as $t \to 0$ for some constant $c \in \mathbb{R}$. Then $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 2c$. In particular, if $\phi_X(t) = 1 + o(t^2)$, then $X = 0$ a.s. As a corollary, $\phi(t) = e^{-|t|^\alpha}$ is not a characteristic function for any $\alpha > 2$.

11. For a sequence $(X_n)$ of random variables, $X_n \xrightarrow{d} 0$ if and only if there is $\delta > 0$ such that $\phi_{X_n}(t) \to 1$ for every $t \in (-\delta, \delta)$.

12. For an integer-valued random variable $X$ and an integer $k$, we have
   \[ \mathbb{P}(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \phi_X(t) \, dt. \]
   (This is Lemma 11.8.)

13. If $X$ is a continuous random variable, then $\phi_X(t) \to 0$ as $t \to \infty$.

14. These help show that certain important functions are characteristic functions:
   a) Show that $\phi(t) = 2\frac{1 - \frac{1}{\sqrt{2\pi}}} {1 + \frac{1}{\sqrt{2\pi}}} t$ is the characteristic function of the triangular distribution with density $(1 - |x|)1_{[-1,1]}(x)$ (the distribution of the sum of two i.i.d. $\text{Unif}[-\frac{1}{2}, \frac{1}{2}]$ random variables).
   b) Using the inverse formula (Theorem 9.9), show that $(1 - \frac{|t|}{\alpha})1_{[-\alpha,\alpha]}(t)$ is a characteristic function.
c) Show that if a function \( \phi : \mathbb{R} \to \mathbb{R} \) satisfies: \( \phi(0) = 1 \), \( \phi \) is even, \( \phi \) is piece-wise linear and \( \phi \) is nonincreasing and convex on \([0, +\infty)\), then \( \phi \) is a characteristic function.

15. Using Exercises 9.5 and 9.14, show the so-called Pólya’s criterion: every function \( \phi \) with \( \phi(0) = 1 \) which is even, nonincreasing and convex on \([0, +\infty)\) is a characteristic function. In particular, \( e^{-|t|^{\alpha}} \) is a characteristic function for \( \alpha \in (0, 1] \).

16. Let \( 0 < \alpha < 2 \) and \( \psi(t) = 1 - (1 - \cos t)^{\alpha/2} \). Using the binomial series argue that

\[
\psi(t) = \sum_{n=1}^{\infty} p_n (\cos t)^n
\]

for nonnegative \( p_1, p_2, \ldots \) with \( \sum_{n=1}^{\infty} p_n = 1 \). Show that

\[
e^{-|t|^\alpha} = \lim_{n \to \infty} \left[ \psi(\sqrt{2}t_{n^{-1/\alpha}}) \right]^n
\]

and conclude that \( e^{-|t|^\alpha} \) is a characteristic function.

17. Let \( X_n \) be a Poisson random variable with parameter \( \lambda_n > 0 \). If \( \lambda_n \to \lambda \) for some \( \lambda > 0 \), then \( X_n \overset{d}{\to} X \) for a Poisson random variable \( X \) with parameter \( \lambda \).

18. Suppose a sequence of random variables \( (X_n) \) converges in law to a random variable \( X \). Suppose sequences of reals \( (a_n) \) and \( (b_n) \) converge to \( a \) and \( b \), respectively. Show that \( a_n X_n + b_n \overset{d}{\to} aX + b \).

19. Let \( (X_n) \) be a sequence of random variables with \( P(X_n = k) = \frac{1}{n^2}, k = 1, \ldots, n^2 \). Does it converge in distribution?

20. Prove that \( \phi(t) = \frac{2}{1 + e^{2t}} \) is not a characteristic function.

21. Let \( X_1, \ldots, X_n \) be i.i.d. Cauchy random variables, that is with density \( \frac{1}{\pi(1 + x^2)} \), \( x \in \mathbb{R} \). Show that for every reals \( a_1, \ldots, a_n \), the weighted sum \( \sum_{j=1}^{n} a_j X_j \) has the same distribution as \( \sum_{j=1}^{n} |a_j| X_1 \).

22. For a random variable \( X \) and \( a \in \mathbb{R} \), we have

\[
P( X = a ) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-iat} \phi_X(t) dt.
\]

23. Let \( X \) and \( Y \) be i.i.d. random variables. Then

\[
P( X = Y ) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi_X(t)|^2 dt.
\]

On the other hand,

\[
P( X = Y ) = \sum_{x \in \mathbb{R}} P( X = x )^2
\]

(of course the set \( \{ x, \ P( X = x ) > 0 \} \) is at most countable).
24. Show that a random variable $X$ has no atoms if and only if

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi_X(t)|^2 dt = 0.$$ 

In particular, if $\phi_X(t) \to 0$ as $t \to \infty$, then $X$ has no atoms. The converse is not true (devil’s staircase).

25. Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs. For every $a = (a_1, a_2, \ldots) \in \ell_2$, i.e., $\sum a_n^2 < \infty$, the series $\sum_{n=1}^{\infty} a_n \varepsilon_n$ converges in probability, hence in distribution (hint: Cauchy’s condition from Exercise 5.28). Show that the distribution of $\sum_{n=1}^{\infty} \varepsilon_n 2^{-n}$ is uniform on $[-1,1]$.

26. Show that for a random variable $X$ the following are equivalent

(a) $X$ is symmetric, that is $X$ and $-X$ have the same distribution
(b) $X$ and $\varepsilon X$ have the same distribution, where $\varepsilon$ is an independent random sign
(c) $X$ and $\varepsilon |X|$ have the same distribution, where $\varepsilon$ is an independent random sign
(d) the characteristic function of $X$ is real valued.

27. For an integrable random variable $X$,

$$\mathbb{E}|X| = \frac{2}{\pi} \int_{0}^{\infty} \frac{1 - \text{Re} \phi_X(t)}{t^2} dt.$$ 

28. Prove Shepp’s inequality: if $X$, $Y$ are i.i.d. integrable random variables with mean 0, then

$$\mathbb{E}|X - Y| \leq \mathbb{E}|X + Y|.$$ 

29. Find an example of two different distributions whose characteristic functions agree on $[-1,1]$.

Hint: Consider $\phi(t) = (1 - |t|)_+$ and $\psi(t)$ defined to be equal to $\phi(t)$ on $[-1,1]$ and 2-periodic.

30. Find an example of 3 random variables $X$, $Y$, $Z$ which are independent such that $Y$ and $Z$ do not have the same distribution, but $X + Y$ and $X + Z$ do.

31. Show that if random variables $X$, $Y$ are independent and $X + Y$ has the same distribution as $X$, then $Y = 0$ a.s.

32. Cramér’s decomposition theorem. If $\xi$ is a Gaussian random variable and $\xi = X + Y$ for some independent random variables $X$ and $Y$, then $X$ and $Y$ are also Gaussian (point masses are assumed to be Gaussian).

Here is a possible, very analytic approach. A function $f : \mathbb{C} \to \mathbb{C}$ is entire if it is holomorphic on $\mathbb{C}$. The order of the entire function $f$ is the infimum of $\rho > 0$
such that $|f(z)| = O(\exp(|z|^\rho))$ as $z \to \infty$. It is a consequence of Hadamard’s factorisation theorem that an entire function $f$ of order $\rho$ without any zeros is of the form $f(z) = e^{g(z)}$, where $g$ is a polynomial of degree at most $\rho$. For instance, $f(z) = e^{z^2}$ is entire of order 2.

(a) Using Hadamard’s factorisation theorem show: if for some random variable $X$ there is $a > 0$ such that $\mathbb{E}e^{aX^2} < \infty$, then $\phi_X$ extends to $\mathbb{C}$, is entire of order at most 2 and if additionally $\phi_X(z) \neq 0$ for all $z \in \mathbb{C}$, then $X$ is Gaussian.

(b) Show that if $X + Y$ is Gaussian for some independent random variables $X$, $Y$, then $\mathbb{E}e^{\delta(X+Y)^2} < \infty$ for some $\delta > 0$.

(c) Deduce from independence that $\mathbb{E}e^{\delta(X+c)^2} < \infty$ for some $c \in \mathbb{R}$ and from (a) that $X$ is Gaussian.
10 Central limit theorem

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}|X_1|^2 < \infty$. By the strong law of large numbers,

$$Y_n = \frac{X_1 + \ldots + X_n}{n} - \mathbb{E}X_1$$

converges to 0 a.s. By our assumption, we can compute

$$\text{Var}(Y_n) = \frac{\text{Var}(X_1 + \ldots + X_n)}{n^2} = \frac{n \text{Var}(X_1)}{n^2} = \frac{\text{Var}(X_1)}{n},$$

so $Y_n$ concentrates around its expectation, which is 0 and in a sense it is not surprising that $Y_n$ goes to 0. What happens if we zoom in, that is rescale appropriately so that the variance of $Y_n$ is fixed, that is when fluctuations of $Y_n$ have a fixed size, as opposed to decaying like $1/n$ as earlier? Consider

$$Z_n = \frac{Y_n}{\sqrt{\text{Var}(Y_n)}} = \sqrt{n} \frac{1}{\sqrt{\text{Var}(X_1)}} \left( \frac{X_1 + \ldots + X_n}{n} - \mathbb{E}X_1 \right)$$

which has variance 1 for all $n$. What “limit distribution” does $Z_n$ have as $n \to \infty$ (if any)? This is addressed by the central limit theorem which says that the weak limit exists and is Gaussian! (If it exists and is universal, that is the same for all i.i.d. sequences, then it has to be Gaussian because when the $X_i$ are standard Gaussian, $Z_n$ is also standard Gaussian.) To establish weak convergence, we shall use characteristic functions.

10.1 Auxiliary elementary lemmas

To handle the convergence of characteristic functions, we shall need several elementary estimates for complex numbers.

10.1 Lemma. If $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ are complex numbers all with modulus at most $\theta$, then

$$\left| \prod_{j=1}^{n} z_j - \prod_{j=1}^{n} w_j \right| \leq \theta^{n-1} \sum_{j=1}^{n} |z_j - w_j|.$$

Proof. We proceed by induction on $n$. For $n = 1$, we have equality. For $n > 1$, we have

$$\left| \prod_{j=1}^{n} z_j - \prod_{j=1}^{n} w_j \right| = \left| z_1 \prod_{j=2}^{n} z_j - w_1 \prod_{j=2}^{n} w_j \right|$$

$$\leq \left| z_1 \prod_{j=2}^{n} z_j - z_1 \prod_{j=2}^{n} w_j \right| + \left| z_1 \prod_{j=2}^{n} w_j - w_1 \prod_{j=2}^{n} w_j \right|$$

$$= \left| z_1 \left( \prod_{j=2}^{n} z_j - \prod_{j=2}^{n} w_j \right) \right| + \left| \prod_{j=2}^{n} w_j \left| z_1 - w_1 \right| \right|$$

$$\leq \theta \left| \prod_{j=2}^{n} z_j - \prod_{j=2}^{n} w_j \right| + \theta^{n-1} |z_1 - w_1|$$
and the inductive assumption allows to finish the proof.

10.2 Lemma. For a complex number \( z \) with \( |z| \leq 1 \), we have

\[
|e^z - (1 + z)| \leq |z|^2.
\]

Proof. Using the power series expansion of \( e^z \), we get

\[
|e^z - (1 + z)| = \left| \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \right| \leq |z|^2 \left( \frac{1}{2!} + \frac{|z|}{3!} + \ldots \right) \leq |z|^2 \left( \frac{1}{2!} + \frac{1}{3!} + \ldots \right) = |z|^2 (e - 2).
\]

10.3 Lemma. If \( (z_n) \) is a sequence of complex numbers such that \( z_n \to z \) for some \( z \in \mathbb{C} \), then

\[
\left( 1 + \frac{z_n}{n} \right)^n \to e^z.
\]

Proof. Fix \( c > |z| \). Then eventually, \( |z_n| < c \) and consequently, \( |1 + \frac{z_n}{n}| \leq 1 + \frac{c}{n} \leq e^{c/n} \) and \( |e^{z_n/n}| = e^{\text{Re}(z_n)/n} \leq e^{c/n} \), so applying Lemma 10.1 with \( \theta = e^{c/n} \), for large \( n \),

\[
\left| \left( 1 + \frac{z_n}{n} \right)^n - e^{z_n} \right| = \left| \prod_{j=1}^{n} \left( 1 + \frac{z_n}{n} \right) - \prod_{j=1}^{n} e^{z_n/n} \right| \leq \left( e^{c/n} \right)^{n-1} \left| 1 + \frac{z_n}{n} - e^{z_n/n} \right|.
\]

Clearly eventually, \( |z_n/n| \leq 1 \), so by Lemma 10.2,

\[
\left| \left( 1 + \frac{z_n}{n} \right)^n - e^{z_n} \right| \leq \left( e^{c/n} \right)^{n-1} \left| \frac{z_n}{n} \right|^2 \leq e^{c^2/n}.
\]

It remains to use continuity, that is that \( e^{z_n} \to e^z \).

10.4 Lemma. For every real number \( t \) and \( n = 0, 1, 2, \ldots \), we have

\[
\left| e^{it} - \sum_{k=0}^{n} \frac{(it)^k}{k!} \right| \leq \frac{|t|^{n+1}}{(n+1)!}.
\]

Proof. We proceed by induction on \( n \). For \( n = 0 \), we have

\[
|e^{it} - 1| = \left| \int_0^t e^{is} ds \right| \leq |t|.
\]

Suppose the assertion holds for \( n \geq 0 \). Then, we have

\[
\left| e^{it} - \sum_{k=0}^{n+1} \frac{(it)^k}{k!} \right| = \left| \int_0^t \left( e^{is} - \sum_{k=0}^{n} \frac{(is)^k}{k!} \right) ds \right| \leq \int_0^t \frac{|s|^{n+1}}{(n+1)!} ds = \frac{|t|^{n+2}}{(n+2)!}.
\]
10.2 Vanilla Central Limit Theorem

10.5 Theorem. Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E|X_1|^2 < \infty$. Then the sequence $(Z_n)$ of normalised sums

$$Z_n = \frac{X_1 + \ldots + X_n - n\bar{E}X_1}{\sqrt{n \text{Var}(X_1)}}$$

converges in distribution to a standard Gaussian random variable.

Proof. Let $\bar{X}_i = X_i - \bar{E}X_i$. Then $E\bar{X}_i = 0$, $E|\bar{X}_i|^2 = 1$, and by independence

$$Z_n = \frac{X_1 + \ldots + X_n - n\bar{E}X_1}{\sqrt{n \text{Var}(X_1)}} = \frac{\bar{X}_1 + \ldots + \bar{X}_n}{\sqrt{n}}.$$

We investigate pointwise convergence of $\phi_{Z_n}(t) = \phi_{\bar{X}_1} \left( \frac{t}{\sqrt{n}} \right) \ldots \phi_{\bar{X}_n} \left( \frac{t}{\sqrt{n}} \right)$. By Theorem 9.3 (v), $\phi_{\bar{X}_1}$ is twice continuously differentiable and we can compute that $\phi_{\bar{X}_1}'(0) = i\bar{E}X_1 = 0$ and $\phi_{\bar{X}_1}''(0) = i^2\bar{E}X_1^2 = -1$. Thus by Taylor’s formula with Lagrange’s remainder

$$\phi_{Z_n}(t) = \left[ \phi_{\bar{X}_1} \left( \frac{t}{\sqrt{n}} \right) \right]^n = \left[ 1 + \frac{t^2}{2} R(t) \right]^n \xrightarrow{n \to \infty} e^{-t^2/2}.$$

By Theorem 9.15, $Z_n$ converges in distribution to a random variable whose characteristic function is $e^{-t^2/2}$, that is a standard Gaussian random variable.

We have two remarks. The first one shows that other notions of convergence are too strong to capture the limit behaviour of sequences of sums of i.i.d. random variables, normalised to have a fixed variance. The second one shows that the finite variance is really a necessary assumption to make for the weak limit to exist. We defer their proofs to exercises.

10.6 Remark. Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with mean 0 and finite variance. Then by the vanilla central limit theorem and Kolmogorov’s 0–1 law,

$$\limsup_{n \to \infty} \frac{X_1 + \ldots + X_n}{\sqrt{n}} = +\infty \quad \text{a.s.}$$

Moreover, the sequence $(\frac{\bar{X}_1 + \ldots + \bar{X}_n}{\sqrt{n}})$, or any of its subsequences, does not converge in probability.
10.7 Remark. Suppose $X_1, X_2, \ldots$ are i.i.d. random variables such that the sequence $(\sqrt{n}X_n)$ converges in distribution. Then $E X_1^2 < \infty$.

In Appendices F and G, we present two other completely different proofs of the vanilla central limit theorem.

10.3 Lindeberg’s Central Limit Theorem

The assumption of identical distribution of the summands in the vanilla central limit theorem can be weakened. The so-called Lindeberg condition is an almost optimal condition under which the central limit theorem holds.

10.8 Theorem. Let $\{X_{n,k}\}_{n \geq 1, 1 \leq k \leq n}$ be a triangular array of random variables with $E X_{n,k}^2 < \infty$ such that for every $n \geq 1$, the variables $X_{n,1}, \ldots, X_{n,n}$ are independent. Let

$$\bar{X}_{n,k} = \frac{X_{n,k} - E X_{n,k}}{\sqrt{\sum_{k=1}^{n} \text{Var}(X_{n,k})}}$$

and for $\varepsilon > 0$, set

$$L_n(\varepsilon) = \sum_{k=1}^{n} E X_{n,k}^2 I\{|X_{n,k}| > \varepsilon\}.$$ 

If the following Lindeberg condition holds:

$$\text{for every } \varepsilon > 0, \quad L_n(\varepsilon) \xrightarrow{n \to \infty} 0 \quad (10.1)$$

then

$$\sum_{k=1}^{n} \bar{X}_{n,k} \xrightarrow{d} Z,$$

where $Z$ is a standard Gaussian random variable.

10.9 Remark. Condition (10.1) implies that

$$\max_{1 \leq k \leq n} \text{Var}(\bar{X}_{n,k}) \xrightarrow{n \to \infty} 0 \quad (10.2)$$

(individual contributions of the summands in $Z_n$ are small). Indeed, for every $\varepsilon > 0$, we have

$$\text{Var}(\bar{X}_{n,k}) \leq E \bar{X}_{n,k}^2 \leq E \bar{X}_{n,k}^2 1_{\{|X_{n,k}| > \varepsilon\}} + \varepsilon \leq L_n(\varepsilon) + \varepsilon.$$

Proof of Theorem 10.8. Denote $Z_n = \sum_{k=1}^{n} \bar{X}_{n,k}$ and $\sigma_{n,k} = \sqrt{\text{Var}(X_{n,k})}$. By the definition of $\bar{X}_{n,k}$, for every $n \geq 1$, we have

$$\sum_{k=1}^{n} \sigma_{n,k}^2 = 1 \quad (10.3)$$

To show $Z_n \xrightarrow{d} Z$, in view of Lévy’s continuity theorem (Theorem 9.15), it is enough to show that for every $t \in \mathbb{R}$, we have

$$\phi_Z(t) \xrightarrow{n \to \infty} e^{-t^2/2}.$$
By independence, (10.3) and Lemma 10.1,
\[
|\phi_{Z_n}(t) - e^{-t^2/2}| = \left| \prod_{k=1}^{n} \phi_{X_{n,k}}(t) - \prod_{k=1}^{n} e^{-\sigma_{n,k}^2 t^2/2} \right| \leq \sum_{k=1}^{n} \left| \phi_{X_{n,k}}(t) - e^{-\sigma_{n,k}^2 t^2/2} \right|.
\]

Denoting
\[
s_n(t) = \sum_{k=1}^{n} \left| \mathbb{E} e^{it X_{n,k}} - 1 + \frac{1}{2} \sigma_{n,k}^2 t^2 \right| = \sum_{k=1}^{n} \left| \mathbb{E} \left[ e^{it X_{n,k}} - 1 - it \bar{X}_{n,k} + \frac{1}{2} t^2 \bar{X}_{n,k}^2 \right] \right|,
\]
and
\[
r_n(t) = \sum_{k=1}^{n} \left| 1 - \frac{1}{2} \sigma_{k,n}^2 t^2 - e^{-\sigma_{n,k}^2 t^2/2} \right|,
\]
by the triangle inequality we thus have
\[
|\phi_{Z_n}(t) - e^{-t^2/2}| \leq s_n(t) + r_n(t).
\]

Fix \( \varepsilon > 0 \). Splitting the expectation in \( s_n(t) \) into two: on \( \{|\bar{X}_{n,k}| \leq \varepsilon\} \) and \( \{|\bar{X}_{n,k}| > \varepsilon\} \), we get
\[
s_n(t) \leq s_n^{(1)}(t) + s_n^{(2)}(t)
\]
with
\[
s_n^{(1)}(t) = \sum_{k=1}^{n} \left| \mathbb{E} \left[ e^{it X_{n,k}} - 1 - it \bar{X}_{n,k} + \frac{1}{2} t^2 \bar{X}_{n,k}^2 \right] 1_{\{|\bar{X}_{n,k}| \leq \varepsilon\}} \right|,
\]
\[
s_n^{(2)}(t) = \sum_{k=1}^{n} \left| \mathbb{E} \left[ e^{it X_{n,k}} - 1 - it \bar{X}_{n,k} + \frac{1}{2} t^2 \bar{X}_{n,k}^2 \right] 1_{\{|\bar{X}_{n,k}| > \varepsilon\}} \right|.
\]

Thanks to Lemma 10.4 (the case \( n = 3 \)),
\[
s_n^{(1)}(t) \leq \sum_{k=1}^{n} \mathbb{E} \left[ \frac{|t|^3 |\bar{X}_{n,k}|^3}{6} 1_{\{|\bar{X}_{n,k}| \leq \varepsilon\}} \right] \leq \frac{|t|^3 \varepsilon}{6} \sum_{k=1}^{n} \mathbb{E} |\bar{X}_{n,k}|^2 = \frac{|t|^3 \varepsilon}{6}.
\]

Thanks to Lemma 10.4 (the case \( n = 2 \)) and the triangle inequality,
\[
s_n^{(2)}(t) \leq \sum_{k=1}^{n} \mathbb{E} \left( \frac{t^2 \bar{X}_{n,k}^2}{2} + \frac{t^2 \bar{X}_{n,k}^2}{2} \right) 1_{\{|\bar{X}_{n,k}| > \varepsilon\}} = t^2 L_n(\varepsilon).
\]

To bound \( r_n(t) \), note that thanks to (10.2), for large enough \( n \) and every \( k \leq n \), we have \( \sigma_{n,k}^2 t^2 \leq 1 \). Thus, thanks to Lemma 10.2,
\[
r_n(t) \leq \sum_{k=1}^{n} \left( \frac{1}{4} \sigma_{n,k}^2 t^2 \right)^2 \leq \frac{t^4}{4} \max_{k \leq n} \sigma_{n,k}^2 \sum_{k=1}^{n} \sigma_{n,k}^2 = \frac{t^4}{4} \max_{k \leq n} \sigma_{n,k}^2.
\]

Putting the bounds on \( s_n(t) \) and \( r_n(t) \) together yields
\[
|\phi_{Z_n}(t) - e^{-t^2/2}| \leq \frac{|t|^3 \varepsilon}{6} + t^2 L_n(\varepsilon) + \frac{t^4}{4} \max_{k \leq n} \sigma_{n,k}^2
\]
which by (10.1) and (10.2) gives \( \phi_{Z_n}(t) \to e^{-t^2/2} \), as desired. \( \square \)
10.10 Remark. Of course Lindeberg’s central limit theorem implies the vanilla one. Indeed, if \( X_1, X_2, \ldots \) are i.i.d. square integrable random variables, say with mean 0 and variance 1, then setting \( X_{n,k} = X_k / \sqrt{n} \), we check that the Lindeberg condition (10.1) holds,

\[
L_n(\varepsilon) = n \mathbb{E} \frac{X_1^2}{n} 1_{\{|X_1| > \varepsilon \sqrt{n}\}} = \mathbb{E} X_1^2 1_{\{|X_1| > \varepsilon \sqrt{n}\}}
\]

which goes to 0 as \( n \) goes to \( \infty \) by Lebesgue’s dominated convergence theorem.

10.11 Remark. The Lindeberg condition (10.1) roughly says that the contribution of each \( X_{n,k}, k = 1, \ldots, n \) to the sum should be “equal and small”. This is not a necessary condition for the central limit theorem to hold as the following simple example shows. Let \( X_{n,1} \sim N(0, 1/2) \) and \( X_{n,k} \sim N(0, \sigma_{n,k}^2), k = 2, \ldots, n \) with arbitrary \( \sigma_{n,k} \) satisfying \( \sum_{k=2}^{n} \sigma_{n,k}^2 = 1/2 \). Then, trivially, \( Z_n \sim N(0, 1) \), but (10.1) does not hold (because even its consequence (10.2) fails). Excluding such situations where one of the summands dominates, it turns out that the Lindeberg condition is necessary for the central limit theorem to hold, as shown by Feller (we defer its proof to Appendix H).

10.12 Theorem. Let \( \{X_{n,k}\}_{n \geq 1, 1 \leq k \leq n} \) be a triangular array of random variables with \( \mathbb{E} X_{n,k}^2 < \infty \) such that for every \( n \geq 1 \), the variables \( X_{n,1}, \ldots, X_{n,n} \) are independent. Let \( \bar{X}_{n,k} \) be the normalised sequence and \( Z_n = \sum_{k=1}^{n} \bar{X}_{n,k} \), as in Theorem (10.8). Assume (10.2). If the sequence \( (Z_n) \) converges in distribution to a standard Gaussian random variable, then (10.1) holds.

10.4 Multidimensional case

Let \( X = (X_1, \ldots, X_d) \) be a random vector in \( \mathbb{R}^d \). We define its characteristic function \( \phi_X : \mathbb{R}^d \to \mathbb{C} \) by

\[
\phi_X(t) = \mathbb{E} e^{i \langle t, X \rangle}, \quad t \in \mathbb{R}^d.
\]

10.13 Example. Let \( X = (X_1, \ldots, X_d) \) be a standard Gaussian random vector in \( \mathbb{R}^d \). Then by the independence of its components,

\[
\phi_X(t) = \prod_{j=1}^{d} \phi_{X_j}(t_j) e^{-\frac{1}{2} \sum_{j=1}^{d} t_j^2}, \quad t \in \mathbb{R}^d.
\]

Let \( Y = AX + b \) be an arbitrary Gaussian random vector in \( \mathbb{R}^m \), where \( A \) is an \( m \times n \) matrix and \( b \) is a vector in \( \mathbb{R}^m \). Then

\[
\phi_Y(t) = \mathbb{E} e^{i \langle AX + b, t \rangle} = e^{i \langle b, t \rangle} \mathbb{E} e^{i \langle X, A^\top t \rangle} = e^{i \langle b, t \rangle} e^{-\frac{1}{2} |AX|^2} = e^{i \langle b, t \rangle} e^{-\frac{1}{2} \langle A A^\top t, t \rangle},
\]

thus, in general, we have

\[
\phi_Y(t) = e^{i \langle b, t \rangle - \frac{1}{2} \langle Qt, t \rangle}, \quad Y \sim N(b, Q).
\]
Working with the multidimensional CDFs in $\mathbb{R}^d$, we can prove a tightness result analogous to Theorem 8.17: a sequence $(\mu_n)_n$ of Borel probability measures on $\mathbb{R}^d$ is tight if and only if its every subsequence has a weakly convergent subsequence. We also have the inversion formula analogous to the one from Theorem 9.7,

$$F_X(x_1,\ldots,x_n) = \lim_{a \to \infty} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\langle s,t \rangle} \phi_X(s) e^{-\frac{|s|^2}{2}} \, ds \, dt.$$

These allow to establish Lévy’s continuity theorem in $\mathbb{R}^d$ in a similar fashion and, consequently, the central limit theorem in $\mathbb{R}^d$.

10.14 Theorem. Let $(X_n)$ be a sequence of random vectors in $\mathbb{R}^d$ such that for every $t \in \mathbb{R}^d$, $\phi_{X_n}(t) \xrightarrow{n \to \infty} \phi(t)$ for some function $\phi : \mathbb{R}^d \to \mathbb{C}$ which is continuous at $t = 0$. Then there is a random vector $X$ in $\mathbb{R}^d$ such that $\phi = \phi_X$ and $X_n \xrightarrow{d} X$.

Proof. Let $e_1, \ldots, e_d$ be the standard basis vectors in $\mathbb{R}^d$. Fix $j \leq d$ and $s \in \mathbb{R}$. By the assumption, $\phi_{X_n,e_j}(s) = \phi_{X_n}(e_js) \to \phi_{X}(e_js) = \phi_{X,e_j}(s)$. Consequently, by the 1-dimensional version of Lévy’s theorem (Theorem 9.15), we get the tightness of the sequence $(X_n,e_j)_n$ of each component of $X_n$, thus of the sequence $(X_n)$. Having the tightness of $(X_n)_n$, we proceed exactly as in the proof of the 1-dimensional case. $\square$

10.15 Theorem (Vanilla version of CLT in $\mathbb{R}^d$). Let $X_1, X_2, \ldots$ be i.i.d. random vectors in $\mathbb{R}^d$ with $\mathbb{E}X_i^2 < \infty$ for each $i$, so that the covariance matrix $Q = \text{Cov}(X)$ is well-defined. Then

$$Z_n = \frac{X_1 + \ldots + X_n - nEX_1}{\sqrt{n}} \xrightarrow{n \to \infty} Z_Q,$$

where $Z_Q$ is a Gaussian random vector in $\mathbb{R}^d$ with mean 0 and covariance matrix $Q$.

Proof. By Lévy’s continuity theorem, it is enough to show that for every $t \in \mathbb{R}^d$, we have

$$\phi_{Z_n}(t) \xrightarrow{n \to \infty} e^{-\langle Qt,t \rangle/2}.$$ (recall (10.4)). Since $\mathbb{E}((t, X_1 - EX_1)^2) = \langle Qt, t \rangle$ and $\phi_{Z_n}(t) = \phi_{(Z_n,t)}(1)$, this follows from the 1-dimensional vanilla CLT (Theorem 10.5) applied to the sequence $\langle (t, Z_n) \rangle_{n=1}^\infty$. $\square$

10.5 Poisson limit theorem

The following result, sometimes called the law of rare events, explains how the Poisson distribution arises as a limit of the binomial distribution when the expected number of successes converges to a constant as the number of Bernoulli trials goes to infinity.

10.16 Theorem (Vanilla Poisson limit theorem). Let a sequence of numbers $p_n \in [0,1]$ be such that $np_n \xrightarrow{n \to \infty} \lambda$ for some $\lambda > 0$. Let $S_n$ be a binomial random variable with parameters $p_n$ and $n$. Then $S_n \xrightarrow{d} X$, where $X$ is a Poisson random variable with parameter $\lambda$. 

111
Proof. For nonnegative integer-valued random variables convergence in distribution is equivalent to the pointwise convergence of the probability mass functions (Exercise 8.4). Thus, \( S_n \overset{d}{\to} X \) if and only if \( \mathbb{P}(S_n = k) \xrightarrow{n \to \infty} \mathbb{P}(X = k) \), for every integer \( k \geq 0 \). Fix then such \( k \) and note that as \( n \to \infty \), we have
\[
\mathbb{P}(S_n = k) = \binom{n}{k} p_n^k (1 - p_n)^{n-k} = \frac{n(n-1) \cdots (n-k+1)}{k!} p_n^k (1 - p_n)^{n-k} = \frac{1 + O(n^{-1})}{k!} (np_n)^k (1 - p_n)^{n-k}.
\]

By the assumption, \( np_n \to \lambda \). In particular, \( p_n \to 0 \). Consequently, \( (1 - p_n)^{-k} \to 1 \) and \( (1 - p_n)^n \to e^{-\lambda} \), so
\[
\mathbb{P}(S_n = k) \xrightarrow{n \to \infty} \frac{1}{k!} \lambda^k e^{-\lambda} = \mathbb{P}(X = k).
\]

\[\square\]

There is a generalisation to triangular arrays of Bernoulli random variables satisfying two assumptions: (i) the means stabilise in the limit and (ii) each random variable has a small contribution.

10.17 Theorem (Poisson limit theorem). Let \( \{X_{n,k}\}_{n \geq 1, 1 \leq k \leq n} \) be a triangular array of Bernoulli random variables such that for every \( n \geq 1 \), the variables \( X_{1,1}, \ldots, X_{n,n} \) are independent. If they satisfy the following two conditions

(i) \( \mathbb{E} \sum_{k=1}^n X_{n,k} \xrightarrow{n \to \infty} \lambda \) for some \( \lambda \in (0, \infty) \),

(ii) \( \max_{1 \leq k \leq n} \mathbb{E} X_{n,k} \xrightarrow{n \to \infty} 0 \),

then
\[
X_{n,1} + \cdots + X_{n,n} \overset{d}{\to} Z,
\]

where \( Z \) is a Poisson random variable with parameter \( \lambda \).

For now we show a Fourier-analytic proof. In the next chapter, we include another proof, based on the total variation distance, which gives some quantitative bounds on the rate of convergence and moreover is quite simple.

1st proof of Theorem 10.17. Let \( S_n = X_{n,1} + \cdots + X_{n,n} \) and let \( p_{n,k} = \mathbb{E} X_{n,k} \) be the parameter of the Bernoulli distribution of \( X_{n,k} \). We have
\[
\phi_{S_n}(t) = \prod_{k=1}^n \phi_{X_{n,k}}(t) = \prod_{k=1}^n (1 + p_{n,k} (e^{it} - 1))
\]
and
\[
\phi_Z(t) = e^{\lambda (e^{it} - 1)}.
\]
Fix $t \in \mathbb{R}$. Assumption (i) is that $\sum_{k=1}^{n} p_{n,k} \xrightarrow{n \to \infty} \lambda$, thus

$$\prod_{k=1}^{n} e^{p_{n,k}(e^{it} - 1)} = e^{\left(\sum_{k=1}^{n} p_{n,k}(e^{it} - 1)\right)} \xrightarrow{n \to \infty} e^{\lambda(e^{it} - 1)} = \phi_Z(t)$$

and it suffices to show that

$$u_n = \prod_{k=1}^{n} (1 + p_{n,k}(e^{it} - 1)) - \prod_{k=1}^{n} e^{p_{n,k}(e^{it} - 1)} \xrightarrow{n \to \infty} 0.$$

For $0 \leq p \leq 1$, we have

$$|e^{p(e^{it} - 1)}| = e^{p\text{Re}(e^{it} - 1)} = e^{p\cos(t - 1)} \leq 1$$

and

$$|1 + p(e^{it} - 1)| = |1 - p + pe^{it}| \leq 1 - p + p|e^{it}| = 1,$$

so by Lemma 10.1 with $\theta = 1$, we get

$$|u_n| = \left| \prod_{k=1}^{n} (1 + p_{n,k}(e^{it} - 1)) - \prod_{k=1}^{n} e^{p_{n,k}(e^{it} - 1)} \right| \leq \sum_{k=1}^{n} \left| 1 + p_{n,k}(e^{it} - 1) - e^{p_{n,k}(e^{it} - 1)} \right|.$$

Thanks to assumption (ii), for large enough $n$, we can use Lemma 10.2 applied to $z = p_{n,k}(e^{it} - 1)$ and thus get

$$|u_n| \leq \sum_{k=1}^{n} p_{n,k}^2 |e^{it} - 1|^2 \leq 4 \sum_{k=1}^{n} p_{n,k}^2 \leq 4 \left( \max_{1 \leq k \leq n} p_{n,k} \right) \sum_{k=1}^{n} p_{n,k} \xrightarrow{n \to \infty} 0.$$ 

\[\square\]
10.6 Exercises

1. Let \( S \) be the number of ones when throwing a fair die 18000 times. Using the central limit theorem, find a Gaussian approximation to \( P(2950 < S < 3050) \).

2. Let \( G \) be a standard Gaussian random vector in \( \mathbb{R}^n \). Let \( \|G\| = \sqrt{G_1^2 + \ldots + G_n^2} \) be its magnitude. Let \( a_n = P(\sqrt{n} - 1 \leq \|G\| \leq \sqrt{n} + 1) \). Find \( a = \lim_{n \to \infty} a_n \).

3. For \( \lambda > 0 \), let \( X_\lambda \) be a Poisson random variable with parameter \( \lambda \). Let \( (\lambda_n) \) be a sequence of positive numbers with \( \lambda_n \to \infty \) as \( n \to \infty \). Find the weak limit of the sequence \( (X_{\lambda_n} - \lambda_n \sqrt{\lambda_n}) \).

4. Show that \( e^{-n \sum_{k=1}^{n} \frac{n^k}{k!}} \to 1 \) as \( n \to \infty \).

5. Let \( X \) be a Poisson random variable with parameter \( n \). Let \( Z \) be a standard Gaussian random variable. Show that

   (i) \( E(X - n \sqrt{n}) = e^{-n \frac{n^{n+1/2}}{n!}} \),
   (ii) \( (X - n \sqrt{n}) \to Z \) as \( n \to \infty \),
   (iii) \( E(X - n \sqrt{n}) \to \frac{n!}{e^{-n n^{1/2}} \sqrt{2\pi}} \) as \( n \to \infty \).

Here, as usual, \( X_- = \max\{-X, 0\} \) denotes the negative part of \( X \).

6. Suppose that a random variable \( X \) with variance one has the following property:

   \( \frac{X + X'}{\sqrt{2}} \) has the same distribution as \( X \), where \( X' \) is an independent copy of \( X \). Show that \( X \sim N(0, 1) \).

7. Suppose that a random vector \( X = (X_1, X_2) \) in \( \mathbb{R}^2 \) is such that \( E X_1^2, E X_2^2 < \infty \), \( X \) is rotationally invariant (\( UX \) has the same distribution as \( X \) for every \( 2 \times 2 \) orthogonal matrix \( U \)) and \( X \) has independent components, that is \( X_1 \) and \( X_2 \) are independent. Show that

   (a) \( X_1 \) has the same distribution as \( X_2 \) and is symmetric (that is has the same distribution as \(-X_1\))
   (b) \( X_1 \) has the same distribution as \( \frac{X_1 + X_2}{\sqrt{2}} \)
   (c) Using Exercise 10.6, deduce that \( X \sim N(0, \sigma^2 I_{2 \times 2}) \).

Generalise this characterisation of multiples of standard Gaussian vectors to \( \mathbb{R}^n \).

8. A roulette wheel has slots numbered 1–36 (18 red and 18 black) and two slots numbered 0 and 00 that are painted green. You can bet $1 that the ball will land in a
red (or black) slot and win $1 if it does. What is the expected value of your winnings after 361 spins of the wheel and what is approximately the probability that it will be positive?

9. A biased coin showing heads with probability $p$ is thrown 2500 times. Using the Poisson or central limit theorem, find approximately the probability of getting no heads when a) $p = \frac{1}{2500}$, b) $p = \frac{1}{5}$? How about the probability of getting 500 heads?


11. Prove Remark 10.7. Here is a suggestion how to proceed.

(a) Considering $X_i - X_i'$ instead of $X_i$, where $X_i'$ is an independent copy of $X_i$ (independent of all the other random variables), show that we can assume the $X_i$ are symmetric (Exercise 7.1 may be of use).

(b) Show that for independent symmetric random variables $X_1, \ldots, X_n$ and every $t, A \geq 0$, we have

$$
\mathbb{P}(X_1 + \cdots + X_n \geq t) \geq \frac{1}{2} \mathbb{P}(X_1 1_{X_1 \leq A} + \cdots + X_n 1_{X_n \leq A} \geq t).
$$

To this end, consider $S = X_1 1_{X_1 \leq A} + \cdots + X_n 1_{X_n \leq A}$, $T = X_1 1_{X_1 > A} + \cdots + X_n 1_{X_n > A}$ and use that $(S, T)$ has the same distribution as $(\pm S, \pm T)$, by symmetry.

(c) Apply (b) with $t = u\sqrt{n}$ and treat the right hand side with the central limit theorem to show that if $\mathbb{E}X_i^2 = +\infty$ ($\mathbb{E}X_i^2 1_{X_i \leq A} \not\to \mathbb{E}X_i^2$ as $A \to \infty$), then $\mathbb{P}(X_1 + \cdots + X_n \geq u\sqrt{n}) > \frac{1}{5}$ for large enough $n$. Choose first $u$, then $A$, then $n$ to reach a contradiction.

12. Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and variance 1. Show that

$$
\sqrt{n}(X_1 + \cdots + X_n) \xrightarrow{d_{n \to \infty}} Z
$$

and

$$
\frac{X_1 + \cdots + X_n}{\sqrt{X_1^2 + \cdots + X_n^2}} \xrightarrow{d_{n \to \infty}} Z,
$$

where $Z$ is a standard Gaussian random variable.

13. Let $X_1, X_2, \ldots$ be i.i.d. nonnegative random variables with mean 1 and variance $\sigma^2$. Show that

$$
2(\sqrt{X_1 + \cdots + X_n} - \sqrt{n}) \xrightarrow{d_{n \to \infty}} Z,
$$

where $Z$ is a Gaussian random variable with mean 0 and variance $\sigma^2$.

14. Let $a > 0$. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that $\mathbb{P}(X_k = a) = \mathbb{P}(X_k = 1/a) = \frac{1}{2}$. Investigate the weak convergence of the sequence $Z_n = (X_1 \cdots X_n)^{1/\sqrt{n}}$, $n \geq 1$. 

115
15. Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and variance $\sigma^2$. Let $f: \mathbb{R} \to \mathbb{R}$ be a function differentiable at 0. Show that
\[
\sqrt{n} \left( f \left( \frac{X_1 + \cdots + X_n}{n} \right) - f(0) \right) \xrightarrow{n \to \infty} Z,
\]
where $Z$ is a Gaussian random variable with mean 0 and variance $\sigma^2 f'(0)^2$.

16. Show that the Lindeberg condition (10.1) implies that for every $\varepsilon > 0$, we have
\[
\max_{1 \leq k \leq n} \mathbb{P}(|\bar{X}_{n,k}| > \varepsilon) \xrightarrow{n \to \infty} 0.
\]

17. Under the notation and assumptions of Theorem 10.8, consider the so-called Lyapunov condition: there is $\delta > 0$ such that $\mathbb{E}|X_{n,k}|^{2+\delta} < \infty$ for all $n, k$ and
\[
\sum_{k=1}^{n} \mathbb{E}|\bar{X}_{n,k}|^{2+\delta} \xrightarrow{n \to \infty} 0.
\]
Show that this implies Lindeberg’s condition (10.1).

18. Let $X_1, X_2, \ldots$ be independent random variables such that for some constant $C > 0$ and all $n$, $|X_n| \leq C$. If $\sum_{k=1}^{n} \text{Var}(X_k) = \infty$, then $\frac{X_1 + \cdots + X_n - \mathbb{E}(X_1 + \cdots + X_n)}{\sqrt{\sum_{k=1}^{n} \text{Var}(X_k)}}$ converges to a standard Gaussian.

19. Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and variance 1. Given positive parameters $\alpha$ and $t$, find
\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{X_1 + \cdots + X_n}{n^\alpha} \right| > t \right).
\]

20. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}(X_k = k) = \mathbb{P}(X_k = -k) = \frac{1}{2}$, $k \geq 1$. Investigate the convergence in distribution of the sequence
\[
\frac{X_1 + \cdots + X_n}{\sqrt{\text{Var}(X_1 + \cdots + X_n)}} \quad n \geq 1.
\]

21. Let $U_1, U_2, \ldots$ be independent random variables with $U_k$ being uniform on $[-a_k, a_k]$. Let $\sigma_n = \sqrt{\text{Var}(X_1 + \cdots + X_n)}$. Investigate the convergence in distribution of the sequence
\[
\frac{U_1 + \cdots + U_n}{\sigma_n}, \quad n \geq 1,
\]
in the following two cases
a) The sequence $(a_n)$ is bounded and $\sigma_n \to \infty$ as $n \to \infty$.

b) $\sum a_n^2 < \infty$.

22. Show that the components $X_j$ of a random vector $X = (X_1, \ldots, X_d)$ in $\mathbb{R}^d$ are independent if and only if $\phi_X(t) = \prod_{k=1}^{n} \phi_{X_k}(t_k)$ for every $t \in \mathbb{R}^d$. 116
23. Show that a random vector $X$ in $\mathbb{R}^d$ is Gaussian with mean $b \in \mathbb{R}^d$ and covariance matrix $Q$ if and only if for every $t \in \mathbb{R}^d$, $\langle t, X \rangle$ is a Gaussian random variable with mean $\langle t, b \rangle$ and variance $\langle Q t, t \rangle$.

24. Let $T_n$ and $Z_n$ be the number of inversions and cycles, respectively, in a permutation chosen uniformly at random from the set of all permutations on an $n$-element set. Show that

$$6 \frac{S_n - n^2/4}{n^{3/2}} \xrightarrow{d} Z$$

and

$$Z_n - \log n \xrightarrow{d} (\log n)^{1/2} Z,$$

where $Z$ is a standard Gaussian random variable.
11 Quantitative versions of the limit theorem*

11.1 Berry-Esseen theorem via Stein’s method

Let $X_1, X_2, \ldots$ be i.i.d. random variables with finite variance. Let $Z_n = \frac{X_1 + \ldots + X_n - n\mathbb{E}X_1}{\sqrt{n\text{Var}(X_1)}}$ and let $Z$ be a standard Gaussian random variable. The central limit theorem asserts that for every $t \in \mathbb{R}$,

$$\lim_{n \to \infty} \mathbb{P}(Z_n \leq t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx.$$ 

For practical purposes, we would like to know what is the error we make when we use $\mathbb{P}(Z \leq t)$ as an approximation to $\mathbb{P}(Z_n \leq t)$ for large $n$. This is possible under an additional assumption (finite third moment) and is settled in the following theorem, discovered independently by Berry and Esseen.

11.1 Theorem (Berry-Esseen theorem). Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mathbb{E}|X_1|^3 < \infty$. Let

$$Z_n = \frac{X_1 + \ldots + X_n - n\mathbb{E}X_1}{\sqrt{n\text{Var}(X_1)}},$$

$$\rho = \mathbb{E} \left| \frac{X_1 - \mathbb{E}X_1}{\sqrt{\text{Var}(X_1)}} \right|^3,$$

and let $Z$ be a standard Gaussian random variable. There is a universal constant $C$ such that for every $n \geq 1$ and every $t \in \mathbb{R}$, we have

$$\left| \mathbb{P}(Z_n \leq t) - \mathbb{P}(Z \leq t) \right| \leq \frac{C\rho}{\sqrt{n}}.$$

11.2 Remark. We present a proof which will give $C = 15.2$, but this value is far from optimal. Currently, the best value is $C = 0.4774$ (established via Fourier analytic methods in [8]). Esseen proved a lower bound: $C \geq \frac{10 + \sqrt{3}}{6\sqrt{2\pi}} = 0.4097 \ldots$

11.3 Remark. The rate $1/\sqrt{n}$ of the error is optimal. Consider i.i.d. symmetric random signs $\varepsilon_1, \varepsilon_2, \ldots$ and let $Z_n = \frac{\varepsilon_1 + \ldots + \varepsilon_n}{\sqrt{n}}$. For even $n$, by symmetry, we have

$$\mathbb{P}(Z_n \leq 0) = \frac{1 + \mathbb{P}(\varepsilon_1 + \ldots + \varepsilon_n = 0)}{2} = \frac{1}{2} + \frac{1}{2} \left( \frac{n}{n/2} \right) \frac{1}{2^n},$$

thus, thanks to Stirling’s formula,

$$\left| \mathbb{P}(Z_n \leq 0) - \mathbb{P}(Z \leq 0) \right| = \left| \mathbb{P}(Z_n \leq 0) - \frac{1}{2} \right| = \frac{1}{2} \left( \frac{n}{n/2} \right) \frac{1}{2^n} \approx \frac{1}{2} \sqrt{\frac{2}{\pi n}},$$

so in this case the error is of the order $1/\sqrt{n}$.

For the proof the Berry-Esseen theorem, we shall need the following elementary tail bound for the standard Gaussian distribution.
11.4 Lemma. For \( x > 0 \), we have
\[
(i) \int_x^\infty e^{-u^2/2} du \leq \sqrt{\frac{2}{\pi}} e^{-x^2/2},
\]
\[
(ii) \int_x^\infty e^{-u^2/2} du \leq \frac{1}{x} e^{-x^2/2}.
\]

Proof. (i) let \( f(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2} - \int_x^\infty e^{-u^2/2} du \). Since \( f'(x) = (1 - x \sqrt{\frac{2}{\pi}}) e^{-x^2/2} \) is first positive, then negative, \( f \) first increases, then decreases. Combined with \( f(0) = 0 \) and \( f(x) \to 0 \) as \( x \to \infty \), this proves that \( f(x) \geq 0 \).

(ii) We have \( \int_x^\infty x e^{-u^2/2} du \leq \int_x^\infty u e^{-u^2/2} du = e^{-u^2/2} \). \hfill \square

Proof of Theorem 11.1. For \( t, x \in \mathbb{R} \) and \( \lambda > 0 \) define functions
\[
h_t(x) = 1_{(-\infty, t]}(x),
\]
and their continuous linear approximations
\[
h_{t,\lambda}(x) = \begin{cases} 1, & x \leq t, \\ 1 - \frac{x-t}{\lambda}, & t < x \leq t + \lambda, \\ 0, & x > t + \lambda. \end{cases}
\]

We will frequently use the following integral representation
\[
h_{t,\lambda}(x) = \int_x^\infty \frac{1}{\lambda} 1_{(t, t+\lambda)}(s) ds.
\]

Given \( \gamma \geq 1 \), define the class of random variables
\[
\mathcal{L}_\gamma = \{X, \ X \text{ is random variable such that } \mathbb{E}X = 0, \ \mathbb{E}X^2 = 1, \ \mathbb{E}|X|^3 = \gamma \}
\]
and for \( n = 1, 2, \ldots \) define two quantities
\[
B_0(\gamma, n) = \sup_{X_1, \ldots, X_n \text{ i.i.d., } X_i \in \mathcal{L}_\gamma} \sup_{t \in \mathbb{R}} |\mathbb{E}h_t(Z_n) - \mathbb{E}h_t(Z)|,
\]
\[
B(\lambda, \gamma, n) = \sup_{X_1, \ldots, X_n \text{ i.i.d., } X_i \in \mathcal{L}_\gamma} \sup_{t \in \mathbb{R}} |\mathbb{E}h_{t,\lambda}(Z_n) - \mathbb{E}h_{t,\lambda}(Z)|.
\]

Plainly, \( \mathbb{P}(X \leq t) = \mathbb{E}1_{X \leq t} = \mathbb{E}h_t(X) \), so to prove the theorem, we would like to show that
\[
\frac{\sqrt{n}}{\gamma} B_0(\gamma, n) \leq C, \quad n \geq 1, \gamma \geq 1.
\]

This is clear for \( n = 1 \) with \( C = 1 \) because \( |\mathbb{E}h_t(Z_n) - \mathbb{E}h_t(Z)| \leq 1 \), so from now on we assume \( n \geq 2 \) and divide the rest of the proof into several steps.

Step 1: regularisation (upper bound for \( B_0 \) in terms of \( B \)). Since \( h_{t-\lambda} \leq h_t \leq h_{t,\lambda} \), we get
\[
\mathbb{E}h_t(Z_n) - \mathbb{E}h_t(Z) \leq \mathbb{E}h_{t,\lambda}(Z_n) - \mathbb{E}h_{t,\lambda}(Z) - \mathbb{E}h_t(Z)
\]
\[
= \mathbb{E}h_{t,\lambda}(Z_n) - \mathbb{E}h_{t,\lambda}(Z) + \mathbb{E}h_{t,\lambda}(Z) - \mathbb{E}h_t(Z)
\]
\[
\leq \mathbb{E}h_{t,\lambda}(Z_n) - \mathbb{E}h_{t,\lambda}(Z) + \mathbb{E}h_{t+\lambda}(Z) - \mathbb{E}h_t(Z).
\]

119
Observe that the first difference is upper bounded by $B(\lambda, \gamma, n)$ by its definition. The second difference is

$$P(t < Z \leq t + \lambda) = \int_t^{t+\lambda} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \leq \int_t^{t+\lambda} \frac{dx}{\sqrt{2\pi}} = \frac{\lambda}{\sqrt{2\pi}}.$$  

Altogether,

$$Eh_t(Z_n) - Eh_t(Z) \leq B(\lambda, \gamma, n) + \frac{\lambda}{\sqrt{2\pi}}.$$  

Similarly,

$$Eh_t(Z_n) - Eh_t(Z) \geq -B(\lambda, \gamma, n) - \frac{\lambda}{\sqrt{2\pi}}.$$  

Thus

$$B_0(\gamma, n) \leq B(\lambda, \gamma, n) + \frac{\lambda}{\sqrt{2\pi}}.$$  

**Step 2: Stein’s method (“encoding” $Eh(Z)$ into a function).** Fix $t \in \mathbb{R}$, $\lambda > 0$ and set $h = h_{t, \lambda}$. Our goal is to upper bound $B$, so to upper bound $Eh_t(Z_n) - Eh_t(Z)$. The heart of Stein’s method is to rewrite this in terms of $Z_n$ only. Let

$$f(x) = e^{x^2/2} \int_{-\infty}^{x} [h(u) - Eh(Z)]e^{-u^2/2}du.$$  

Then

$$f'(x) - xf(x) = h(x) - Eh(Z),$$  

so

$$Eh(Z_n) - Eh(Z) = \mathbb{E} [f'(Z_n) - Z_nf(Z_n)].$$  

**Step 3: Estimates for $f$ and $f'$.** For every $x \in \mathbb{R}$, we have

$$|f(x)| \leq \sqrt{\frac{\pi}{2}}, \quad |xf(x)| \leq 1, \quad |f'(x)| \leq 2$$  

and for every $x, y \in \mathbb{R}$, we have

$$|f'(x + y) - f'(x)| \leq |y| \left( \sqrt{\frac{\pi}{2}} + 2|x| + \frac{1}{\lambda} \int_0^1 1_{(t, t+\lambda)}(x + vy)dv \right).$$  

Indeed, since $h$ takes values in $[0, 1]$, we have $|h(u) - h(v)| \leq 1$ for any $u$ and $v$, so for $x < 0$,

$$|f(x)| \leq e^{x^2/2} \int_{-\infty}^{x} |h(u) - Eh(Z)|e^{-u^2/2}du \leq e^{x^2/2} \int_{-\infty}^{x} e^{-u^2/2}du = e^{x^2/2} \int_{-x}^{\infty} e^{-u^2/2}du,$$

which by Lemma 11.4 (i) is upper bounded by $\sqrt{\pi}$. For $x > 0$, notice that $\int_{-\infty}^{\infty} |h(u) - Eh(Z)|e^{-u^2/2}du = 0$, so

$$f(x) = -e^{x^2/2} \int_{-x}^{\infty} |h(u) - Eh(Z)|e^{-u^2/2}du.$$
and as above we get the bound $|f(x)| \leq \sqrt{\frac{n}{2}}$. To bound $xf(x)$ we proceed the same way but use Lemma 11.4 (ii). Finally, since $f'(x) = xf(x) + h(x) - \mathbb{E}h(Z)$ (Step 2), we get

$$|f'(x)| \leq |xf(x)| + |h(x) - \mathbb{E}h(Z)| \leq 1 + 1 = 2.$$  

This establishes (11.2). To prove (11.3), we use the formula for $f'$ from Step 2 and write

$$|f'(x + y) - f'(x)| = |(x + y)f(x + y) + h(x + y) - xf(x) - h(x)|$$

$$= |yf(x + y) + x(f(x + y) - f(x)) + h(x + y) - h(x)|$$

$$\leq |y| \sqrt{\frac{\pi}{2}} + 2|x||y| + |h(x + y) - h(x)|,$$

where in the last inequality we used the mean value theorem writing $f(x + y) - f(x) = f'(\xi)y$ and then estimating $|f'(\xi)| \leq 2$. Finally, by the integral representation for $h$,

$$|h(x + y) - h(x)| = \left| \frac{1}{\lambda} \int_x^{x+y} 1_{(t,t+\lambda]}(u)du \right| = \left| \frac{y}{\lambda} \int_0^1 1_{(t,t+\lambda]}(x + vy)dv \right|$$

which after plugging back in the previous inequality finishes the proof of (11.3).

**Step 4:** Estimates for $B(\lambda, \gamma, n)$ via (11.1). To estimate $B(\lambda, \gamma, n)$, we need to upper bound $\mathbb{E}h(Z_n) - \mathbb{E}h(Z) = \mathbb{E}[f'(Z_n) - Z_n f(Z_n)]$ (recall (11.1) from Step 2). Here we exploit that $Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$ is a sum of i.i.d. random variables. Since the $X_i$ have the same distribution, by linearity,

$$\mathbb{E}Z_n f(Z_n) = \mathbb{E} \frac{\sum X_i}{\sqrt{n}} f(Z_n) = \sqrt{n} \mathbb{E} X_n f(Z_n).$$

Note also that $Z_n = \sqrt{\frac{n-1}{n}} Z_{n-1} + \frac{X_n}{\sqrt{n}}$ and thus

$$\mathbb{E}[f'(Z_n) - Z_n f(Z_n)] = \mathbb{E}[f'(Z_n) - \sqrt{n} X_n f(Z_n)]$$

$$= \mathbb{E} \left[ f'(Z_n) - \sqrt{n} X_n \int_0^1 \frac{d}{du} f \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + u \frac{X_n}{\sqrt{n}} \right) du \right]$$

$$- \sqrt{n} X_n f \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right)$$

By independence and $\mathbb{E}X_n = 0$ the last term vanishes and after computing the derivative we get

$$\mathbb{E}[f'(Z_n) - Z_n f(Z_n)] = \mathbb{E} \left[ f'(Z_n) - X_n^2 \int_0^1 f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + u \frac{X_n}{\sqrt{n}} \right) du \right]$$

$$= \mathbb{E} \left[ f'(Z_n) - f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right) \right]$$

$$+ \mathbb{E} \left[ - X_n^2 \int_0^1 \left\{ f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + u \frac{X_n}{\sqrt{n}} \right) - f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right) \right\} du \right].$$
where in the last equality we used independence and \( \mathbb{E}X_n^2 = 1 \). We bound the two terms separately.

**Step 4.1: First term.** Using \( Z_n = \sqrt{\frac{n-1}{n}} Z_{n-1} + \frac{X_n}{\sqrt{n}} \) and (11.3),
\[
\left| \mathbb{E} \left[ f'(Z_n) - f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right) \right] \right| \\
\leq \mathbb{E} \left| \frac{X_n}{\sqrt{n}} \right| \left( \sqrt{\frac{\pi}{2}} + 2 \sqrt{\frac{n-1}{n}} |Z_{n-1}| + \frac{1}{\lambda} \int_0^1 1_{(t,t+\lambda)} \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + \frac{X_n}{\sqrt{n}} \right) \, dt \right)
\]

Since \( \mathbb{E}|X_n| \leq \sqrt{\mathbb{E}|X_n|^2} = 1 \) and similarly \( \mathbb{E}|Z_{n-1}| \leq 1 \), as well as trivially \( \sqrt{\frac{n-1}{n}} \leq 1 \), we get
\[
\left| \mathbb{E} \left[ f'(Z_n) - f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right) \right] \right| \\
\leq \frac{1}{\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 \frac{1}{\lambda \sqrt{n}} \mathbb{E}X_n \right) |X_n| \int_0^1 \mathbb{E}Z_{n-1} 1_{(t,t+\lambda)} \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + \frac{X_n}{\sqrt{n}} \right) \, dt,
\]

where in the last term we used the independence of \( X_n \) and \( Z_{n-1} \). Note that
\[
\mathbb{E}Z_{n-1} 1_{(t,t+\lambda)} \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + \frac{X_n}{\sqrt{n}} \right) \\
= \mathbb{P}_{Z_{n-1}} \left( \left( t - \frac{X_n}{\sqrt{n}} \right) \sqrt{\frac{n-1}{n}} < Z_{n-1} < \left( t - \frac{X_n}{\sqrt{n}} \right) \sqrt{\frac{n-1}{n}} + \lambda \sqrt{\frac{n}{n-1}} \right).
\]

Denoting \( a = \left( t - \frac{X_n}{\sqrt{n}} \right) \sqrt{\frac{n-1}{n}} \) and estimating \( \frac{n}{n-1} \leq 2 \), we get that this probability is upper bounded by
\[
\mathbb{P} \left( a < Z_{n-1} < a + \lambda \sqrt{2} \right)
\]
which we rewrite in order to upper bound it in terms of \( B_0 \),
\[
\mathbb{P} \left( a < Z_{n-1} < a + \lambda \sqrt{2} \right) = \mathbb{P} \left( Z_{n-1} < a + \lambda \sqrt{2} \right) - \mathbb{P} \left( Z < a + \lambda \sqrt{2} \right) \\
+ \mathbb{P} \left( Z \leq a \right) - \mathbb{P} \left( Z_{n-1} \leq a \right) + \mathbb{P} \left( a \leq Z \leq a + \lambda \sqrt{2} \right)
\]
where the last term was crudely bounded using the maximum of standard Gaussian density. Plugging this back yields
\[
\left| \mathbb{E} \left[ f'(Z_n) - f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right) \right] \right| \\
\leq \frac{1}{\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 \frac{1}{\lambda \sqrt{n}} \mathbb{E}X_n \right) \left( 2B_0(\gamma, n-1) + \frac{\lambda}{\sqrt{\pi}} \right)
\]
\[
\leq \frac{1}{\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 \frac{2B_0(\gamma, n-1) + 1}{\lambda} \right)
\]
\[
\leq \frac{\gamma}{\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 \frac{2B_0(\gamma, n-1) + 1}{\lambda} \right).
\]

122
Step 4.2: Second term. Using again (11.3) and independence,

\[
\begin{align*}
\left| \mathbb{E} \left[ -X_n^2 \int_0^1 \left\{ f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + u \frac{X_n}{\sqrt{n}} \right) - f' \left( \sqrt{\frac{n-1}{n}} Z_{n-1} \right) \right\} du \right] \right| \\
&\leq \mathbb{E}X_n^2 \frac{1}{\sqrt{n}} \int_0^1 u \left( \sqrt{\frac{\pi}{2}} + 2 \sqrt{\frac{n-1}{n}} |Z_{n-1}| \\
&\quad + \frac{1}{\lambda} \int_0^1 \mathbb{E}_{Z_{n-1}} \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + uv \frac{X_n}{\sqrt{n}} \right) du \right) \\
&\leq \mathbb{E} \frac{X_n^3}{\sqrt{n}} \int_0^1 u \left( \sqrt{\frac{\pi}{2}} + 2 \mathbb{E}_{Z_{n-1}} |Z_{n-1}| \\
&\quad + \frac{1}{\lambda} \int_0^1 \mathbb{E}_{Z_{n-1}} 1_{(t,t+\lambda)} \left( \sqrt{\frac{n-1}{n}} Z_{n-1} + uv \frac{X_n}{\sqrt{n}} \right) du \right) \\
&\leq \mathbb{E} \frac{X_n^3}{\sqrt{n}} \int_0^1 u \left( \sqrt{\frac{\pi}{2}} + 2 + \frac{1}{\lambda} \left( \frac{2B_0(\gamma,n-1)}{\lambda} + \frac{\lambda}{\sqrt{\pi}} \right) \right) du \\
&= \frac{\gamma}{2} \sqrt{n} \left( \sqrt{\frac{\pi}{2}} + 2 + \frac{2B_0(\gamma,n-1)}{\lambda} + \frac{1}{\sqrt{\pi}} \right).
\end{align*}
\]

Putting Steps 4.1 and 4.2 together yields

\[
\left| \mathbb{E}[f'(Z_n) - Z_n f(Z_n)] \right| \leq \frac{3\gamma}{2\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 + \frac{2B_0(\gamma,n-1)}{\lambda} + \frac{1}{\sqrt{\pi}} \right).
\]

By Step 2, this gives

\[
B(\lambda, \gamma, n) \leq \frac{3\gamma}{2\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 + \frac{2B_0(\gamma,n-1)}{\lambda} + \frac{1}{\sqrt{\pi}} \right).
\]

Step 5: Optimisation of parameters and end of proof. The previous inequality and Step 1 yield

\[
B_0(\gamma,n) \leq \frac{3\gamma}{2\sqrt{n}} \left( \sqrt{\frac{\pi}{2}} + 2 + \frac{2B_0(\gamma,n-1)}{\lambda} + \frac{1}{\sqrt{\pi}} \right) + \frac{\lambda}{\sqrt{2\pi}}
\]

Set \( \lambda = \alpha \frac{3\gamma}{\sqrt{n}} \), \( \alpha > 0 \) and multiply both sides by \( \frac{\sqrt{\pi}}{\gamma} \) to get

\[
B_0(\gamma,n) \frac{\sqrt{n}}{\gamma} \leq \frac{3}{\gamma} \left( \sqrt{\frac{\pi}{2}} + 2 + \frac{1}{\sqrt{\pi}} \right) + \frac{\gamma}{\alpha} B_0(\gamma,n-1) \frac{\sqrt{n}}{\gamma} + \frac{\alpha}{\sqrt{2\pi}}.
\]

Let

\[
B = \sup_{\gamma \geq 1, n \geq 2} B_0(\gamma,n) \frac{\sqrt{n}}{\gamma}.
\]

For \( n \geq 2 \), we have

\[
B_0(\gamma,n-1) \frac{\sqrt{n}}{\gamma} = B_0(\gamma,n-1) \frac{\sqrt{n-1}}{\gamma} \frac{n}{n-1} \leq \max \left\{ \sqrt{2}, B \sqrt{\frac{3}{2}} \right\}
\]

123
(recall that trivially $B_0(\gamma,1)\frac{1}{\gamma} \leq 1$). If $B > \frac{2}{\sqrt{3}}$, we thus obtain

$$B \leq \frac{3}{2} \left( \frac{\pi}{2} + 2 + \frac{1}{\sqrt{\pi}} \right) + \frac{3}{\alpha} B \frac{\sqrt{3}}{2} + \frac{\alpha}{\sqrt{2\pi}}.$$  

For $\alpha > 3\sqrt{\frac{3}{2}}$ this gives

$$B \leq \frac{\alpha}{\alpha - 3\sqrt{\frac{3}{2}}} \left( \frac{\pi}{2} + 2 + \frac{1}{\sqrt{\pi}} \right) + \frac{\alpha^2}{\alpha - 3\sqrt{\frac{3}{2}}} \frac{1}{\sqrt{2\pi}}.$$  

The choice of $\alpha$ which equates the two terms on the right hand side gives

$$B < 15.4.$$  

Optimising over $\alpha$ (which requires more computations) gives a slightly better estimate

$$B < 15.2.$$  

\[\Box\]

11.5 Remark. The proof presented here is from [2]. The heart of the argument is based on Stein’s method (Step 2), introduced by Charles Stein, who developed this influential technique for teaching purposes of the central limit theorem for his course in statistics.

11.6 Example. Let us apply the Berry-Esseen theorem to i.i.d. Bernoulli random variables $X_1, \ldots, X_n$ with parameter $0 < p < 1$. We have $\mathbb{E}X_i = p$, $\text{Var}(X_i) = p(1 - p)$ and we obtain for every real $t$ and every integer $n \geq 1$

$$\left| \mathbb{P}\left( \frac{X_1 + \ldots + X_n - np}{\sqrt{np(1 - p)}} \leq t \right) - \mathbb{P}(Z \leq t) \right| \leq C \frac{\rho}{\sqrt{n}},$$

where

$$\rho = \mathbb{E} \left| \frac{X_1 - p}{\sqrt{p(1 - p)}} \right|^3 = \frac{p(1 - p)^3 + (1 - p)p^3}{\sqrt{p(1 - p)^3}} = \frac{1 - 2p(1 - p)}{\sqrt{p(1 - p)}}.$$  

In particular, when $np$ is of the constant order for large $n$, the Berry-Esseen theorem is not useful at all because the bound of the error, $C \frac{\rho}{\sqrt{n}}$, is of the order $\frac{C}{\sqrt{np(1 - p)}}$ which is constant. This might suggest that the Gaussian approximation is not valid in this case, which is in fact true in view of the Poisson limit theorem (Theorem 10.16).

11.2 Local central limit theorem

In applications we often need to address the following: suppose $X_1, X_2, \ldots$ are i.i.d. discrete, say integer-valued random variables and we would like to know for large $n$ what is the approximate value of $\mathbb{P}(X_1 + \ldots + X_n = x_n)$ for some $x_n \in \mathbb{Z}$. If $\mathbb{E}X_i^2 < \infty$, 

"
\[ \mu = \mathbb{E}X_1, \sigma^2 = \text{Var}(X_1) \text{ and } \frac{x_n - n\mu}{\sqrt{n}} \approx y \text{ is of constant order for large } n, \text{ by the central limit theorem,} \]

\[
\mathbb{P}(X_1 + \ldots + X_n = x_n) \\
= \mathbb{P}\left(\frac{x_n - \frac{1}{2} - 1}{\sqrt{n}} < X_1 + \ldots + X_n - \frac{1}{2} \right) \\
= \mathbb{P}\left(\frac{x_n - n\mu}{\sqrt{n}} - \frac{1}{2\sqrt{n}} < \frac{X_1 + \ldots + X_n - n\mu}{\sqrt{n}} \right) < \frac{x_n - n\mu}{\sqrt{n}} + \frac{1}{2}\sqrt{n} \\
\approx \frac{1}{\sqrt{2\pi}\sigma}\int_{\frac{x_n - n\mu}{\sqrt{n}} - \frac{1}{2}\sqrt{n}}^{\frac{x_n - n\mu}{\sqrt{n}} + \frac{1}{2}\sqrt{n}} e^{-\frac{t^2}{2\sigma^2}} dt \\
\approx \frac{1}{\sqrt{n\sqrt{2\pi}\sigma}} e^{-\frac{x^2}{2\sigma^2}},
\]

obtaining the approximation for \( \mathbb{P}(X_1 + \ldots + X_n = x_n) \) by the Gaussian density. To control the error in this approximation, we cannot simply use the Berry-Esseen theorem here because its error bound \( O\left(\frac{1}{\sqrt{n}}\right) \) is of the same order as the value of our approximation \( \frac{1}{\sqrt{n\sqrt{2\pi}\sigma}} e^{-\frac{x^2}{2\sigma^2}} \). The local central limit theorem addresses this deficiency. We only discuss the discrete case. There are also versions which give approximations to densities of sums of i.i.d. continuous random variables.

We shall use the common notation \( a + b\mathbb{Z} \) for the set \( \{a + bx, x \in \mathbb{Z}\} \).

**11.7 Theorem** (Local central limit theorem). Let \( X_1, X_2, \ldots \) be i.i.d. integer-valued random variables such that \( \mathbb{E}X_1^2 < \infty \). Suppose \( X_i \) is not supported on any proper subprogression of \( \mathbb{Z} \), that is there are no \( r > 1, a \in \mathbb{R} \) such that \( \mathbb{P}(X_i \in a + r\mathbb{Z}) = 1 \).

Denote \( \mu = \mathbb{E}X_1, \sigma = \sqrt{\text{Var}(X_1)} \) and

\[
p_n(x) = \mathbb{P}\left(\frac{X_1 + \ldots + X_n - n\mu}{\sqrt{n}} = x\right), \quad x \in \mathbb{Z} - n\mu/\sqrt{n}.
\]

Then

\[
\sup_{x \in \mathbb{Z} - n\mu/\sqrt{n}} \left|\sqrt{n}p_n(x) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\right| \xrightarrow{n \to \infty} 0.
\]

**11.8 Lemma.** For an integer-valued random variable \( X \) and an integer \( k \), we have

\[
\mathbb{P}(X = k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk}\phi_X(t)dt.
\]

**Proof.** Note that for two integers \( k \) and \( l \), we have

\[
1_{l=k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(l-k)} dt.
\]

Thus

\[
\mathbb{P}(X = k) = \mathbb{E}1_{X=k} = \mathbb{E} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{it(X-k)} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk}\mathbb{E}e^{itX} dt \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-itk}\phi_X(t)dt.
\]

\[\square\]
Proof of Theorem 11.7. Applying Lemma 11.8 to $X_1 + \ldots + X_n$ and changing the variables yields

$$p_n(x) = \mathbb{P}(X_1 + \ldots + X_n = x \sqrt{n} + n\mu)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it(x\sqrt{n} + n\mu)} \phi_{X_1 + \ldots + X_n}(t)dt$$

$$= \frac{1}{\sqrt{n}} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi \sqrt{n}} e^{-itx} \left[ e^{-itx} \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) \right]^n dt.$$

Using that the characteristic function of a centred Gaussian random variable with variance $1/\sigma^2$ is $e^{-\frac{x^2}{2\sigma^2}}$, we have

$$e^{-\frac{x^2}{2\sigma^2}} = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-t^2\sigma^2/2} dt,$$

which gives (by symmetry, we can write $e^{-itx}$ instead of $e^{itx}$)

$$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-t^2\sigma^2/2} dt.$$

Therefore,

$$\left| \sqrt{n} p_n(x) - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \right| \leq \frac{1}{2\pi} \int_{|t| \leq \pi \sqrt{n}} \left| \phi_{X_1 - \mu} \left( \frac{t}{\sqrt{n}} \right) \right|^n e^{-t^2\sigma^2/2} dt$$

$$+ \frac{1}{2\pi} \int_{|t| \geq \pi \sqrt{n}} e^{-t^2\sigma^2/2} dt.$$

Since the right hand side does not depend on $x$, we need to show that it converges to 0 as $n \to \infty$. The second integral clearly does. To deal with the first integral, we change the variables

$$\int_{|t| \leq \pi \sqrt{n}} \left| \phi_{X_1 - \mu} \left( \frac{t}{\sqrt{n}} \right) \right|^n e^{-t^2\sigma^2/2} dt = \frac{1}{\sigma} \int_{|t| \leq \pi \sqrt{n}} \left| \phi_{X_1 - \mu} \left( \frac{t}{\sqrt{n}} \right) \right|^n e^{-t^2/2} dt,$$

let $X_1 = \frac{X_1 - \mu}{\sigma}$ (which has mean 0 and variance 1) and break it into two pieces

$$\int_{|t| \leq \sqrt{n}} \left| \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) \right|^n e^{-t^2/2} dt + \int_{\sqrt{n} \leq |t| \leq \pi \sqrt{n}} \left| \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) \right|^n e^{-t^2/2} dt. \quad (11.4)$$

Recall from the proof of the central limit theorem that

$$\phi_{X_1} \left( \frac{t}{\sqrt{n}} \right)^n \to e^{-t^2/2}$$

and by Taylor’s formula,

$$\phi_{X_1}(t) = 1 - \frac{t^2}{2} + t^2 R(t),$$

for some (complex-valued) function $R$ such that $R(t) \to 0$ as $t \to 0$. Choose $\varepsilon < 1$ such that $|R(t)| < \frac{1}{4}$ for all $|t| < \varepsilon$. Then for $|t| \leq \varepsilon \sqrt{n}$,

$$\left| \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) \right| \leq 1 - \frac{t^2}{2n} + \frac{t^2}{4n} = 1 - \frac{t^2}{2n} \leq e^{-\frac{t^2}{2n}},$$

and for $|t| > \varepsilon \sqrt{n}$,

$$\left| \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) \right| \leq \frac{1}{4}.$$
so
\[
\left| \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) - e^{-t^2/2} \right| \leq e^{-t^2/4} + e^{-t^2/2}.
\]

By Lebesgue’s dominated convergence theorem, the first piece in (11.4) converges to 0 as \( n \to \infty \). Finally, to handle the second piece, we claim that: \( |\phi_{X_1}(t)| < c_\varepsilon \) for all \( \varepsilon \leq |t| \leq \pi \sigma \) for some constant \( c_\varepsilon < 1 \). This suffices because then
\[
\int_{\varepsilon \sqrt{\pi} \leq |t| \leq \pi \sqrt{\pi}} \left| \phi_{X_1} \left( \frac{t}{\sqrt{n}} \right) - e^{-t^2/2} \right| dt \leq \int_{\varepsilon \sqrt{\pi} \leq |t| \leq \pi \sqrt{\pi}} (c_\varepsilon + e^{-t^2/2}) dt
\]
and the right hand side clearly goes to 0 as \( n \to \infty \). Now we use that \( X_1 \) is integer-valued, not concentrated on any proper subprogresion to show the claim. Since \( X_1 \) is integer-valued, \( \phi_{X_1} \) is \( 2\pi \)-periodic and in particular \( \phi_{X_1}(2\pi) = 1 \). Moreover, \( |\phi_{X_1}(t)| < 1 \) for all \( 0 < t < 2\pi \). Otherwise, if \( |\phi_{X_1}(t_0)| = 1 \) for some \( 0 < t_0 < 2\pi \), then \( e^{it_0 X_1} \) is constant, say equal to \( e^{i\alpha} \). Consequently, \( X_1 \in \frac{\alpha}{t_0} + \frac{2\pi}{t_0} \mathbb{Z} \), which contradicts the assumption. By periodicity and continuity, there is \( c_\varepsilon < 1 \) such that \( |\phi_{X_1}(t)| < c_\varepsilon \) for all \( \varepsilon < |t| \leq \pi \).

Since \( \phi_{X_1}(t) = e^{-it^2/2}\phi_{X_1}(\frac{t}{\pi}) \), the claim follows.

Of course, in the proof it was not important that the \( X_i \) are integer-valued because by rescaling we could assume that they take values in \( a + r\mathbb{Z} \) for some \( a, r \in \mathbb{R} \). Such random variables are said to have a \textit{lattice distribution}. We finish this section by summarising periodicity properties of their characteristic functions, which played a crucial role in the proof of the local central limit theorem.

11.9 Lemma. For a random variable \( X \) with characteristic function \( \phi_X \) the following are equivalent

(i) \( \phi_X(s) = 1 \) for some \( s \neq 0 \),

(ii) \( \mathbb{P}(X \in \frac{2\pi}{s} \mathbb{Z}) = 1 \),

(iii) \( \phi_X \) is \( |s| \) periodic.

Proof. (i) \( \Rightarrow \) (ii): Since \( 1 = \phi_X(s) = \mathbb{E}(e^{isX}) + i\mathbb{E}(\sin(sX)) \), we have \( 0 = \mathbb{E}(1 - \cos(sX)) \). Since \( 1 - \cos(sX) \) is a nonnegative random variable whose expectation is 0, we have \( \mathbb{P}(\cos(sX) = 1) = 1 \) (see Theorem E.2 (c)), equivalently \( \mathbb{P}(sX \in 2\pi\mathbb{Z}) = 1 \).

(ii) \( \Rightarrow \) (iii): We have
\[
\phi_X(t + 2\pi |s|) = \mathbb{E}e^{i(t+2\pi|s|)X} = \sum_{k \in \mathbb{Z}} e^{i(t+2\pi|s|)\frac{2\pi k}{|s|}} \mathbb{P}\left(X = \frac{2\pi}{|s|} k\right) = \sum_{k \in \mathbb{Z}} e^{it \frac{2\pi}{|s|} k} \mathbb{P}\left(X = \frac{2\pi}{|s|} k\right) = \phi_X(t).
\]

(iii) \( \Rightarrow \) (i): Plainly, \( \phi_X(s) = \phi_X(0) = 1 \). \( \square \)
11.10 Lemma. Let $X$ be a random variable with characteristic function $\phi_X$. There are only 3 possibilities

(i) $|\phi_X(t)| < 1$ for every $t \neq 0$,

(ii) $|\phi_X(s)| = 1$ for some $s > 0$ and $|\phi_X(t)| < 1$ for all $0 < t < s$ and then $\phi_X$ is $s$-periodic and $X \in a + \frac{2\pi}{s} \mathbb{Z}$ a.s. for some $a \in \mathbb{R}$,

(iii) $|\phi_X(t)| = 1$ for every $t \in \mathbb{R}$ and then we have that $\phi_X(t) = e^{ita}$ for some $a \in \mathbb{R}$, that is $X = a$ a.s.

If (ii) holds, $X$ has a lattice distribution and since $|\phi_X(t)| < 1$ for all $0 < t < s$, by Lemma 11.9, $s$ is the largest $r > 0$ such that $\mathbb{P}(X \in a + r\mathbb{Z}) = 1$. We sometimes call $s$ the span of the distribution of $X$.

Proof. Let us first explain the implication in (ii). Suppose $|\phi_X(s)| = 1$ for some $s > 0$. Then $\phi_X(s) = e^{ia}$ for some $a \in \mathbb{R}$. Since $1 = e^{-ia}\phi_X(s) = \phi_{X-a}(s)$, by Lemma 11.9 applied to $X-a$, we get that $X-a \in \frac{2\pi}{s} \mathbb{Z}$ a.s. and $\phi_{X-a}$ is $s$-periodic, so $\phi_X = e^{ia}\phi_{X-a}$ is $s$-periodic.

To prove the trichotomy, suppose (i) and (ii) do not hold. Then there is a positive sequence $t_n \to 0$ such that $|\phi_X(t_n)| = 1$. Consequently, by what we just proved, there are $a_n \in \mathbb{R}$ such that $X \in a_n + \frac{2\pi}{t_n} \mathbb{Z}$ a.s. and $\phi_X$ is $t_n$-periodic. Without loss of generality, we can pick $a_n \in (-\frac{\pi}{t_n}, \frac{\pi}{t_n})$. Since $t_n \to 0$, we have $\mathbb{P}\left(X \in (-\frac{\pi}{t_n}, \frac{\pi}{t_n})\right) \to 1$, which combined with $X \in a_n + \frac{2\pi}{t_n} \mathbb{Z}$ and $a_n \in (-\frac{\pi}{t_n}, \frac{\pi}{t_n})$ gives $\mathbb{P}(X = a_n) \to 1$. Consequently, there is $n_0$ such that for all $n \geq n_0$, $\mathbb{P}(X = a_n) > 3/4$, but then all $a_n$, $n \geq n_0$ have to be equal, say $a_n = a$ and $\mathbb{P}(X = a_n) \to 1$ finally gives $\mathbb{P}(X = a) = 1$. Then $\phi_X(t) = e^{ita}$, consequently (iii) holds.

\hfill \Box

11.3 Poisson limit theorem

For probability measures $\mu$ and $\nu$ supported on $\mathbb{Z}$, the total variation distance between $\mu$ and $\nu$ is

$$\|\mu - \nu\|_{TV} = \sup_{A \subseteq \mathbb{Z}} |\mu(A) - \nu(A)|,$$

where the supremum is over all subsets $A$ of $\mathbb{Z}$.

Our goal is to prove the following quantitative version of Theorem 10.17.

11.11 Theorem. Let $\{X_{n,k}\}_{n \geq 1, 1 \leq k \leq n}$ be a triangular array of Bernoulli random variables, $X_{n,k} \sim \text{Ber}(p_{n,k})$, such that for every $n \geq 1$, the variables $X_{n,1}, \ldots, X_{n,n}$ are independent. Let $S_n = X_{n,1} + \cdots + X_{n,n}$ and let $Z_n$ be a Poisson random variable with parameter $E S_n = \sum_{k=1}^{n} p_{n,k}$. Then

$$\|\mu_n - \nu_n\|_{TV} \leq \sum_{k=1}^{n} p_{n,k}^2,$$
where $\mu_n$ is the law of $S_n$ and $\nu_n$ is the law of $Z_n$.

11.12 Remark. This theorem quantitatively shows when and how close the sum of Bernoullis $S_n$ is to a Poisson random variable with parameter $\lambda_n = \mathbb{E}S_n$. For instance, in the case of Theorem 10.16, when $p_{n,k} = p_n$, we see that the distribution of $S_n$ is close to Poiss($\lambda_n$), as long as $\sum_{k=1}^n p_{n,k}^2 = np_n^2 = \frac{\lambda_n^2}{n} \to 0$; in particular, if $\lambda_n \to \infty$ with $\lambda_n = o(\sqrt{n})$.

En route to proving Theorem 11.11, we need to develop a few facts related to the total variation distance.

11.13 Theorem. For two probability measures $\mu$, $\nu$ on $\mathbb{Z}$, we have

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_{x \in \mathbb{Z}} |\mu(\{x\}) - \nu(\{x\})|.$$  

Proof. For a subset $A$ of $\mathbb{Z}$, by the triangle inequality, we have

$$2|\mu(A) - \nu(A)| = |\mu(A) - \nu(A)| + |\mu(A^c) - \nu(A^c)|$$

$$= \left| \sum_{x \in A} \left( \mu(\{x\}) - \nu(\{x\}) \right) \right| + \left| \sum_{x \in A^c} \left( \mu(\{x\}) - \nu(\{x\}) \right) \right|$$

$$\leq \sum_{x \in \mathbb{Z}} |\mu(\{x\}) - \nu(\{x\})|$$

with equality for $A = \{x \in \mathbb{Z}, \mu(\{x\}) \geq \nu(\{x\})\}$.  

As a corollary, we easily see that the total variation distance is a metric. Convergence in the total variation distance is equivalent to pointwise convergence on atoms (this can be seen as an analogy in this discrete setup to Scheffé’s lemma from Exercise 8.8; the proofs are of course identical).

11.14 Theorem. Let $(\mu_n)_n$ be a sequence of probability measures on $\mathbb{Z}$. For a probability measure $\mu$ on $\mathbb{Z}$, we have

$$\|\mu_n - \mu\|_{TV} \xrightarrow{n \to \infty} 0 \quad \text{if and only if} \quad \forall x \in \mathbb{Z} \quad \mu_n(\{x\}) \xrightarrow{n \to \infty} \mu(\{x\}).$$

Proof. To ease the notation, let $p_{n,x} = \mu_n(\{x\})$ and $p_x = \mu(\{x\})$. In view of Theorem 11.13, we want to show that

$$\sum_{x \in \mathbb{Z}} |p_{n,x} - p_x| \xrightarrow{n \to \infty} 0 \quad \text{if and only if} \quad \forall x \in \mathbb{Z} \quad p_{n,x} \xrightarrow{n \to \infty} p_x.$$  

Implication “$\Rightarrow$” is clear. For the other one, note that

$$|p_x - p_{n,x}| = (p_x - p_{n,x})_+ + (p_x - p_{n,x})_-.$$
Since  
\[ 0 = 1 - 1 = \sum_{x \in \mathbb{Z}} (p_x - p_{n,x}) = \sum_{x \in \mathbb{Z}} (p_x - p_{n,x})_+ - \sum_{x \in \mathbb{Z}} (p_x - p_{n,x})_-, \]
we get  
\[ \sum_{x \in \mathbb{Z}} |p_x - p_{n,x}| = 2 \sum_{x \in \mathbb{Z}} (p_x - p_{n,x})_+. \]
Moreover, \( (p_x - p_{n,x})_+ \leq p_x \), so Lebesgue’s dominated convergence theorem finishes the argument. □

This in turn easily leads to the conclusion that the weak convergence is equivalent to convergence in total variation distance.

11.15 Corollary. Let \((\mu_n)_n\) be a sequence of probability measures on \(\mathbb{Z}\). For a probability measure \(\mu\) on \(\mathbb{Z}\), we have  
\[ \|\mu_n - \mu\|_{TV} \xrightarrow{n \to \infty} 0 \quad \text{if and only if} \quad \mu_n \xrightarrow{d \to \infty} \mu. \]

Proof. “⇒”: Fix a continuous bounded function \(f : \mathbb{R} \to \mathbb{R}\) with \(M = \sup |f|\). Then,  
\[ \left| \int f \, d\mu_n - \int f \, d\mu \right| = \sum_{x \in \mathbb{Z}} |f(x)(\mu(\{x\}) - \nu(\{x\}))| \leq M \sum_{x \in \mathbb{Z}} |\mu(\{x\}) - \nu(\{x\})| = 2M\|\mu_n - \mu\|_{TV}. \]

“⇐”: Fix \(x \in \mathbb{Z}\). Let \(f : \mathbb{R} \to \mathbb{R}\) be continuous bounded such that \(f(t) = 0\) for all \(t \in \mathbb{Z} \setminus \{x\}\) and \(f(x) = 1\). Then  
\[ \left| \int f \, d\mu_n - \int f \, d\mu \right| = |\mu_n(\{x\}) - \mu(\{x\})| \]
and we conclude by Theorem 11.14. □

Of course, the definition of \(\|\cdot\|_{TV}\) and the above results extend verbatim from \(\mathbb{Z}\) to any countable set.

To prove a quantitative version of the Poisson limit theorem, we need three lemmas.

11.16 Lemma. Let \(\mu_1, \mu_2, \nu_1, \nu_2\) be probability measures on \(\mathbb{Z}\). Then  
\[ \|\mu_1 \otimes \mu_2 - \nu_1 \otimes \nu_2\|_{TV} \leq \|\mu_1 - \nu_1\|_{TV} + \|\mu_2 - \nu_2\|_{TV}. \]

Proof. By Theorem 11.13 and the triangle inequality, we have  
\[ 2\|\mu_1 \otimes \mu_2 - \nu_1 \otimes \nu_2\|_{TV} = \sum_{x,y \in \mathbb{Z}} |\mu_1(\{x\})\mu_2(\{y\}) - \nu_1(\{x\})\nu_2(\{y\})| \]
\[ \leq \sum_{x,y \in \mathbb{Z}} |\mu_1(\{x\})\mu_2(\{y\}) - \nu_1(\{x\})\mu_2(\{y\})| + \sum_{x,y \in \mathbb{Z}} |\nu_1(\{x\})\mu_2(\{y\}) - \nu_1(\{x\})\nu_2(\{y\})| \]
\[ = \sum_{y \in \mathbb{Z}} \mu_2(\{y\}) \sum_{x \in \mathbb{Z}} |\mu_1(\{x\}) - \nu_1(\{x\})| + \sum_{x \in \mathbb{Z}} \nu_1(\{x\}) \sum_{y \in \mathbb{Z}} |\mu_2(\{y\}) - \nu_2(\{y\})| \]
\[ = 2\|\mu_1 - \nu_1\|_{TV} + 2\|\mu_2 - \nu_2\|_{TV}. \] □
Recall that the convolution of two measures $\mu$ and $\nu$ on $\mathbb{Z}$ is defined as a measure

$$(\mu * \nu)(A) = \sum_{y \in \mathbb{Z}} \mu(A - y) \nu(y), \quad A \subset \mathbb{Z}.$$ 

It is the distribution of the sum $X + Y$ of two independent $\mathbb{Z}$-valued random variables $X, Y$ with law $\mu, \nu$ respectively.

**11.17 Lemma.** Let $\mu_1, \mu_2, \nu_1, \nu_2$ be probability measures on $\mathbb{Z}$. Then

$$\|\mu_1 * \mu_2 - \nu_1 * \nu_2\|_{TV} \leq \|\mu_1 - \nu_1\|_{TV} + \|\mu_2 - \nu_2\|_{TV}.$$ 

**Proof.** By Theorem 11.13 and the triangle inequality, we have

$$2\|\mu_1 * \mu_2 - \nu_1 * \nu_2\|_{TV} = \sum_{x \in \mathbb{Z}} |(\mu_1 * \mu_2)(\{x\}) - (\nu_1 * \nu_2)(\{x\})|$$

$$= \sum_{x \in \mathbb{Z}} \left| \sum_{y \in \mathbb{Z}} \mu_1(\{x - y\}) \mu_2(y) - \sum_{y \in \mathbb{Z}} \nu_1(\{x - y\}) \nu_2(y) \right|$$

$$\leq \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} |\mu_1(\{x - y\}) \mu_2(y) - \nu_1(\{x - y\}) \nu_2(y)|$$

$$= \sum_{x \in \mathbb{Z}} \sum_{y \in \mathbb{Z}} |\nu_1(\{x\}) \nu_2(y) - \nu_1(\{x\}) \nu_2(y)|$$

$$= 2\|\mu_1 - \nu_1\|_{TV} + \|\mu_2 - \nu_2\|_{TV}. \quad \square$$

**11.18 Lemma.** Let $p \in [0, 1]$. Let $\mu$ be the Bernoulli distribution with parameter $p$ and let $\nu$ be the Poisson distribution with parameter $p$. Then

$$\|\mu - \nu\|_{TV} \leq p^2.$$ 

**Proof.** By Theorem 11.13, we have

$$2\|\mu - \nu\|_{TV} = |\mu(\{0\}) - \nu(\{0\})| + |\mu(\{1\}) - \nu(\{1\})| + \sum_{k \geq 2} \nu(\{k\})$$

$$= |1 - p - e^{-p}| + |p - pe^{-p}| + 1 - e^{-p}(1 + p)$$

$$= e^{-p} - 1 + p + p - pe^{-p} + 1 - e^{-p}(1 + p) = 2p(1 - e^{-p}) \leq 2p^2$$

(we use $e^{-p} \geq 1 - p$). \quad \square

**Proof of Theorem 11.11.** Let $\mu_n$ be the distribution of $S_n = X_{n,1} + \cdots + X_{n,n}$ and let $\mu_{n,k}$ be the distribution of $X_{n,k}$ which is Bernoulli with parameter $p_{n,k}$. Then

$$\mu_n = \mu_{n,1} * \cdots * \mu_{n,n}. \quad \square$$
Let \( \nu_n \) be the Poisson distribution with parameter \( \lambda_n = \mathbb{E}S_n = \sum_{k=1}^n p_{n,k} \) and let \( \nu_{n,k} \) be the Poisson distribution with parameter \( p_{n,k} \). Since sums of independent Poisson random variables are Poisson, we have

\[
\nu_n = \nu_{n,1} \cdots \nu_{n,n}.
\]

Thus, by Lemmas 11.17, 11.16 and 11.18,

\[
\|\mu_n - \nu_n\|_{TV} \leq \sum_{k=1}^n p_{n,k}^2.
\]

(11.5)

11.19 Remark. We can quickly deduce Theorem 10.17 from (11.5). Let \( \nu \) be the Poisson distribution with parameter \( \lambda \). As we saw in the proof of Theorem 10.17, the right hand side of (11.5) goes to 0 as \( n \to \infty \). Moreover, by Theorem 11.14, \( \|\nu_n - \nu\|_{TV} \to 0 \) because \( \lambda_n \to \lambda \). Thus, \( \|\mu_n - \nu\|_{TV} \to 0 \), as desired.

11.20 Remark. When \( p_{n,k} = \frac{1}{n} \), we have \( \lambda_n = \lambda = 1 \) and from the second proof,

\[
\|\mu_n - \nu\|_{TV} \leq \frac{1}{n}.
\]

On the other hand, since \( 1 - x \geq e^{-x-x^2} \) and \( 1 - e^{-x} \geq \frac{x}{2} \) for \( x \in [0, \frac{1}{2}] \), we have for \( n \geq 2 \),

\[
|\mu_n(\{0\}) - \nu(\{0\})| = e^{-1} - \left(1 - \frac{1}{n}\right)^n \geq e^{-1} - e^{-1-1/n} = e^{-1} \left(1 - e^{-1/n}\right) \geq \frac{1}{2e\pi},
\]

which shows that the rate \( \frac{1}{n} \) is optimal here.
11.4 Exercises

1. Using the Berry-Esseen theorem, how can you bound the error you make in your approximation in Exercise 10.1?

2. In Exercise 10.2, using the Berry-Esseen theorem, show additionally that $|a_n - a| \leq \frac{15}{\sqrt{n}}$ for all $n \geq 1$.

3. For two probability measures on $\mathbb{Z}$, $\|\mu - \nu\|_{TV} \leq \delta$ if and only if there are random variables $X, Y$ with distributions $\mu, \nu$, respectively such that $\mathbb{P}(X \neq Y) \leq \frac{\delta}{2}$.
12 Conditional expectation

We begin with a motivating example. Let \((\Omega, \mathcal{F}, P)\) be a probability space and suppose we have two discrete random variables

\[ X: \Omega \rightarrow \{x_1, \ldots, x_m\}, \]
\[ Z: \Omega \rightarrow \{z_1, \ldots, z_n\}, \]

taking finitely many values. Recall the conditional probabilities

\[ P(X = x_i \mid Z = z_j) = \frac{P(X = x_i, Z = z_j)}{P(Z = z_j)} \]

and conditional expectations

\[ y_j = E(X \mid Z = z_j) = \sum x_i P(X = x_i \mid Z = z_j). \]

In this simple situation, the conditional expectation of \(X\) given \(Z\) is a random variable \(Y = E(X \mid Z)\) defined as

\[ Y(\omega) = y_j = E(X \mid Z = z_j) \quad \text{on} \quad \{Z = z_j\}. \]

In other words, “knowing” the value of \(Z\) amounts to the partitioning \(\Omega = \bigcup \{Z = z_j\}\) and \(Y\) is set to be constant \(y_j\) on the atoms \(\{Z = z_j\}\) of this partitioning. We point out two features of \(Y\) which will be central in the general definition. Let \(G = \sigma(Z)\) be the \(\sigma\)-algebra generated by \(Z\). Here \(G\) is generated by the partitioning \(\{Z = z_j\}\). Since \(Y\) is constant on the atoms of \(G\),

1. \(Y\) is \(G\)-measurable.

Note also that

\[ \mathbb{E}Y1_{\{Z = z_j\}} = y_j \mathbb{P}(Z = z_j) = \sum x_i P(X = x_i \mid Z = z_j) \mathbb{P}(Z = z_j) \]
\[ = \sum x_i P(X = x_i, Z = z_j) = E(X)1_{\{Z = z_j\}}. \]

Since every element in \(G\) is a union of the atoms \(\{Z = z_j\}\), by linearity, we thus have

2. \(\forall G \in G \quad \mathbb{E}Y1_G = E(X)1_G.\)

12.1 Construction

The following theorem due to Kolmogorov (1933) gives a way to define conditional expectations.

12.1 Theorem. Let \(X\) be a random variable on a probability space \((\Omega, \mathcal{F}, P)\) with \(E|X| < \infty\). Let \(G\) be a sub-\(\sigma\)-algebra of \(\mathcal{F}\). There is a random variable \(Y\) with \(E|Y| < \infty\) such that
(1) $Y$ is $\mathcal{G}$-measurable,

(2) $\forall G \in \mathcal{G}, \ EY 1_G = EX 1_G$.

Moreover, if $\tilde{Y}$ is another random variable with $E|\tilde{Y}| < \infty$ satisfying (1) and (2), then $\tilde{Y} = Y$ a.s.

This random variable $Y$ is called (a version) of the conditional expectation of $X$ given $\mathcal{G}$, denoted $E(X|\mathcal{G})$. We then write $Y = E(X|\mathcal{G})$. For a random variable $Z$, the conditional expectation of $X$ given $Z$, $E(X|Z)$, is defined as $E(X|\sigma(Z))$. More generally, we define $E(X|Z_1, Z_2, \ldots) = E(X|\sigma(Z_1, Z_2, \ldots))$. Of course, we also adopt the natural definition of the conditional probability of an event $A$ given $\mathcal{G}$, $P(A|\mathcal{G}) = E(1_A|\mathcal{G})$ (which is a random variable!).

The intuitive meaning of conditional expectations:

a) $E(X|A)$, that is given an event $A$: an experiment has been performed and all we know is whether $\omega \in A$ or not and we recalculate the expectation according to $P(\cdot|A)$

b) $E(X|Z)$, that is given a discrete random variable $Z$: an experiment has been performed and all we know about $\omega$ is in which set $\{Z = z_j\}$ it is, so $E(X|Z)(\omega)$ is still a random quantity, but constant on these sets

c) $E(X|\mathcal{G})$, that is given a sub-$\sigma$-algebra $\mathcal{G}$: an experiment has been performed and all we know about $\omega$ is $\{Z(\omega): Z$ is $\mathcal{G}$-measurable\}, so $E(X|\mathcal{G})(\omega)$ is a random quantity being the average of $X$ given this information.

The richer $\mathcal{G}$ is, the more we “know” about $X$, so that $E(X|\mathcal{G})$ more accurately describes $X$. In the two extreme cases:

a) if $\mathcal{G} = \{\emptyset, \Omega\}$ (a trivial $\sigma$-algebra, so no knowledge), then $E(X|\mathcal{G}) = EX$ (constant)

b) if $\mathcal{G} = \mathcal{F}$ (full knowledge), then $E(X|\mathcal{G}) = X$.

These can be simply verified by checking that the claimed variables $EX$ and $X$ respectively satisfy (1) and (2).

**Proof of Theorem 12.1.** We break the proof into several steps.

**Step 1: uniqueness.** Let $Y, \tilde{Y}$ be two integrable $\mathcal{G}$-measurable versions of $E(X|\mathcal{G})$, that is satisfying (1) and (2). Since $\{Y - \tilde{Y} > \frac{1}{n}\} \in \mathcal{G}$, we get from (2),

$$EY 1_{\{Y - \tilde{Y} > \frac{1}{n}\}} = EX 1_{\{Y - \tilde{Y} > \frac{1}{n}\}} = E\tilde{Y} 1_{\{Y - \tilde{Y} > \frac{1}{n}\}},$$

or,

$$0 = E(Y - \tilde{Y}) 1_{\{Y - \tilde{Y} > \frac{1}{n}\}} \geq \frac{1}{n} P(Y - \tilde{Y} > \frac{1}{n}),$$

135
which shows that \( P\left( Y - \tilde{Y} > \frac{1}{n}\right) = 0 \) and consequently, taking the union of these events over \( n \), \( P\left( Y - \tilde{Y} > 0\right) = 0 \). Swapping the roles of \( Y \) and \( \tilde{Y} \), we also get that \( P\left( \tilde{Y} - Y > 0\right) = 0 \), so \( P\left( Y \neq \tilde{Y}\right) = 0 \).

**Step 2: existence for** \( X \in L_2 \). Let \( X \in L_2(\Omega,\mathcal{F},P) \) and consider the subspace

\[
\mathcal{H} = L_2(\Omega,\mathcal{G},P) \subset L_2(\Omega,\mathcal{F},P)
\]

which is complete (as being an \( L_2 \) space – see Theorem 6.10), so by the existence of the orthogonal projection, Theorem 6.11, there is a random variable \( Y \in \mathcal{H} \) which is closest to \( X \), that is it satisfies

\[
E(X - Y)^2 = \inf\{E(X - W)^2, \ W \in \mathcal{H}\},
\]

or, equivalently,

\[
X - Y \perp \mathcal{H}, \text{ that is } \forall W \in \mathcal{G} \ E(X - Y)W = 0.
\]

We claim that this \( Y \) satisfies (1) and (2). Since \( Y \in \mathcal{H} \), it is \( \mathcal{G} \) measurable. For \( G \in \mathcal{G} \), letting \( W = 1_G \in \mathcal{H} \), we get \( E(X - Y)W = 0 \), that is (2).

**Step 3: basic and key properties of** \( E(X|\mathcal{G}) \) **for** \( X \in L_2 \).

1. **Linearity:** \( E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G}), \ a, b \in \mathbb{R} \), \( X, Y \in L_2 \)
2. **Monotonicity:** if \( X \in L_2 \), \( X \geq 0 \) a.s., then \( E(X|\mathcal{G}) \geq 0 \) a.s.
3. **If** \( X_1, X_2 \in L_2 \), \( X_1 \geq X_2 \) a.s., then \( E(X_1|\mathcal{G}) \geq E(X_2|\mathcal{G}) \) a.s.

Property (i) is clear because the orthogonal projection is a linear map. Property (ii) follows by an argument identical to the one from Step 1. Property (iii) follows from (i) and (ii).

**Step 4: existence of** \( E(X|\mathcal{G}) \) **for arbitrary** \( X \in L_1 \). We decompose \( X = X^+ - X^- \) and set \( E(X|\mathcal{G}) = E(X^+|\mathcal{G}) - E(X^-|\mathcal{G}) \).
12.2 Important properties

12.2 Theorem. All random variables are assumed to be integrable, \( \mathcal{G} \) is sub-\( \sigma \)-algebra.

(a) If \( Y = \mathbb{E}(X|\mathcal{G}) \), then \( \mathbb{E}Y = \mathbb{E}X \).

(b) If \( X \) is \( \mathcal{G} \)-measurable, then \( \mathbb{E}(X|\mathcal{G}) = X \).

(c) Linearity: \( \mathbb{E}(a_1X_1 + a_2X_2|\mathcal{G}) = a_1\mathbb{E}(X_1|\mathcal{G}) + a_2\mathbb{E}(X_2|\mathcal{G}) \) a.s., \( a_1, a_2 \in \mathbb{R} \) (understood as: if \( Y_i \) is a version of \( \mathbb{E}(X_i|\mathcal{G}) \), then \( a_1Y_1 + a_2Y_2 \) is a version of \( \mathbb{E}(a_1X_1 + a_2X_2|\mathcal{G}) \)).

(d) Positivity: if \( X \geq 0 \) a.s., then \( \mathbb{E}(X|\mathcal{G}) \geq 0 \) a.s. If \( X_1 \geq X_2 \), then \( \mathbb{E}(X_1|\mathcal{G}) \geq \mathbb{E}(X_2|\mathcal{G}) \) a.s.

(e) Lebesgue’s monotone convergence theorem: if \( 0 \leq X_n \uparrow X \) a.s., then \( \mathbb{E}(X_n|\mathcal{G}) \uparrow \mathbb{E}(X|\mathcal{G}) \) a.s.

(f) Fatou’s lemma: if \( X_n \geq 0 \), then

\[
\mathbb{E}(\liminf X_n|\mathcal{G}) \leq \liminf \mathbb{E}(X_n|\mathcal{G}) \quad \text{a.s.}
\]

(g) Lebesgue’s dominated convergence theorem: if \( \forall n \ |X_n| \leq V \) for some \( V \in L_1 \) and \( X_n \to X \), then

\[
\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G}) \quad \text{a.s.}
\]

(h) Jensen’s inequality: if \( f : \mathbb{R} \to \mathbb{R} \) is convex and \( f(X) \in L_1 \), then

\[
\mathbb{E}(f(X)|\mathcal{G}) \geq f(\mathbb{E}(X|\mathcal{G})) \quad \text{a.s.}
\]

(i) Tower property: if \( \mathcal{H} \subset \mathcal{G} \) is a sub-\( \sigma \)-algebra, then

\[
\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{H})|\mathcal{G}) \quad \text{a.s.}
\]

(j) “Taking out what is known”: if \( Z \) is a \( \mathcal{G} \)-measurable bounded random variable, then

\[
\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G}) \quad \text{a.s.}
\]

(k) “Role of independence”: if \( \mathcal{H} \) is a \( \sigma \)-algebra independent of \( \sigma(X, \mathcal{G}) \), then

\[
\mathbb{E}(X|\sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(X|\mathcal{G}) \quad \text{a.s.}
\]

In particular, if \( X \) is independent of \( \mathcal{H} \), for \( \mathcal{G} = \{\emptyset, \Omega\} \), we get

\[
\mathbb{E}(X|\mathcal{H}) = \mathbb{E}X \quad \text{a.s.}
\]

Proof. (a) We use (2) with \( G = \Omega \) to get \( \mathbb{E}X = \mathbb{E}X1_{\Omega} = \mathbb{E}Y1_{\Omega} = \mathbb{E}Y \).
(b) Obvious because \( Y = X \) clearly satisfies (1) and (2).

(c) It follows from the linearity of \( \mathbb{E}(\cdot)1_G \) for every fixed \( G \in \mathcal{G} \).

(d) It is done in a similar way as in Step 1 of the proof of Theorem 12.1.

(e) It is done in a similar way as in Step 3 of the proof of Theorem 12.1. We let \( Y_n = \mathbb{E}(Y_n|G) \), use monotonicity to put \( Y = \lim Y_n \) and argue through the usual Lebesgue’s monotone convergence theorem.

(f) Use (e) to deduce it as in the unconditional case.

(g) Use (f) to deduce it as in the unconditional case (see Appendix E).

(h) It is done as in the unconditional case in the proof of Theorem 6.2. We write \( f(x) = \sup_{\ell \in \mathcal{L}} \ell(x) \) for a countable family of linear functions \( \mathcal{L} \). Fix \( \ell \in \mathcal{L} \). Since \( f(X) \geq \ell(X) \), by monotonicity and linearity, \( \mathbb{E}(f(X)|G) \geq \mathbb{E}(\ell(X)|G) = \ell(\mathbb{E}(X|G)) \) a.s. Since \( \mathcal{L} \) is countable, this also holds for all \( \ell \in \mathcal{L} \) a.s. Taking the supremum over \( \ell \) finishes the argument.

(i) Let \( Y = \mathbb{E}(X|\mathcal{H}) \). It is \( \mathcal{H} \)-measurable, so also \( \mathcal{G} \)-measurable and by (b),
\[
\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(Y|\mathcal{G}) = Y.
\]

To check that
\[
\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = Y,
\]

it suffices to show that for every \( H \in \mathcal{H} \), \( \mathbb{E}(\mathbb{E}(X|\mathcal{G})1_H) = \mathbb{E}Y1_H \). Since \( H \in \mathcal{G} \), by the definition of \( \mathbb{E}(X|\mathcal{G}) \), the left hand side is \( \mathbb{E}X1_H \). Since \( H \in \mathcal{H} \), by the definition of \( \mathbb{E}(X|\mathcal{H}) \), the right hand side is also \( \mathbb{E}X1_H \).

(j) Thanks to linearity, without loss of generality we can assume that \( X \geq 0 \). Then the standard argument of complicating \( Z \) works because we can use monotone convergence (first we check the claim for \( Z = 1_G, G \in \mathcal{G} \), then by linearity we have it for simple \( Z \) and then for all \( \mathcal{G} \)-measurable bounded \( Z \)).

(k) Thanks to linearity, without loss of generality we can assume that \( X \geq 0 \). Let \( Y = \mathbb{E}(X|\mathcal{G}) \). We want to show that for every \( A \in \sigma(\mathcal{G}, \mathcal{H}) \), \( \mathbb{E}Y1_A = \mathbb{E}X1_A \). By Dynkin’s theorem, it suffices to this for all \( A \) in a \( \pi \)-system generating \( \sigma(\mathcal{G}, \mathcal{H}) \), so for every \( A = G \cap H \) with \( G \in \mathcal{G} \) and \( H \in \mathcal{H} \). Then, we have
\[
\mathbb{E}Y1_{G \cap H} = \mathbb{E}(Y1_G1_H) = \mathbb{E}Y1_G1_H
\]
because \( Y1_G \) is \( \mathcal{G} \)-measurable, hence independent of \( \mathcal{H} \). By the definition of \( \mathbb{E}(X|\mathcal{G}) \), \( \mathbb{E}Y1_G = \mathbb{E}X1_G \). Since \( X1_G \) and \( 1_H \) are independent, we conclude that
\[
\mathbb{E}Y1_{G \cap H} = \mathbb{E}X1_G1_H = \mathbb{E}X1_{G \cap H}.
\]

\( \square \)
12.3 Basic examples

12.3 Example. Let \((X, Z)\) be a continuous random vector in \(\mathbb{R}^2\) with density \(f\). Then
\[
f_X(x) = \int_{\mathbb{R}} f(x, z)dz \quad \text{is the density of } X,
\]
\[
f_Z(z) = \int_{\mathbb{R}} f(x, z)dx \quad \text{is the density of } Z.
\]
Recall we define the conditional density \(f_{X|Z}\) of \(X\) given \(Z\) as
\[
f_{X|Z}(x|z) = \begin{cases} f(x, z) / f_Z(z), & \text{if } f_Z(z) > 0, \\ 0, & \text{otherwise.} \end{cases}
\]
Let \(h: \mathbb{R} \to \mathbb{R}\) be a Borel function such that \(E|h(X)| < \infty\). Let
\[
g(z) = \int_{\mathbb{R}} h(x)f_{X|Z}(x|z)dx.
\]

Claim. \(Y = g(Z)\) is a version of \(E(h(X)|Z)\).

Proof. Clearly \(Y\) is \(Z\)-measurable (i.e. \(\sigma(Z)\)-measurable), so it suffices to check that for every \(A \in \sigma(Z)\), \(Eg(Z)1_A = Eh(X)1_A\). We have \(A = \{Z \in B\}\) for some Borel set \(B\). Then, by the definition of \(g\),
\[
Eg(Z)1_A = Eg(Z)1_{z \in B} = \int g(z)1_B(z)f_Z(z)dz = \int h(x)1_B(z)f(x, z)dx dz = Eh(X)1_{z \in B}.
\]

\(\square\)

12.4 Example. Let \(X_1, \ldots, X_n\) be independent random variables. Let \(h: \mathbb{R}^n \to \mathbb{R}\) be a bounded Borel function. Then
\[
E(h(X_1, \ldots, X_n)|X_1) = g(X_1),
\]
where \(g(x) = Eh(x, X_2, \ldots, X_n), x \in \mathbb{R}\). Clearly \(g(X_1)\) is \(X_1\)-measurable. That condition (2) holds, follows from Fubini’s theorem.

12.5 Example. Let \(X_1, \ldots, X_n\) be i.i.d. integrable random variables and let \(S_n = X_1 + \cdots + X_n\). Let \(G_n = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots)\). By the “Role of independence” property (point (k) of Theorem 12.2), we have
\[
E(X_1|G_n) = E(X_1|S_n).
\]
To find the latter, we use symmetry, which gives
\[
Y = E(X_1|S_n) = E(X_2|S_n) = \ldots = E(X_n|S_n).
\]
By linearity, \(nY = E(S_n|S_n) = S_n\), thus \(Y = \frac{S_n}{n}\), that is
\[
E(X_1|S_n) = \frac{S_n}{n}.
\]
We finish by remarking that in all of these examples $E(X|Z)$ is of the form $g(Z)$ for some Borel function $g$. This holds in general and immediately follows from the following lemma.

**12.6 Lemma.** If $Z$ is a random vector in $\mathbb{R}^n$ and $X$ is a $Z$-measurable random variable, then $X = g(Z)$ for some Borel function $g: \mathbb{R}^n \to \mathbb{R}$.

A proof by a standard complication of $X$ is left as an exercise. Thus we immediately get the following general observation about conditional expectations.

**12.7 Theorem.** Let $Z$ be a random vector in $\mathbb{R}^n$ and let $X$ be an integrable random variable. Then there is a Borel function $g: \mathbb{R}^n \to \mathbb{R}$ such that $g(Z)$ is a version of $E(X|Z)$.

This in particular gives a way around defining conditioning on events of probability 0. We set $E(X|Z = z) = g(z)$, $z \in \mathbb{R}^n$, where $g$ is the function provided by Theorem 12.7.
12.4 Exercises

1. We toss a fair coin 10 times. Let $X$ be the number of heads altogether and $Y$ in the first 4 tosses. Find $E(X|Y)$ and $E(Y|X)$.

2. Give an example of random variables $X$ and $Y$ which are not independent, but $E(X|Y) = E(X)$.

3. Let $(X,Y)$ be a centred Gaussian random vector in $\mathbb{R}^2$. Show that $E(X|Y) = \frac{XY}{2Y}$.

4. Let $\rho \in (-1,1)$ and let $(U,V)$ be a random vector in $\mathbb{R}^2$ with density
   
   $$f(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} (u^2 - 2\rho uv + v^2) \right\}, \quad (u,v) \in \mathbb{R}^2.$$  

   Find $E(U|V)$.

5. Let $X_1,\ldots,X_n$ be i.i.d. random variables uniform on $[0,1]$. Find the conditional expectation $E(X_1|\max\{X_1,\ldots,X_n\})$.

6. Let $X$ be a nonnegative integrable random variable and let $\mathcal{G}$ be a sub-$\sigma$-algebra. Show that
   
   a) $E(X|\mathcal{G}) = \int_0^\infty P(X > t|\mathcal{G}) \, dt,$
   
   b) $P(X > t|\mathcal{G}) \leq t^{-p}E(X^p|\mathcal{G})$, $p,t > 0$, provided that $X \in L_p$.

7. Let $X$ and $Y$ be independent random variables uniform on $[-1,1]$. Find $E(X|X^2+Y^2)$ and $E(X^2|X+Y)$.

8. Suppose $X,Y$ are integrable random variables such that $E(X|Y) = Y$ a.s. and $E(Y|X) = X$ a.s. Then $X = Y$ a.s.

9. Prove Lemma 12.6
13 Martingales I

13.1 Definitions and basic examples

By a process $X = (X_n)_{n \geq 0} = (X_0, X_1, X_2, \ldots)$ we just mean a sequence of random variables $X_0, X_1, \ldots$ (index $n$ is thought of as time which is discrete here). A filtration $\{F_n\}_{n \geq 0}$ on a probability space $(\Omega, F, \mathbb{P})$ is a nondecreasing sequence of sub-$\sigma$-algebras of $F$,

$$F_0 \subset F_1 \subset F_2 \subset \ldots \subset F.$$

We set

$$F_\infty = \sigma(\bigcup_{n \geq 0} F_n),$$

which is also a sub-$\sigma$-algebra of $F$. Sometimes $(\Omega, F, \{F_n\}_{n \geq 0}, \mathbb{P})$ is then referred to as a filtered probability space.

Intuitively, $F_n$ carries information available at time $n$.

Given a process $X = (X_n)_{n \geq 0}$, its natural filtration is given by

$$F_n = \sigma(X_0, X_1, \ldots, X_n), \quad n \geq 0.$$

The process $X$ is called adapted (to $\{F_n\}$) if for every $n \geq 0$, $X_n$ is $F_n$-measurable. It is called predictable or previsible if for every $n \geq 1$, $X_n$ is $F_{n-1}$-measurable and $X_0$ is constant. Intuitively, if $X$ is adapted, then at time $n$ we know $X_n(\omega)$ (and $X_0(\omega), \ldots, X_{n-1}(\omega)$); if it is predictable, at time $n$ we additionally know $X_{n+1}(\omega)$.

A process $X = (X_n)_{n \geq 0}$ is a martingale (with respect to a filtration $\{F\}_{n \geq 0}$) if

(i) $X$ is adapted,

(ii) $E|X_n| < \infty$, for every $n \geq 0$,

(iii) $E(X_n | F_{n-1}) = X_{n-1}$ a.s., for every $n \geq 1$.

It is called a supermartingale (respectively submartingale) if (iii) is replaced by: $E(X_n | F_{n-1}) \leq X_{n-1}$ (resp. $\geq X_{n-1}$) a.s., for every $n \geq 1$.

Submartingales correspond to subharmonic functions ($f$ on $\mathbb{R}^n$ is subharmonic if and only if $f(B)$ is a local submartingale, where $B$ is a standard Brownian motion on $\mathbb{R}^n$).

13.1 Remark. If $X$ is a supermartingale, then

$$EX_n = E(E(X_n | F_{n-1})) \leq EX_{n-1},$$

so its averages do not increase over time. For martingales, the averages are of course constant.
The word ‘martingale’ originates from the Spanish word ‘almâtaga’ = fastening. In everyday use, a martingale refers to a horse gear item which is a strap attached to the reins of horse used to prevent the horse from raising its head too high (P. Halmos allegedly sent J. Doob such a martingale as a “gift”; J. Hammersley, after his lecture in 1965 was made aware of this meaning of the word martingale, thought that the mathematical term originated from its equestrian meaning and started calling his martingale process, “harness process”). As a mathematical term, it was first used in French by J. Ville in his thesis in 1939, where initially he writes “game system or martingale” and then just continues with “martingale”. It was J. Doob, the godfather of martingale theory, who after reviewing Ville’s thesis, coined and used the term ‘martingale’.

13.2 Example. Let $X_1, X_2, \ldots$ be independent integrable random variables with mean 0, $E X_k = 0$ for every $k \geq 1$. Let
\[
S_0 = 0,
S_n = X_1 + \cdots + X_n, \quad n \geq 1
\]
and
\[
F_0 = \{\emptyset, \Omega\},
F_n = \sigma(X_1, \ldots, X_n), \quad n \geq 1.
\]
Then $(S_n)_{n \geq 0}$ is a martingale with respect to the natural filtration $\{F_n\}$. Indeed, because $S_n = S_{n-1} + X_n$ and $S_{n-1}$ is $F_{n-1}$ measurable, whereas $X_n$ is independent of $F_{n-1}$, we have
\[
E(S_n|F_{n-1}) = E(S_{n-1}|F_{n-1}) + E(X_n|F_{n-1}) = S_{n-1} + E X_n = S_{n-1}.
\]

13.3 Example. Let $X_1, X_2, \ldots$ be independent nonnegative integrable random variables with $E X_k = 1$ for every $k \geq 1$. Let
\[
M_0 = 1,
M_n = X_1 \cdots X_n, \quad n \geq 1
\]
and
\[
F_0 = \{\emptyset, \Omega\},
F_n = \sigma(X_1, \ldots, X_n), \quad n \geq 1.
\]
Then $(M_n)_{n \geq 0}$ is a martingale with respect to the natural filtration $\{F_n\}$, thanks to the “taking out what is known property”.

13.4 Example. Let $X$ be an integrable random variable and let $\{F_n\}_{n \geq 0}$ be a filtration. Let
\[
X_n = E(X|F_n), \quad n \geq 0.
\]

143
Then \((X_n)_{n \geq 0}\) is a martingale (with respect to \(\mathcal{F}_n\)), by the tower property.

Intuitively, martingales model fair gambling games. If \(X_n\) is your capital at time \(n\), then \(X_n - X_{n-1}\) are your net winnings in game \(n\). Note that \((X_n)\) is a martingale if and only if \(\mathbb{E}(X_n - X_{n-1} | \mathcal{F}_{n-1}) = 0\), that is the game is fair (with all the knowledge at time \(n-1\), the net winnings for game \(n\) are 0 on average).

### 13.2 Martingale transforms and stopping times

Let \(X = (X_n)_{n \geq 0}\) be an adapted process and let \(H = (H_n)_{n \geq 0}\) be a predictable process. Define

\[
Y_0 = 0, \\
Y_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}), \quad n \geq 1.
\]

The process \(Y = (Y_n)_{n \geq 0}\), denoted \(Y = H \cdot X\), is called the martingale transform of \(X\) by \(H\). (It is a discrete analogue of the stochastic integral \(\int H \, dX\)).

Intuitively, thinking of \(H_n\) as your stake on game \(n\), your total winnings at time \(n\) are \(Y_n = (H \cdot X)_n\).

The fundamental lemma says that martingale transforms of martingales are of course martingales.

#### 13.5 Lemma

Let \(H = (H_n)\) be a bounded predictable process, that is for some constant \(K\), \(|H_n| \leq K\) a.s. for every \(n\). Let \(X = (X_n)\) be an adaptable process.

(i) If \(H\) is nonnegative, that is for every \(n\), \(H_n \geq 0\) and \(X\) is a supermartingale, then \(H \cdot X\) is a supermartingale.

(ii) If \(X\) is a martingale, then \(H \cdot X\) is a martingale.

**Proof.** (i): Let \(Y = H \cdot X\). Since \(H\) is bounded, \(Y\) is in \(L_1\). Moreover, \(Y_n - Y_{n-1} = H_n (X_n - X_{n-1})\) and since \(H_n\) is \(\mathcal{F}_{n-1}\)-measurable (as being predictable), we have

\[
\mathbb{E}(Y_n - Y_{n-1} | \mathcal{F}_{n-1}) = \mathbb{E}(H_n (X_n - X_{n-1}) | \mathcal{F}_{n-1}) = H_n \mathbb{E}(X_n - X_{n-1} | \mathcal{F}_{n-1}) \leq 0
\]

because \(H_n \geq 0\) and \(\mathbb{E}(X_n - X_{n-1} | \mathcal{F}_{n-1})\) (\(X\) is a supermartingale). The proof of (ii) is the same.

A function \(\tau: \Omega \to \{0, 1, 2, \ldots\} \cup \{+\infty\}\) is a **stopping time** if for every \(n \geq 0\), \(\{\tau \leq n\} \in \mathcal{F}_n\) (equivalently, \(\{\tau = n\} \in \mathcal{F}_n\)). Intuitively, \(\tau\) tells you when to stop playing.
13.6 Example. Let $X = (X_n)$ be an adapted process and let $B \in \mathcal{B}(\mathbb{R})$ be a Borel set. The time of the first entry of $X$ into $B$,

$$\tau = \inf\{n \geq 0, \ X_n \in B\}$$

is a stopping time. Indeed,

$$\{\tau \leq n\} = \bigcup_{k \leq n} \{X_k \in B\} \in \sigma\left(\bigcup_{k \leq n} F_k\right) \subset F_n.$$ 

On the other hand, in general, $\eta = \sup\{n \leq 10, \ X_n \in B\}$ is not a stopping time (why?).

We shall often use notation

$$a \wedge b = \min\{a, b\}, \quad a \vee b = \max\{a, b\}, \quad a, b \in \mathbb{R}.$$ 

13.7 Example. If $\sigma$, $\tau$ are stopping times, then $\sigma \wedge \tau$, $\sigma \vee \tau$, $\sigma + \tau$ are also stopping times.

For a process $X = (X_n)_{n \geq 0}$, we set $X_{\tau} = (X_{\tau \wedge n})_{n \geq 0}$ which is called the stopped process.

The following, often called the optional sampling (or stopping) lemma, says that the stopped process of a supermartingale is a supermartingale (without any extra assumptions).

13.8 Lemma (Doob’s optional sampling lemma). If $X$ is a supermartingale and $\tau$ is a stopping time, then the stopped process $X_{\tau}$ is a supermartingale.

Proof. Let $H_n = 1_{\tau \geq n}$ (we bet 1 until we quit the game and then bet 0). This process takes values in $\{0, 1\}$, so it is in particular nonnegative and bounded. To check that it is predictable, it thus suffices to check that $\{H_n = 0\}$ is in $F_{n-1}$ which is clear because $\{H_n = 0\} = \{\tau \leq n - 1\}$. Finally,

$$(H \cdot X)_n = \sum_{k=1}^{n} H_k (X_k - X_{k-1}) = \sum_{1 \leq k \leq \tau} 1_{k \leq \tau} (X_k - X_{k-1})$$

$$= \sum_{1 \leq k \leq \tau \wedge n} (X_k - X_{k-1})$$

$$= X_{\tau \wedge n} - X_0,$$

so $H \cdot X = X_{\tau} - X_0$, the stopped process is a martingale transform. We are done by Lemma 13.5. \[\square\]

13.9 Example. Let $X = (X_n)_{n \geq 0}$ be a supermartingale and let $\tau$ be a stopping time. Then, for every $n \geq 0$,

$$\mathbb{E}X_{\tau \wedge n} \leq \mathbb{E}X_0$$
and
\[ \mathbb{E}X_{\tau \wedge n} \geq \mathbb{E}X_n. \]
The first inequality follows because \( X^\tau \) is a supermartingale, so in particular,
\[ \mathbb{E}X_{\tau \wedge n} = \mathbb{E}X_n^\tau \leq \mathbb{E}X_0^\tau = \mathbb{E}X_{\tau \wedge 0} = \mathbb{E}X_0. \]
The second inequality follows because \( \{\tau = k\} \in \mathcal{F}_k \), so for every \( k \leq n \), \( \mathbb{E}X_k 1_{\{\tau = k\}} \geq \mathbb{E}X_n 1_{\{\tau = k\}} \) (since \( X_k \geq \mathbb{E}(X_n|\mathcal{F}_k) \)), hence
\[
\mathbb{E}X_{(\tau \wedge n)} = \sum_{k=0}^{n} \mathbb{E}X_k 1_{\{\tau = k\}} + \mathbb{E}X_n 1_{\{\tau > n\}} \geq \sum_{k=0}^{n} \mathbb{E}X_n 1_{\{\tau = k\}} + \mathbb{E}X_n 1_{\{\tau > n\}} = \mathbb{E}X_n.
\]

13.10 Lemma (Doob’s optional sampling lemma – continuation). Let \( X \) be a supermartingale and let \( \tau \) be a stopping time. Then
\[ X_\tau \in L_1 \quad \text{and} \quad \mathbb{E}X_\tau \leq \mathbb{E}X_0, \]
if one of the following conditions holds

(i) \( \tau \) is bounded,

(ii) \( X \) is bounded (say, \( |X_n| \leq K \) for every \( n \geq 0 \)) and \( \tau < \infty \) a.s.

(iii) \( \mathbb{E}\tau < \infty \) and \( X \) has bounded increments: \( |X_n - X_{n-1}| \leq K \) for every \( n \geq 1 \),

(iv) \( X \) is nonnegative and \( \tau < \infty \) a.s.

Moreover, if \( X \) is martingale and one of the conditions (i)-(iii) holds, then
\[ X_\tau \in L_1 \quad \text{and} \quad \mathbb{E}X_\tau = \mathbb{E}X_0. \]

Proof. Let \( X \) be a supermartingale. We have
\[ |X_{\tau \wedge n} - X_0| \leq \sum_{1 \leq k \leq \tau \wedge n} |X_k - X_{k-1}|. \]
If (i) holds, say \( \tau \leq T \) a.s. for some positive integer \( T \), then applying the above inequality to \( n = T \) gives \( |X_\tau| \leq |X_0| + \sum_{k \leq T} |X_k - X_{k-1}| \) showing that \( X_\tau \in L_1 \). If (ii) holds, we trivially have \( |X_\tau| \leq K \), so \( X_\tau \in L_1 \). If (iii) holds, we get \( |X_{\tau \wedge n}| \leq |X_0| + \sum_{1 \leq k \leq \tau} |X_k - X_{k-1}| \leq |X_0| + \tau K \) which is in \( L_1 \), so taking the expectation, letting \( n \to \infty \) and using Lebesgue’s dominated convergence theorem shows that \( \mathbb{E}|X_\tau| \leq \mathbb{E}|X_0| + K\mathbb{E}\tau \). If (iv) holds, since \( X \) is assumed to be nonnegative, \( X_\tau \in L_1 \) follows from the inequality \( \mathbb{E}X_\tau \leq \mathbb{E}X_0 \) argued below.

Now we show \( \mathbb{E}X_\tau \leq \mathbb{E}X_0 \). From Lemma 13.8 we know that \( X_{\tau \wedge n} \in L_1 \) and \( \mathbb{E}X_{\tau \wedge n} \leq \mathbb{E}X_0 \). For (i), say \( \tau \leq N \) a.s., simply take \( n = N \). For (ii), take \( n \to \infty \) and use Lebesgue’s dominated convergence theorem. For (iii), as noted earlier, \( X_{\tau \wedge n} \)}
is dominated by \( |X_0| + K \tau \), so we can use Lebesgue’s dominated convergence theorem again. For (iv), we use Fatou’s lemma,

\[
E X_\tau = E \liminf X_{\tau \wedge n} \leq \liminf E X_{\tau \wedge n} \leq E X_0.
\]

Finally, if \( X \) is a martingale, in each of the cases (i)-(iii), we use the previous part for \( X \) and \(-X\).

13.11 Example. Let \( X \) be a simple random walk on \( \mathbb{Z} \), that is \( X_0 = 0 \), \( X_n = \varepsilon_1 + \cdots + \varepsilon_n \), where \( \varepsilon_1, \varepsilon_2, \ldots \) are i.i.d. symmetric random signs, so \( X \) is a martingale (with respect to the natural filtration). Let \( \tau = \inf \{ n \geq 1, X_n = 1 \} \) be the first moment of visiting 1. It is known that \( P ( \tau < \infty ) = 1 \). However, \( E X_\tau = E 1 = 1 \) and \( E X_0 = 0 \), so in this case \( E X_\tau \neq E X_0 \). Since \( X \) has bounded increments, in view of Lemma 13.10 (iii), we have \( E \tau = +\infty \).

We shall now determine the distribution of \( \tau \) using a martingale argument. Fix \( \lambda > 0 \) and let \( M_n = e^{\lambda X_n} / (E e^{\lambda \varepsilon_1})^n \). By Example 13.3, \( M \) is a martingale. We have

\[
E e^{\lambda \varepsilon_1} = \frac{e^\lambda + e^{-\lambda}}{2} = \cosh \lambda,
\]

so

\[
M_n = (\cosh \lambda)^{-n} e^{\lambda X_n}.
\]

By Doob’s optional sampling lemma,

\[
E [(\cosh \lambda)^{-\tau \wedge n} e^{\lambda X_{\tau \wedge n}}] = E M_{\tau \wedge n} = 1.
\]

Since \( \lambda > 0 \) and \( X_{\tau \wedge n} \leq 1 \), we have that \( e^{\lambda X_{\tau \wedge n}} \) is bounded by \( e^{\lambda} \). Clearly, we have \( (\cosh \lambda)^{-\tau \wedge n} \leq 1 \), so by Lebesgue’s dominated convergence theorem, letting \( n \to \infty \) yields

\[
E (\cosh \lambda)^{-\tau} e^{\lambda} = 1
\]

because

\[
(\cosh \lambda)^{-\tau \wedge n} e^{\lambda X_{\tau \wedge n}} \to \begin{cases} 
(\cosh \lambda)^{-\tau} e^{\lambda}, & \text{on } \{ \tau < \infty \}, \\
0, & \text{on } \{ \tau = \infty \}
\end{cases}
\]

(\( X_{\tau \wedge n} \) stays bounded by 1 and if \( \tau < \infty \) clearly converges to \( X_\tau \)). Letting \( \lambda \to 0^+ \) in a decreasing way, we have \( (\cosh \lambda)^{-\tau} \to \mathbb{1}_{\{ \tau < \infty \}} \), thus \( P ( \tau < \infty ) = E \mathbb{1}_{\{ \tau < \infty \}} = 1 \). Finally, letting \( x = (\cosh \lambda)^{-1} \) and using \( e^{-\lambda} = \frac{1 - \sqrt{1 - x^2}}{x} \), we find the generating function of \( \tau \),

\[
E x^\tau = e^{-\lambda} = \frac{1 - \sqrt{1 - x^2}}{x} = \sum_{k=1}^{\infty} (-1)^{k+1} \binom{1/2}{k} x^{2k-1},
\]

hence the distribution of \( \tau \), \( P ( \tau = 2k - 1 ) = (-1)^{k+1} \binom{1/2}{k} \), \( k \geq 1 \).
13.3 Convergence theorem

The main martingale convergence theorem is due to Doob.

13.12 Theorem (Doob’s “forward” convergence theorem). Let $X$ be a supermartingale bounded in $L_1$, that is for some constant $K$, $E|X_n| \leq K$ for all $n$. Then there is an integrable random variable $X_\infty$ such that $X_n \xrightarrow{a.s.} X_\infty$.

13.13 Corollary. If $X$ is a nonnegative supermartingale, then there is an integrable random variable $X_\infty$ such that $X_n \xrightarrow{a.s.} X_\infty$.

Proof. We have $E|X_n| = EX_n \leq EX_0$ (Remark 13.1), so $X$ is bounded in $L_1$.

13.14 Remark. Since each $X_n$ is $\mathcal{F}_\infty$ measurable, so is $X_\infty$. Suppose $X$ is nonnegative. Then, for a fixed index $m$, by Fatou’s lemma,

$$E(X_\infty|\mathcal{F}_m) = E(\liminf X_n|\mathcal{F}_m) \leq \liminf E(X_n|\mathcal{F}_m) \leq EX_m,$$

so the extended sequence $(X_0, X_1, X_2, \ldots, X_\infty)$ satisfies the supermartingale property (with respect to $\{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_\infty\}$). In particular,

$$EX_\infty \leq \ldots \leq EX_1 \leq EX_0.$$

Figure 1: The number of upcrossings is 3.

The proof (undoubtedly from the book) of Doob’s convergence theorem relies on the so-called upcrossings. For $a < b$, a sequence $(x_n)_{n \geq 0}$ and an index $N \geq 0$, we define the number of upcrossings by time $N$ as

$$U_N(a, b) = \text{largest } k \geq 1 \text{ such that there are indices } 0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < s_k < t_k \leq N$$

with $x_{s_i} < a$ and $x_{t_i} > b$ for each $i = 1, \ldots, k$. 

148
See Figure 1.

13.15 Lemma (Doob’s upcrossing inequality). Let $X$ be a supermartingale, let $a < b$, $N \geq 0$. Let $U_N(a, b)$ be the number of upcrossings by time $N$ of $(X_n)_{n \geq 0}$. Then

$$(b - a)\mathbb{E}U_N(a, b) \leq \mathbb{E}(X_N - a)^-.$$

Proof. Define

$$H_1 = 1_{\{X_0 < a\}},$$

$$H_{n+1} = 1_{\{H_n = 1\}}1_{\{X_n \leq b\}} + 1_{\{H_n = 0\}}1_{\{X_n < a\}}, \quad n \geq 1.$$ 

Figure 2: The process $Y$ increases along upcrossings and $(X_N - a)^-$ bounds the loss in the last interval of play \{n, $H_n = 1$\}.

This is a predictable process taking values in \{0, 1\}. Informally, $H = 1$ on the ways to upcross, or as a gambling strategy: “wait until $X$ gets below $a$, play stake 1 until $X$ gets above $b$, repeat”.

Let $Y = H \cdot X$ be the martingale transform of $X$ by $H$. The crucial observation is that

$$Y_N \geq (b - a)U_N(a, b) - (X_N - a)^-.$$ 

Explanation: every upcrossing increases $Y$ by at least $(b - a)$ and the nonpositive term $-(X_N - a)^-$ offsets the potential loss incurred in the last interval of play (see Figure 2).

Taking the expectation and using that $\mathbb{E}Y_N \leq \mathbb{E}Y_0 = 0$ ($Y$ is a supermartingale, see Lemma 13.5) finishes the proof. □
Proof of Theorem 13.12. Consider the event

\[ E = \{ \omega, \; X_n(\omega) \text{ does not converge to a limit in } [-\infty, \infty] \} \]

\[ = \{ \lim \inf X_n < \lim \sup X_n \} \]

\[ = \bigcup \limits_{a \leq b} \{ \lim \inf X_n < a < b < \lim \sup X_n \}. \]

Fix \( a < b \). Note that by the definition of \( \lim \inf \) and \( \lim \sup \),

\[ \{ \lim \inf X_n < a < b < \lim \sup X_n \} \subset \{ U_\infty(a, b) = \infty \}, \]

where \( U_\infty(a, b) \) is defined (pointwise) as the limit \( \lim_{N \to \infty} U_N(a, b) \in [0, +\infty] \) (which exists by the monotonicity of \( U_N(a, b) \)). By Lemma 13.15,

\[ (b - a)\mathbb{E}U_N(a, b) \leq \mathbb{E}(X_N - a) - \leq \mathbb{E}|X_N| + |a| \leq K + |a|, \]

so in particular, letting \( N \to \infty \), we get (by Lebesgue’s monotone convergence theorem) that \( \mathbb{E}U_\infty(a, b) < \infty \) and thus \( \mathbb{P}(U_\infty(a, b) = \infty) = 0 \). As a result, \( \mathbb{P}(E) = 0 \). On \( E^c \), we can define

\[ X_\infty = \lim X_n \in [-\infty, \infty]. \]

By Fatou’s lemma,

\[ \mathbb{E}|X_\infty| = \mathbb{E}\lim \inf |X_n| \leq \lim \inf \mathbb{E}|X_n| \leq K, \]

so \( X_\infty \) is integrable (and thus \( X_\infty \in (-\infty, \infty) \)).
13.4 Exercises

1. Verify the claim made in Example 13.3.

2. Verify the claim made in Example 13.4.

3. Let $\tau$, $\sigma$ be stopping times (relative to $(\Omega, \{F_n\}, F, P)$). Prove that $\tau \land \sigma$, $\tau \lor \sigma$ and $\tau + \sigma$ are also stopping times. Are $\tau + 1$, $\tau - 1$ stopping times as well?

Here and throughout: $a \land b = \min\{a, b\}$, $a \lor b = \max\{a, b\}$, $a, b \in \mathbb{R}$.

4. Let $X = (X_n)_{n \geq 0}$ be an adaptable process (to a filtration $\{F_n\}_{n \geq 0}$). Let $B$ be a Borel subset of $\mathbb{R}$. Define

\[
\tau_1 = \inf\{n : X_n \in B\} \quad \text{the first visit in } B,
\]

\[
\tau_k = \inf\{n > \tau_{k-1} : X_n \in B\} \quad \text{the } k\text{th visit in } B, \ k \geq 2.
\]

We know that $\tau_1$ is a stopping time. Show that each $\tau_k$ is also a stopping time.

5. Let $X = (X_n)_{n \geq 0}$ be a martingale and let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that $E|f(X_n)| < \infty$ for every $n \geq 0$. Show that $(f(X_n))_{n \geq 0}$ is a submartingale.

6. Let $X_0, X_1, \ldots$ be i.i.d. square-integrable random variables with mean 0. Let $Y_0 = 0$ and $Y_n = X_0X_1 + X_1X_2 + \cdots + X_{n-1}X_n$, $n \geq 1$. Show that $(Z_n)_{n \geq 0}$ is a martingale relative to the natural filtration $\{F_n\}_{n \geq 0}$ of $(X_n)_{n \geq 0}$.

Hint: First show that $E X_n = E X_0$ for every $n \geq 0$. Then, given $n \geq 0$ and $A \in F_n$, consider $\tau = n1_A + (n + 1)1_{A^c}$.

7. Let $X = (X_n)_{n \geq 0}$ be an integrable adapted process (to a filtration $\{F_n\}_{n \geq 0}$). Show that $X$ is a martingale if and only if for every bounded stopping time $\tau$, we have

$E X_\tau = EX_0$.

Deduce that $P(\sup_n X_n \geq t) \leq \frac{EX_0}{t}$, for $t > 0$.

8. Let $X$ be a nonnegative supermartingale and let $\tau$ be a stopping time. Show that

$EX_\tau 1_{\tau < \infty} \leq EX_0$.

9. Pólya’s urn. At time 0, an urn contains 1 black ball and 1 white ball. At each time $n = 1, 2, 3, \ldots$ a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Just after time $n$, there are therefore $n + 2$ balls in the urn, of which $B_n + 1$ are black, where $B_n$ denotes the number of black balls chosen by time $n$, $B_0 = 0$. Let $M_n = \frac{B_n + 1}{n + 2}$, $n \geq 0$. Prove that

(a) $M$ is a martingale (relative to the natural filtration $\mathcal{F}_n = \sigma(B_0, \ldots, B_n)$),

(b) $P(B_n = k) = \frac{1}{n+1}$, for $0 \leq k \leq n$,
(c) \( M_n \) converges a.s., say to \( M_\infty \)

(d) \( M_\infty \) is uniform on \([0,1]\),

(e) \( X_n = \frac{(n+1)!}{n!(n-2_0)!} \theta^{B_n} (1-\theta)^{n-B_n} \) is a martingale, where \( 0 < \theta < 1 \) is fixed.

10. **Bellman’s Optimality Principle.** Let \( \frac{1}{2} < p < 1 \). Let \( X_1, X_2, \ldots \) be i.i.d. random variables with \( P(X_n = 1) = p = 1 - P(X_n = -1) \). Your winnings per unit stake on game \( n \) are \( X_n \). Your stake \( H_n \) on game \( n \) must lie between 0 and \( Y_{n-1} \), where \( Y_n \) is your fortune at time \( n \), \( Y_0 \) is a positive constant and \( Y_n = Y_{n-1} + H_n X_n \). Show that if \( H \) is a predictable process, then \( (\log Y_n - n \alpha)_{n \geq 0} \) is a supermartingale, where \( \alpha = p \log p + (1-p) \log(1-p) + \log 2 \). Deduce that \( E \log(Y_n/Y_0) \leq n \alpha \). Find the best strategy \( H \), that is the one that gives a martingale, hence equality.

11. Suppose that \( \tau \) is a stopping time (relative to \( \{F_n\}_{n \geq 0} \)) such that for some \( N \geq 1 \) and \( \varepsilon > 0 \), we have for every \( n \),

\[
P(\tau \leq n + N | F_n) > \varepsilon \text{ a.s.}
\]

Show that for \( k = 1, 2, \ldots \), we have \( P(\tau > kN) \leq (1-\varepsilon)^k \) and deduce that \( E\tau < \infty \).

12. **ABRACADABRA.** At each of times 1, 2, 3, …, a monkey types a capital letter at random, the sequence of letters typed forming an i.i.d. sequence of random variables each chosen uniformly from the 26 capital letters. Let \( \tau \) be the first time by which the monkey has produced the consecutive sequence ‘ABRACADABRA’. Using martingale theory, show that

\[
E\tau = 26^{11} + 26^4 + 26.
\]

13. **Gambler’s ruin.** Let \( 0 < p < 1 \), \( p \neq \frac{1}{2} \). Let \( X_1, X_2, \ldots \) be i.i.d. random variables with \( P(X_n = 1) = p = 1 - P(X_n = -1) \) for every \( n \). Let \( a \) and \( b \) be integers with \( 0 < a < b \). Let \( S_0 = a, S_n = a + X_1 + \ldots + X_n, n \geq 1 \) and \( \tau = \inf\{n \geq 0 : S_n \in \{0, b\}\} \). Show that \( E\tau < \infty \). Let \( X_n = S_n - n(2p-1) \) and \( Y_n = \left( \frac{1-p}{p} \right)^{S_n}, n \geq 0 \). Show that \( X \) and \( Y \) are martingales. Deduce the values of \( P(S_\tau = 0) \) and \( ES_\tau \). What about the symmetric case \( p = \frac{1}{2} \)?

14. **Wald’s identity.** Let \( \{F_n\}_{n \geq 0} \) be a filtration and let \( X \) be an integrable adaptable process such that the \( X_k \) have the same distribution and \( X_{k+1} \) is independent of \( F_k \) for every \( k \geq 0 \). Let \( S_n = X_1 + \ldots + X_n \). Let \( \tau \) be a stopping time with \( E\tau < \infty \). Show that

\[
E(X_1 + \cdots + X_\tau) = E\tau \cdot E X_1.
\]

If additionally \( E X_1^2 < \infty \), then

\[
E(S_\tau - \tau E X_1)^2 = (E\tau) Var(X_1).
\]
15. Baby version of Kakutani’s theorem. Let $X_1, X_2, \ldots$ be i.i.d. nonnegative random variables with mean 1 such that $P(X_i = 1) < 1$. Let $M_n = X_1 \cdots X_n$, $n \geq 1$. Show that $M_n$ converges a.s. but not in $L_1$.

Hint. Use Corollary 13.13 ($M$ is a nonnegative martingale). To tackle convergence in $L_1$, first using a Cauchy condition in $L_1$, show that $M_n \to M_\infty$ in $L_1$ implies that $\lim_n M_n = 1$ a.s. Then consider a.s. convergence of $\tilde{M}_n = \frac{\sqrt{X_1 \cdots \sqrt{X_n}}}{(E\sqrt{X_1})^{n/2}}$ using again a martingale argument.
14 Martingales II

14.1 $L_2$ martingales

Whenever possible, one of the easiest ways to prove that a martingale is bounded in $L_1$ is to show that it is bounded in $L_2$ and use $E|X| \leq (E|X|^2)^{1/2}$. For martingales, verifying $L_2$-type conditions can be done very efficiently. Moreover, bounded $L_2$ martingales admit better convergence results and have nice applications to classical topics such as series of independent random variables.

We begin with a basic condition saying that a martingale is bounded in $L_2$ if and only if the series of the squares of the $L_2$-norms of its increments is convergent.

14.1 Theorem. Let $M = (M_n)_{n \geq 0}$ be a square-integrable martingale, that is $E M_n^2 < \infty$ for every $n \geq 0$. Then

$M$ is bounded in $L_2$, that is $\exists K > 0 \forall n \geq 0 \ E M_n^2 \leq K$ \hfill (14.1)

if and only if

$$\sum_{k \geq 1} E (M_k - M_{k-1})^2 < \infty.$$ \hfill (14.2)

Moreover, when this holds, there exists a random variable $M_\infty$ which is in $L_2$ such that

$$M_n \xrightarrow{n \to \infty} M_\infty \text{ a.s. and in } L_2.$$ 

Proof. For $i < j$, $E(M_j | F_i) = M_i$, that is $E(M_j - M_i)1_A = 0$ for every $A \in F_i$, or, in other words, $M_j - M_i$ is orthogonal to the subspace $L_2(F_i)$. In particular, if $k < l \leq m < n$, then

$$E(M_l - M_k)(M_n - M_m) = 0.$$ 

Consequently, writing $M$ as the sum of its increments,

$$M_n = M_0 + \sum_{k=1}^n (M_k - M_{k-1}),$$

we find that all the terms are orthogonal, so in particular,

$$E M_n^2 = E M_0^2 + \sum_{k=1}^n E (M_k - M_{k-1})^2.$$ 

This identity explains the equivalence of (14.1) and (14.2).

Suppose these conditions hold. By Doob’s convergence theorem for martingales bounded in $L_1$ (Theorem 13.12), there is an integrable random variable $M_\infty$ such that $M_n \to M_\infty$ a.s. To see that $M_\infty$ is in fact square-integrable and the convergence is also in $L_2$, first note that, by orthogonality, for $n, r \geq 0$, we have

$$E(M_{n+r} - M_n)^2 = \sum_{k=n+1}^{n+r} E(M_k - M_{k-1})^2,$$
so letting \( r \to \infty \) and using Fatou’s lemma, we conclude
\[
\mathbb{E}(M_\infty - M_n)^2 \leq \sum_{k \geq n+1} \mathbb{E}(M_k - M_{k-1})^2 < \infty,
\]
hence \( M_\infty \in L_2 \). Moreover, since tails of convergent series go to 0, this bound also shows that
\[
\mathbb{E}(M_\infty - M_n)^2 \xrightarrow{n \to \infty} 0,
\]
that is \( M_n \to M_\infty \) in \( L_2 \).

We introduce Doob’s decomposition, which will be particularly fruitful to study \( L_2 \) martingales.

14.2 Theorem (Doob’s decomposition). Let \( X = (X_n)_{n \geq 0} \) be an adapted, integrable process. It admits the decomposition
\[
X_n = X_0 + M_n + A_n, \quad n \geq 0,
\]
where \( M = (M_n)_{n \geq 0} \) is a martingale with \( M_0 = 0 \) and \( A = (A_n)_{n \geq 0} \) is a predictable process with \( A_0 = 0 \). Moreover, if \( X = X_0 + \tilde{M} + \tilde{A} \) is another such decomposition, then
\[
\mathbb{P} \left( \forall n \geq 0, M_n = \tilde{M}_n, A_n = \tilde{A}_n \right) = 1.
\]
Moreover, \( X \) is a supermartingale if and only if the process \( A \) is nonincreasing, that is \( A_n \geq A_{n+1} \) for all \( n \geq 0 \) a.s.

Proof. Let
\[
A_0 = 0,
\]
\[
A_n = \sum_{k=1}^{n} \mathbb{E}(X_k - X_{k-1}|\mathcal{F}_{k-1}), \quad n \geq 1
\]
\[
M_n = X_n - X_0 - A_n, \quad n \geq 0.
\]
Plainly, \( A \) satisfies the desired properties. Moreover, \( M \) is integrable and adaptable and it satisfies the martingale property because
\[
M_{n+1} - M_n = X_{n+1} - X_n - (A_{n+1} - A_n) = X_{n+1} - X_n - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)
\]
\[
= X_{n+1} - \mathbb{E}(X_{n+1}|\mathcal{F}_n),
\]
so
\[
\mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) - \mathbb{E}(X_{n+1}|\mathcal{F}_n) = 0.
\]
Moreover, if \( X = X_0 + \tilde{M} + \tilde{A} \) for a martingale \( \tilde{M} \) and a predictable process \( \tilde{A} \), then for every \( n \geq 1 \),
\[
\mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(\tilde{M}_n - \tilde{M}_{n-1}|\mathcal{F}_{n-1}) + \mathbb{E}(\tilde{A}_n - \tilde{A}_{n-1}|\mathcal{F}_{n-1})
\]
\[
= 0 + \tilde{A}_n - \tilde{A}_{n-1} = \tilde{A}_n - \tilde{A}_{n-1},
\]
155
which gives that \( \tilde{A} = A \) a.s. and consequently \( \tilde{M} = M \) a.s. Finally, by the identity,
\[
\mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) = A_n - A_{n-1},
\]
we have that \( \mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) \leq 0 \) if and only if \( A_n \leq A_{n-1} \).

We note that Doob’s decomposition of the stopped process comes from stopping Doob’s decomposition processes.

**14.3 Lemma.** Let \( X \) be an adapted, integrable process with Doob’s decomposition \( X = X_0 + M + A \) with \( M \) being a martingale and \( A \) being a predictable process. Let \( \tau \) be a stopping time. Then
\[
X_\tau = X_0 + M_\tau + A_\tau
\]
is Doob’s decomposition of the stopped process \( X_\tau \).

**Proof.** By Doob’s optional sampling lemma, we know that \( M_\tau \) is a martingale, so it suffices to show that \( A_\tau \) is predictable. This can be seen from the identity
\[
A_\tau^n = A_{\tau \wedge n} = A_n 1_{\tau \geq n} + \sum_{k<n} A_k 1_{\tau = k}
\]
because \( \{\tau \geq n\} \) is in \( \mathcal{F}_{n-1} \).

Let \( M \) be a martingale in \( L_2 \), that is \( \mathbb{E}M_n^2 < \infty \) for every \( n \geq 0 \). We define its quadratic variation process (the angle-brackets process), denoted \( \langle M \rangle = (\langle M \rangle_n)_{n \geq 0} \) as
\[
\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}(M_k^2 - M_{k-1}^2|\mathcal{F}_{k-1}),
\]
that is \( \langle M \rangle = A \), where
\[
M^2 = M_0^2 + N + A
\]
is Doob’s decomposition of \( M^2 \) into a martingale \( N \) and a predictable process \( A \).

**14.4 Remark.** Since \( \mathbb{E}(M_kM_{k-1}|\mathcal{F}_{k-1}) = M_{k-1} \mathbb{E}(M_k|\mathcal{F}_{k-1}) = M_{k-1}^2 \), we have
\[
\langle M \rangle_n = \sum_{k=1}^{n} \mathbb{E}((M_k - M_{k-1})^2|\mathcal{F}_{k-1}).
\]
In particular, \( \langle M \rangle \) is a nonnegative process. Moreover,
\[
\langle M \rangle_n - \langle M \rangle_{n-1} = \mathbb{E}(M_n^2 - M_{n-1}^2|\mathcal{F}_{n-1}) = \mathbb{E}((M_n - M_{n-1})^2|\mathcal{F}_{n-1}) \geq 0,
\]
hence \( \langle M \rangle = A \) is nondecreasing (which also follows from the fact that \( M^2 \) is a submartingale – by Jensen’s inequality, see Exercise 13.5). Consequently, we can define (point-wise)
\[
\langle M \rangle_\infty = \lim_{n \to \infty} \langle M \rangle_n \in [0, + \infty].
\]
14.5 Example. Let \(X_1, X_2, \ldots\) be independent random variables which are in \(L^2\). Let \(M_0 = 0\) and \(M_n = X_1 + \cdots + X_n - (\mathbb{E}X_1 + \cdots + \mathbb{E}X_n)\). We know this is a martingale (Example 13.2). We have,
\[
\langle M \rangle_n = \sum_{k=1}^{n} \text{Var}(X_k)
\]
(exercise). Thus, \(M_n^2 - \text{Var}(M_n)\) is a martingale.

14.6 Remark. From Doob’s decomposition, \(\mathbb{E}M_n^2 = \mathbb{E}M_0^2 + \mathbb{E}N_n + \mathbb{E}\langle M \rangle_n = \mathbb{E}M_0^2 + \mathbb{E}\langle M \rangle_n\), thus \(M\) is bounded in \(L^2\) if and only if \(\mathbb{E}\langle M \rangle_\infty < \infty\) \((14.3)\).

In view of Theorem 14.1 and the above remark, if \(\mathbb{E}\langle M \rangle_\infty < \infty\), then \(M_n\) converges a.s. and in \(L^2\). The next theorem refines that and describes very precisely convergence of \(L^2\) martingales in a general situation.

14.7 Theorem. Let \(M\) be a martingale with \(M_0 = 0\) which is in \(L^2\), that is \(\mathbb{E}M_n^2 < \infty\) for every \(n\).

(i) On the event \(\{\langle M \rangle_\infty < \infty\}\), we have “\(\lim_n M_n\) exists and is finite”.

(ii) If \(M\) has additionally bounded increments, that is there is a constant \(K > 0\) such that \(|M_n - M_{n-1}| \leq K\) for all \(n \geq 0\) a.s., then the converse holds: on the event \(\{\lim_n M_n\) exists and is finite\}, we have \(\langle M \rangle_\infty < \infty\).

(iii) On the event \(\{\langle M \rangle_\infty = \infty\}\), we have \(\frac{M_n}{\langle M \rangle_n} \rightarrow 0\).

(All the inclusions hold modulo sets of measure 0).

14.8 Remark. Part (iii) can be thought of a strong law of large numbers for \(L^2\) martingales. To illustrate this point, let \(X_1, X_2, \ldots\) be i.i.d. random variables which are in \(L^2\). We consider the sum martingale \(M_n = X_1 + \cdots + X_n - n\mathbb{E}X_1\), for which \(\langle M \rangle_n = n\text{Var}(X_1)\) (see Example 14.5). Thus, \(\{\langle M \rangle_\infty = \infty\}\) = \(\Omega\), so, a.s.,
\[
\frac{X_1 + \cdots + X_n - n\mathbb{E}X_1}{n\text{Var}(X_1)} = \frac{M_n}{\langle M \rangle_n} \rightarrow 0,
\]
equivalently,
\[
\frac{X_1 + \cdots + X_n}{n} \rightarrow \mathbb{E}X_1 \quad \text{a.s.}
\]

Proof. (i): For \(l = 1, 2, \ldots\), we define
\[
\tau_l = \inf\{n \geq 0 : \langle M \rangle_{n+1} > l\}.
\]
Since \(\langle M \rangle\) is predictable, this is a stopping time. Moreover, by its definition,
\[
\langle M \rangle_{\tau_l} \leq l.
\]
Thus, for the stopped process $M^{\tau_l}$,

$$\langle M^{\tau_l}\rangle_\infty = \langle M\rangle_\infty = \langle M\rangle_{\tau_l} \leq l$$

(in the first equality we use Lemma 14.3). In view of (14.1), the stopped process $M^{\tau_l}$ is a bounded in $L_2$ martingale, so $\lim_n M^{\tau_l}_n$ exists (a.s.) and is in $L_2$. Therefore,

$$\{\langle M\rangle_\infty < \infty\} = \bigcup_{l \geq 1} \{\tau_l = \infty\} \subset \{\lim_n M_n \text{ exists and is finite}\}$$

because on $\{\tau_l = \infty\}$, we have $M^{\tau_l}_n = M_n$. This shows (a).

(ii): For $l = 1, 2, \ldots$, we define

$$\sigma_l = \inf\{n \geq 0 : |M_n| > l\}.$$  

This is a stopping time. Since $M$ has bounded increments, the stopped process is bounded,

$$|M^{\sigma_l}_n| = |M_{\sigma_l \wedge n}| \leq |M_{(\sigma_l \wedge n) - 1}| + K \leq l + K.$$  

Thus $M^{\sigma_l}$ is bounded in $L_2$. Therefore $\mathbb{E}\langle M^{\sigma_l}\rangle_\infty < \infty$ and

$$\mathbb{E}\langle M\rangle_{\sigma_l} = \mathbb{E}\langle M\rangle^{\sigma_l}_\infty = \mathbb{E}\langle M^{\sigma_l}\rangle_\infty < \infty.$$  

In particular, on $\{\sigma_l = \infty\}$, we have $\langle M\rangle_{\sigma_l} = \langle M\rangle_\infty$ and thus $\langle M\rangle_\infty < \infty$ a.s. on $\{\sigma_l = \infty\}$. Since convergent sequences are bounded, by the definition of $\sigma_l$, this finishes the proof of (b),

$$\{\lim_n M_n \text{ exists and is finite}\} \subset \{\sup_n |M_n| < \infty\} = \sum_{l \geq 1} \{\sigma_l = \infty\} \subset \{\langle M\rangle_\infty < \infty\}.$$  

(iii): We define the process $Y$ with $Y_0 = 0$ and

$$Y_n = \sum_{k=1}^n \frac{1}{1 + \langle M \rangle_k} (M_k - M_{k-1}), \quad n \geq 1,$$

which is the martingale transform of $M$ by the predictable process $\frac{1}{1 + \langle M \rangle}$ (which is bounded because $\langle M \rangle$ is nonnegative). Thus $Y$ is a martingale (in $L_2$) and since $\langle M \rangle$ is nondecreasing, we have,

$$\mathbb{E}(Y_n - Y_{n-1})^2|F_{n-1}) = \frac{\langle M \rangle_n - \langle M \rangle_{n-1}}{(1 + \langle M \rangle_n)^2} \leq \frac{\langle M \rangle_n - \langle M \rangle_{n-1}}{(1 + \langle M \rangle_n)(1 + \langle M \rangle_{n-1})} = \frac{1}{1 + \langle M \rangle_{n-1}} - \frac{1}{1 + \langle M \rangle_n}.$$  

Therefore,

$$\mathbb{E}Y_n^2 = \sum_{k=1}^n \mathbb{E}(Y_k - Y_{k-1})^2 = \mathbb{E}\left(1 - \frac{1}{1 + \langle M \rangle_n}\right) \leq 1,$$  

158
so \( Y \) is bounded in \( L_2 \) and, consequently, \( \lim Y_n \) exists and is finite a.s. On the event 
\[ \{(M)_n \not\to \infty\} = \{(M)_\infty = \infty\}, \]
we get \( \frac{M_n}{M_{M_n}} \to 0 \), by the following standard Cesàro-type lemma. \hfill \( \square \)

**14.9 Lemma (Kronecker).** Let \( (b_n) \) be a sequence of positive real numbers with \( b_n \not\to \infty \).
Let \( (x_n) \) be a sequence of real numbers. Then,
\[
\text{if } \sum \frac{x_n}{b_n} \text{ converges, then } \frac{x_1 + \cdots + x_n}{b_n} \to 0.
\]

**Proof.** Let \( s_n = \sum_{k=1}^{n} \frac{x_k}{b_k} \). Then \( s_1 = \frac{x_1}{b_1} \), \( s_n - s_{n-1} = \frac{x_n}{b_n} \), \( n \geq 2 \), so
\[
x_1 + \cdots + x_n = \frac{b_1 s_1 + b_2 (s_2 - s_1) + b_3 (s_3 - s_2) + \cdots + b_n (s_n - s_{n-1})}{b_n} = \frac{(b_1 - b_2) s_1 + (b_2 - b_3) s_2 + \cdots + (b_{n-1} - b_n) s_{n-1} + b_n s_n}{b_n}.
\]

Fix \( \epsilon > 0 \). Since \( (s_n) \) is a convergent sequence, it is bounded, say \( |s_n| \leq M \) for every \( n \), and by the Cauchy criterion, there is \( N \) such that for \( n, m > N \), we have \( |s_n - s_m| < \epsilon \). Consequently, for \( n > N \),
\[
\left| \frac{x_1 + \cdots + x_n}{b_n} \right| \leq \frac{(b_2 - b_1) 2M + \cdots + (b_N - b_{N-1}) 2M + (b_{N+1} - b_N) \epsilon + \cdots + (b_n - b_{n-1}) \epsilon}{b_n} + \frac{b_1 M}{b_n} = 2M \frac{b_N - b_1}{b_n} + \frac{b_n - b_N}{b_n} \epsilon + \frac{b_1 M}{b_n} \leq 2M \frac{b_N - b_1}{b_n} + \epsilon + \frac{b_1 M}{b_n}
\]

which is less than, say \( 2 \epsilon \) for \( n \) large enough. \hfill \( \square \)

**14.2 Uniformly integrable martingales**

We say that a family of random variables \( \{X_t\}_{t \in T} \) is **uniformly integrable** if for every \( \epsilon > 0 \), there is \( K > 0 \) such that for all \( t \in T \), we have \( \mathbb{E}[|X_t| 1_{\{|X_t| > K\}}] \leq \epsilon \). We refer to Appendix I for basic results. We recall one: for \( p > 0 \), we have
\[
X_n \xrightarrow{\mathcal{L}_p} X \quad \text{if and only if} \quad X_n \xrightarrow{\mathbb{P}} X \text{ and } \{|X_n|^p\} \text{ is uniformly integrable}
\]
(see Theorem 1.6)

In the context of martingales, the following construction of a uniformly integrable family is rather important.

**14.10 Lemma.** If \( X \) is an integrable random variable and \( \{\mathcal{F}_t\}_{t \in T} \) is a family of sub-\( \sigma \)-algebras, then the family \( \{X_t = \mathbb{E}(X | \mathcal{F}_t)\}_{t \in T} \) is uniformly integrable.
Proof. Fix $\varepsilon > 0$. We choose $\delta > 0$ such that for every event $A$ with $P(A) < \delta$, we have $E|X_1|_A < \varepsilon$ (see Remark I.5). First note that, by Jensen’s inequality,

$$|X_t| = |E(X|F_t)| \leq E(|X| |F_t)$$

and, consequently,

$$E|X_t| \leq E|X|.$$ 

For the event $A = \{|X_t| > \kappa\}$, if $\kappa$ is large enough, we have, by Markov’s inequality,

$$P(A) \leq \frac{1}{\kappa} E|X_t| \leq \frac{1}{\kappa} E|X| < \delta,$$

so, using that $A \in F_t$,

$$E|X_t| 1_{\{\|X_t\| > \kappa\}} \leq E|X_t| 1_A \leq E(|X| 1_A |F_t) = E|X| 1_A \leq \varepsilon,$$

which is the definition of uniform integrability.

Uniformly integrable martingales are bounded in $L_1$ (see (i) of Theorem I.4). Thus such martingales converge a.s. Moreover, the following basic result gives $L_1$ convergence and says that such martingales are of the tower form (see Example 13.4).

**14.11 Theorem.** Let $M$ be an uniformly integrable martingale. Then there is an integrable random variable $M_\infty$ such that $M_n \to M_\infty$ a.s. and in $L_1$. Moreover,

$$M_n = E(M_\infty |F_n), \quad n \geq 0.$$

**Proof.** As we said, the existence of $M_\infty \in L_1$ such that $M_n \to M_\infty$ a.s. follows from Doob’s convergence theorem (Theorem 13.12). Since $\{M_n\}$ is uniformly integrable and converges in probability to $M_\infty$, by (14.4), $M_n \to M_\infty$ also in $L_1$. It remains to show that $M_n = E(M_\infty |F_n)$ for every $n$. To this end, we fix $n \geq 0$, fix an event $A \in F_n$ and argue that $EM_n 1_A = EM_\infty 1_A$. By the martingale property, for every $r \geq n$, we have $EM_r 1_A = EM_n 1_A$. Moreover,

$$|EM_r 1_A - EM_\infty 1_A| \leq E|M_r - M_\infty| \to 0,$$

so $EM_n 1_A = EM_r 1_A \to EM_\infty 1_A$ as $r \to \infty$ and thus, $EM_n 1_A = EM_\infty 1_A$, as desired.

For tower-type martingales, we have two refinements.

**14.12 Theorem** (Lévy’s “upward” convergence theorem). Let $X$ be an integrable random variable and let $\{F_n\}_{n \geq 0}$ be a filtration, $F_\infty = \sigma(\bigcup_{n \geq 0} F_n)$. Let $X_n = E(X|F_n), \quad n \geq 0$. We have,

$$X_n \to E(X|F_\infty) \quad a.s. \text{ and in } L_1.$$
Proof. By Lemma 14.10, the martingale \((X_n)_{n \geq 0}\) is uniformly integrable, so by Theorem 14.11, there is \(X_\infty \in L_1\) such that \(X_n \to X_\infty\) a.s. and in \(L_1\). It remains to show that \(X_\infty = \mathbb{E}(X|\mathcal{F}_\infty)\). Since \(X_\infty\) is \(\mathcal{F}_\infty\)-measurable, it suffices to argue that for every event \(A \in \mathcal{F}_\infty\), we have \(\mathbb{E}X_\infty 1_A = \mathbb{E}X 1_A\). All events satisfying this form a \(\lambda\)-system. By Dynkin’s theorem, it thus suffices to show that it contains the \(\pi\)-system \(\bigcup_{n \geq 0} \mathcal{F}_n\).

Suppose \(A \in \mathcal{F}_n\) for some \(n\). Then,

\[
\mathbb{E}X 1_A = \mathbb{E}X_n 1_A = \mathbb{E}X_\infty 1_A,
\]

where the first equality holds because \(X_n = \mathbb{E}(X|\mathcal{F}_n)\) and the second one holds because \(X_n = \mathbb{E}(X_\infty|\mathcal{F}_n)\), as provided by Theorem 14.11. This finishes the proof. \(\square\)

**14.13 Theorem** (Lévy’s “downward” convergence theorem). Let \({\mathcal{G}_{-n}, n = 1, 2, \ldots}\) be sub-\(\sigma\)-algebras such that \(\mathcal{G}_{-1} \supset \mathcal{G}_{-2} \supset \cdots \supset \mathcal{G}_{-\infty}\) with \(\mathcal{G}_{-\infty} = \bigcap_{n \geq 1} \mathcal{G}_{-n}\). Let \(X\) be an integrable random variable and \(X_{-n} = \mathbb{E}(X|\mathcal{G}_{-n})\), \(n \geq 1\). Then \(X_{-n} \xrightarrow{n \to \infty} \mathbb{E}(X|\mathcal{G}_{-\infty})\) a.s. and in \(L_1\).

Proof. We consider the martingale \((X_{-N}, X_{-N+1}, \ldots, X_{-1})\). By Doob’s upcrossing inequality from Lemma 13.15,

\[
(b - a)\mathbb{E}U_N(a, b) \leq \mathbb{E}(X_{-1} - a)_- < \infty,
\]

so as in the proof of Doob’s convergence theorem, \(X_{-\infty} = \lim_n X_{-n}\) exists and is finite a.s. By the uniform integrability of \(\{X_{-n}\}_{n \geq 1}\), we get that also \(X_n \to X_{-\infty}\) in \(L_1\). As in the proof of Lévy’s upward convergence theorem, \(X_{-\infty} = \mathbb{E}(X|\mathcal{G}_{-\infty})\). \(\square\)

**14.3 Maximal inequalities**

Maximal inequalities concern tail and moment bounds for the maximum \(\max_{k \leq n} X_k\) of a process \(X\).

**14.14 Theorem** (Doob’s maximal inequality). Let \((X_n)_{n \geq 0}\) be a submartingale. Then for every \(t > 0\), we have

\[
\mathbb{P}\left(\max_{k \leq n} X_k \geq t\right) \leq \frac{1}{t} \mathbb{E}X_n 1_{\{\max_{k \leq n} X_k \geq t\}} \leq \frac{1}{t} \mathbb{E}X_n^+, \quad (14.5)
\]

\[
\mathbb{P}\left(\min_{k \leq n} X_k \leq -t\right) \leq \frac{1}{t} \left(\mathbb{E}X_n 1_{\{\min_{k \leq n} X_k \leq -t\}} - \mathbb{E}X_0\right) \leq \frac{1}{t} (\mathbb{E}X_n^+ - \mathbb{E}X_0), \quad (14.6)
\]

\[
\mathbb{P}\left(\max_{k \leq n} |X_k| \geq t\right) \leq \frac{1}{t} (2\mathbb{E}X_n^+ - \mathbb{E}X_0). \quad (14.7)
\]

161
Proof. For the first inequality, consider the stopping time \( \tau = \inf \{ n \geq 0 : X_n \geq t \} \). By Example 13.9,

\[ E X_n \geq E X_{\tau \wedge n} \]

and

\[ E X_{\tau \wedge n} = E X_\tau 1_{\{\tau \leq n\}} + E X_n 1_{\{\tau > n\}} \geq t P(\tau \leq n) + E X_n 1_{\{\tau > n\}}, \]

hence

\[ t P(\tau \leq n) \leq E X_n 1_{\tau \leq n}, \]

so \( \{\tau \leq n\} = \{\max_{k \leq n} X_k \geq t\} \) finishes the argument.

For the second inequality, we consider the stopping time \( \sigma = \inf \{ n \geq 0 : X_n \leq -t \} \), use \( E X_0 \leq E X_{\tau \wedge n} \) and proceed analogously (we leave the details as an exercise).

The third inequality follows from \( \max_k |X_k| = \max\{\max_k X_k, \max_k (-X_k)\} \), the union bound and applying the previous two inequalities (we leave the details as an exercise).

14.15 Corollary. Let \( f : \mathbb{R} \to \mathbb{R} \) be convex and let \( X \) be a martingale such that \( E|f(X_n)| < \infty \) for every \( n \). Then for every \( t > 0 \), we have

\[ P(\max_{k \leq n} f(X_k) \geq t) \leq \frac{1}{t} E|f(X_n)|. \]

Proof. It follows from (14.5) applied to the submartingale \( (f(X_n)) \) (see also Exercise 13.5).

14.16 Corollary (Kolmogorov’s maximal inequality). Let \( X_1, \ldots, X_n \) be independent square-integrable random variables, each with mean \( 0 \) and let \( S_k = X_1 + \ldots + X_k \), \( 1 \leq k \leq n \). Then

\[ P\left(\max_{k \leq n} |S_k| \geq t\right) \leq \frac{1}{t^2} E S_n^2, \quad t > 0. \]

Proof. We have,

\[ P\left(\max_{k \leq n} |S_k| \geq t\right) = P\left(\max_{k \leq n} |S_k|^2 \geq t^2\right) \]

and the result follows from the previous corollary applied to the sum martingale \( (S_0 = 0, S_1, \ldots, S_n) \) and convex function \( f(x) = x^2 \).

14.17 Theorem. (Doob’s maximal inequality in \( L_p \)) Let \( p > 1 \) and let \( X = (X_n)_{n \geq 0} \) be a nonnegative submartingale. Then

\[ \left( E \max_{k \leq n} X_k^p \right)^{1/p} \leq \frac{p}{p-1} (E X_n^p)^{1/p}. \]

Proof. If the right hand side is \( +\infty \), there is nothing to prove. Suppose that it is finite.

Then, by a trivial bound,

\[ E \max_{k \leq n} X_k^p \leq E \sum_{k \leq n} X_k^p = \sum_{k=1}^n E X_k^p \leq n E X_n^p, \]

162
(we use $\mathbb{E}X_k^p \leq \mathbb{E}(\mathbb{E}(X_{k+1}|\mathcal{F}_k))^p \leq \mathbb{E}(\mathbb{E}(X_{k+1}^p|\mathcal{F}_k)) = \mathbb{E}X_{k+1}^p$), we conclude that the left hand side is also finite. By (14.5), we have

$$\mathbb{E}\max_{k \leq n} X_k^p = \int_0^\infty pt^{p-1} \mathbb{P}(\max_{k \leq n} X_k > t) \, dt \leq \int_0^\infty pt^{p-2} \mathbb{E}X_n \mathbb{1}_{\{\max_{k \leq n} X_k \geq t\}} \, dt.$$  

By Fubini’s theorem (all the terms are nonnegative), we have

$$\int_0^\infty pt^{p-2} \mathbb{E}X_n \mathbb{1}_{\{\max_{k \leq n} X_k \geq t\}} \, dt = p\mathbb{E}X_n \int_0^\infty t^{p-2} \, dt = \frac{p}{p-1} \mathbb{E}X_n (\max_{k \leq n} X_k)^{p-1} \leq \frac{p}{p-1} (\mathbb{E}X_n^p)^{1/p} (\mathbb{E}\max_{k \leq n} X_k^p)^{1-1/p},$$

where in the last estimate we use H"older’s inequality. Thus

$$\mathbb{E}\max_{k \leq n} X_k^p \leq \frac{p}{p-1} (\mathbb{E}X_n^p)^{1/p} (\mathbb{E}\max_{k \leq n} X_k^p)^{1-1/p},$$

so dividing by $(\mathbb{E}\max_{k \leq n} X_k^p)^{1-1/p}$ finishes the proof (it is finite; if it is 0, the inequality is trivial).

\[\square\]

### 14.4 Martingales bounded in $L_p$, $p > 1$


(i) If $X = (X_n)_{n \geq 0}$ is a nonnegative submartingale bounded in $L_p$, then there is a random variable $X_\infty \in L_p$ such that $X_n \to X_\infty$ a.s. and in $L_p$. Moreover, $\|X_n\|_p \nearrow \|X_\infty\|_p$.

(ii) If $M = (M_n)_{n \geq 0}$ is a martingale bounded in $L_p$, then there is a random variable $M_\infty \in L_p$ such that $M_n \to M_\infty$ a.s. and in $L_p$.

**Proof.** Let $X_\ast = \sup_{k \geq 0} X_k$. From Doob’s maximal inequality in $L_p$, Theorem 14.17, and Lebesgue’s monotone convergence theorem, we obtain

$$\|X_\ast\|_p \leq \frac{p}{p-1} \sup_{k \geq 0} \|X_k\|_p$$

and the right hand side is finite by the assumption. Since $-X$ is a supermartingale, which is bounded in $L_1$, thanks to Doob’s convergence theorem, there exists a random variable $X_\infty \in L_1$ such that $X_n \to X_\infty$ a.s. By the triangle inequality,

$$|X_n - X_\infty|^p \leq (2X_\ast)^p.$$  

Consequently, $X_\infty \in L_p$ and by Lebesgue’s dominated convergence theorem, from the pointwise convergence we can conclude that $X_n \to X_\infty$ in $L_p$. Finally, as we saw in the proof of Doob’s maximal inequality in $L_p$, $\|X_n\|_p$ is a nondecreasing sequence, so $\|X_n\|_p \nearrow \|X_\infty\|_p$. This finishes the proof of (i).
To prove (ii), we apply (i) to $X = |M|$. The existence of $M_\infty \in L_1$ such that $M_n \to M_\infty$ a.s. is guaranteed by Doob’s convergence theorem. Since

$$|M_n - M_\infty|^p \leq (|M_n| + |M_\infty|)^p \leq (2X_*)^p,$$

we get the convergence of $M_n$ to $M_\infty$ in $L_p$, as before, thanks to Lebesgue’s dominated convergence theorem. \qed
14.5 Exercises

1. Find the quadratic variation process of the sum martingale (see Example 14.5).

2. Give an example of a family of random variables \( \{X_n\}_{n=1,2,...} \) which is uniformly integrable and \( \mathbb{E} \sup_n |X_n| = \infty. \)

3. Prove inequality (14.6) and then deduce inequality (14.7).

4. Show that there is no positive finite constant \( C \) such that for every nonnegative submartingale \( (X_n)_{n \geq 0} \), we have
   \[
   \mathbb{E} \max_{k \leq n} X_k \leq C \mathbb{E} X_n.
   \]

5. Show that for every nonnegative submartingale \( (X_n)_{n \geq 0} \), we have
   \[
   \mathbb{E} \max_{k \leq n} X_k \leq \frac{e}{e-1} \left( 1 + \mathbb{E} X_n \log_+ X_n \right),
   \]
   where \( \log_+ x = \max\{\log x, 0\}, x \geq 0. \)
   
   **Hint.** For \( a \geq 0, b > 0, \) we have \( a \log b \leq a \log_+ a + \frac{b}{e}. \)

6. Prove that the constant \( \frac{p}{p-1} \) in Doob’s maximal inequality from Theorem 14.17 is optimal.

7. Azuma’s inequality. Let \( (M_n)_{n \geq 0} \) be a martingale with \( M_0 = 0 \) and \( |M_k - M_{k-1}| \leq a_k \) for every \( k \geq 1 \) for some positive constants \( a_1, a_2, \ldots \). Then for every \( n \geq 1 \) and \( t > 0, \)
   \[
   \mathbb{P} \left( \max_{k \leq n} M_k \right) \leq \exp \left\{ -\frac{t^2}{2 \sum_{k=1}^n a_k^2} \right\}.
   \]
   
   **Hint.** Follow the proof of Bernstein’s inequality (Exercise 6.19). Using convexity, show that for a random variable \( X \) with \( |X| \leq a \) for some \( a > 0 \) and \( \mathbb{E} X = 0, \) we have \( \mathbb{E} e^{tX} \leq \cosh(ta), t \in \mathbb{R}. \)
15 Applications of martingale theory

15.1 Series of independent random variables

We begin with applications of the $L_2$ theory to series of independent random variables. The main classical result is Kolmogorov's three-series test which gives necessary and sufficient conditions for a series $\sum X_i$ of independent random variables to be convergent a.s. Combined with Kronecker's lemma, it leads to the strong law of numbers. An advantage of this approach is that it gives a way of obtaining rates of convergence. First, we need a lemma which is a direct consequence of the convergence result for $L_2$ bounded martingales. Historically, it was established by means of Kolmogorov's maximal inequality.

15.1 Lemma. Let $X_1, X_2, \ldots$ be independent random variables with $E X_k^2 < \infty$ and $E X_k = 0$ for every $k$. If $\sum_{n=1}^{\infty} \text{Var}(X_n) < \infty$, then $\sum_{n=1}^{\infty} X_n$ converges a.s.

Proof. Of course, we consider the sum martingale $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Since

$$\sup_n \mathbb{E} S_n^2 = \sum_{k \geq 1} \text{Var}(X_k) < \infty,$$

the martingale $(S_n)$ is bounded in $L_2$, so $S_n$ converges a.s. (and in $L_2$), by Theorem 14.1.

15.2 Remark. Alternatively, we can say that since $\langle S \rangle_\infty = \sum_{k=1}^{\infty} < \infty$, we get the assertion by (i) of Theorem 14.7.

15.3 Remark. If the variables $X_n$ are all bounded, that is there is a constant $K > 0$ such that $|X_n| \leq K$ for every $n$, then $(S_n)$ has bounded increments and the converse holds: if $\sum_{n=1}^{\infty} X_n$ converges, then $\sum_{n=1}^{\infty} \text{Var}(X_n) < \infty$. This follows immediately from (ii) of Theorem 14.7. We shall strengthen it soon by removing the assumption of mean 0.

15.4 Theorem (Kolmogorov’s three-series test). Let $X_1, X_2, \ldots$ be independent random variables. Then $\sum X_n$ converges a.s. if for some $K > 0$ (equivalently, every $K > 0$) the following three conditions hold

(i) $\sum P(|X_n| > K) < \infty$,  
(ii) $\sum E X_n I_{|X_n| \leq K}$ converges,  
(iii) $\sum \text{Var}(X_n I_{|X_n| \leq K}) < \infty$.

Conversely, if $\sum X_n$ converges a.s., then (i)-(iii) hold for every $K > 0$. 

166
Proof of sufficiency. Suppose that for some positive $K$ conditions (i), (ii) and (iii) hold. Let $\tilde{X}_n = X_n 1_{\{X_n \leq K\}}$. First note that, thanks to (i),
\[ \sum \mathbb{P} \left( X_n \neq \tilde{X}_n \right) = \sum \mathbb{P} \left( |X_n| > K \right) < \infty, \]
so by the first Borel-Cantelli lemma,
\[ \mathbb{P} \left( X_n = \tilde{X}_n \text{ for all but finitely many } n's \right) = 1, \]
so it is enough to show that $\sum \tilde{X}_n$ converges a.s. Thanks to (ii), it is enough to show that $\sum (X_n - \mathbb{E}\tilde{X}_n)$ converges a.s. which in view of Lemma 15.1 follows from (iii). \qed

Proof of necessity. Fix $K > 0$ and suppose that $\sum X_n$ converges a.s. In particular, $X_n \to 0$ a.s., so $\mathbb{P} \left( |X_n| > K \text{ for infinitely many } n's \right) = 0$, so by the second Borel-Cantelli lemma, $\sum \mathbb{P} \left( |X_n| > K \right) < \infty$, that is (i) holds. As in the proof of sufficiency, this in turn gives that $X_n = \tilde{X}_n$ eventually, a.s., so we also know that $\sum \tilde{X}_n$ converges a.s. The following lemma applied to the sequence $(\tilde{X}_n)$ finishes the proof. \qed

15.5 Lemma. Let $(X_n)$ be a sequence of independent random variables bounded by some positive constant $K$, that is $|X_n| \leq K$ for every $n$. If $\sum X_n$ converges, then $\sum \mathbb{E}X_n$ and $\sum \text{Var}(X_n)$ converge.

Proof. We shall use characteristic functions. Let $Y_n = X_n - \mathbb{E}X_n$. First note that, $|Y_n| \leq 2K$ and, plainly, $\phi_{Y_n}(t) = e^{-i t X_n} \phi_{X_n}(t)$, so
\[ |\phi_{Y_n}(t)| = |\phi_{Y_n}(t)|. \]

Denote $\sigma_n^2 = \text{Var}(X_n) = \mathbb{E}Y_n^2$. By Lemma 10.4,
\[ |\phi_{Y_n}(t) - (1 - \frac{1}{2} \sigma_n^2 t^2)| = \mathbb{E} \left[ e^{\imath t Y_n} - (1 + \imath t Y_n - \frac{1}{2} t^2 Y_n^2) \right] \leq \mathbb{E} \left[ |t|^3 \left| Y_n \right|^3 \right] \leq \frac{|t|^3 (2K) \mathbb{E}Y_n^2}{6} = \frac{|t|^3 K \sigma_n^2 t^2}{6}. \]

Consequently, for all $|t| < \frac{3}{4K}$, we have
\[ |\phi_{Y_n}(t)| \leq 1 - \frac{1}{2} \sigma_n^2 t^2 + \frac{1}{4} \sigma_n^2 t^2 = 1 - \frac{1}{4} \sigma_n^2 t^2. \]

Since $S_n = X_1 + \cdots + X_n$ converges a.s., say to $S$, we have $\phi_{S_n}(t) \to \phi_S(t)$ for every $t$. By continuity, $|\phi_S(t)| > \frac{1}{2}$ for all $t$ sufficiently small. Fix one such positive $t$ with $t < \frac{3}{4K}$. Then, for all $n$ large enough,
\[ \frac{1}{4} < |\phi_{S_n}(t)| = \prod_{k=1}^{n} |\phi_{X_k}(t)| = \prod_{k=1}^{n} |\phi_{Y_k}(t)| \leq e^{-\frac{1}{4} t^2 \sum_{k=1}^{n} \sigma_k^2}, \]
which gives $\sum_{k=1}^{\infty} \sigma_k^2 < \infty$, as desired. Finally, by Lemma 15.1 applied to $Y_n$, we get that the series $\sum Y_n = \sum (X_n - \mathbb{E}X_n)$ converges a.s., which together with $\sum X_n$ being convergent a.s., gives that $\sum \mathbb{E}X_n$ converges. \qed
We refer to the exercises for applications of the basic lemma 15.1 combined with Kronecker’s Lemma 14.9 allowing to obtain strong laws of large numbers with rates of convergence.

### 15.2 Kolmogorov’s 0 − 1 law and strong law of large numbers

Here we present applications of Lévy’s convergence theorems.

**Martingale proof of Theorem 3.17 (Kolmogorov’s 0 − 1 law).** Let \( F_n = \sigma(X_1, \ldots, X_n) \).
Let \( A \in \mathcal{T} \) and \( X = 1_A \). Clearly \( X \) is independent of \( F_n \) for every \( n \) and \( X \) is \( F_\infty = \sigma(\bigcup_n F_n) \)-measurable. Thus, \( \mathbb{E}(X|F_n) = \mathbb{E}X = \mathbb{P}(A) \) and \( \mathbb{E}(X|F_\infty) = X \).
By Theorem 14.12, \( \mathbb{E}(X|F_n) \xrightarrow{a.s.} \mathbb{E}(X|F_\infty) \).
so \( \mathbb{P}(A) \rightarrow X \) a.s. and because \( X \in \{0, 1\} \), this gives \( \mathbb{P}(A) \in \{0, 1\} \).

**Strong law of large numbers: martingale proof.** Suppose \( X_1, X_2, \ldots \) are i.i.d. random variables with \( \mathbb{E}|X_1| < \infty \). Then
\[
\frac{X_1 + \cdots + X_n}{n} \xrightarrow{n \to \infty} \mathbb{E}X_1 \quad \text{a.s. and in } L_1
\]
(compare this to Etemadi’s strong law, Theorem 7.16, where we only assume pairwise independence but only conclude a.s. convergence).

For the proof, let \( S_n = X_1 + \cdots + X_n \), \( G_{-n} = \sigma(S_n, S_{n+1}, \ldots) \), \( n \geq 1 \) and as in Lévy’s downward convergence theorem, \( G_{-\infty} = \bigcap_n G_{-n} \). By Example 12.5,
\[
\mathbb{E}(X_1|G_{-n}) = \frac{S_n}{n}.
\]
Thus, by Lévy’s theorem (Theorem 14.13), there is an integrable random variable \( Y \) such that
\[
\frac{S_n}{n} \rightarrow Y \quad \text{a.s. and in } L_1.
\]
Observe that for every fixed \( m \), \( Y = \lim_n \frac{S_n}{n} = \lim_n \frac{X_{m+1} + \cdots + X_n}{n} \). As a result, \( Y \) is \( \sigma(X_{m+1}, X_{m+2}, \ldots) \)-measurable, thus it is \( \mathcal{T} \)-measurable, where \( \mathcal{T} \) is the tail \( \sigma \)-algebra.

By Kolmogorov’s 0 − 1 law, \( Y \) is therefore constant a.s., say \( Y = c \) a.s. for some \( c \in \mathbb{R} \).
By the \( L_1 \) convergence, \( \mathbb{E}X_1 = \lim_n \mathbb{E} \frac{S_n}{n} = \mathbb{E}Y = c. \)

### 15.3 Kakutani’s theorem

The next result is obtained as a basic application of Doob’s convergence theorem to product martingales and Doob’s maximal inequality in \( L_2 \).

**15.6 Theorem** (Kakutani’s theorem on product martingales). Let \( X_1, X_2, \ldots \) be independent nonnegative random variables, each one with mean 1. Let \( M_0 = 1 \) and
Then there is an integrable random variable $M_\infty$ such that $M_n \to M_\infty$ a.s. and the following conditions are equivalent

(i) $\mathbb{E}M_\infty = 1$

(ii) $M_n \to M_\infty$ in $L_1$

(iii) $M = (M_n)_{n \geq 0}$ is uniformly integrable

(iv) $\prod_{k=1}^\infty \mathbb{E}X_k^{1/2} > 0$

(v) $\sum_{k=1}^\infty (1 - \mathbb{E}X_k^{1/2}) < \infty$.

If they fail to hold, then $M_\infty = 0$ a.s.

**Proof.** Here is how we proceed: (ii) $\Leftrightarrow$ (iii), (iv) $\Leftrightarrow$ (v), (iv) $\Rightarrow$ (i)-(iii), NOT(iv) $\Rightarrow M_\infty = 0$ a.s. and as a result NOT(i), NOT(ii). These suffice.

Since $M$ is a nonnegative martingale (see Example 13.3), the existence of $M_\infty \in L_1$ with $M_n \to M_\infty$ a.s. immediately follows from Doob’s convergence theorem.

Note that (ii) and (iii) are equivalent because of the characterisation of $L_p$ convergence in terms of uniform integrability, Theorem I.6.

Let $a_k = \mathbb{E}X_k^{1/2}$ which is positive (because $\mathbb{E}X_k = 1$). By Jensen’s inequality, $\mathbb{E}X_k^{1/2} \leq (\mathbb{E}X_k)^{1/2} = 1$, so in fact $a_k \in (0, 1]$. Then the equivalence of (iv) and (v), $\prod a_k > 0 \Leftrightarrow \sum (1 - a_k) < \infty$, is a straightforward consequence of the inequalities $1 - x \leq e^{-x}$, $x \in \mathbb{R}$ and, say $1 - x \geq e^{-2x}$, $x \in [0, \frac{1}{2}]$.

Suppose (iv) holds. Consider

$$Y_0 = 1, \quad Y_n = \frac{X_1^{1/2}}{a_1} \cdots \frac{X_n^{1/2}}{a_n}.$$  

This is a nonnegative martingale, bounded in $L_2$ because

$$\mathbb{E}Y_n^2 = \frac{1}{\prod_{k=1}^n a_k^2} \leq \frac{1}{\prod_{k=1}^\infty a_k^2} < \infty.$$  

Note that $M_n = Y_n^2 (a_1 \cdots a_n)^2 \leq Y_n^2$. Therefore, by Doob’s maximal $L_2$ inequality,

$$\mathbb{E}\sup_{n \geq 1} M_n \leq \mathbb{E}\sup_{n \geq 1} Y_n^2 \leq 4\sup_{n \geq 1} \mathbb{E}Y_n^2 < \infty.$$  

Letting $M_\ast = \sup_{n \geq 1} M_n$, which is in $L_1$ by the above, we have $M_n \leq M_\ast$, showing that $(M_n)$ is uniformly integrable (Lemma I.1). Thus (iii) holds, hence (ii), too. Of course, (ii) and $\mathbb{E}M_\infty = 1$ implies (i).

Suppose (iv) does not hold, that is $\prod_{k=1}^n a_k \to 0$. Then, since $Y_n \to Y_\infty$ a.s. for some integrable random variable $Y_\infty$ ($Y$ is a nonnegative martingale!), we have

$$M_n = Y_n^2 \left(\prod_{k=1}^n a_k\right)^2 \xrightarrow{n \to \infty} 0$$  

that is $M_\infty = 0$ a.s. Consequently, neither (i) nor (ii) do hold. □
In particular, when the \( X_i \) are i.i.d., unless they are constant a.s., the product \( X_1 \cdots X_n \) converges a.s. but not in \( L_1 \), to \( M_\infty = 0 \) a.s. (see also Exercise 13.15).

### 15.4 The law of the iterated logarithm for Gaussians

**15.7 Theorem.** Let \( X_1, X_2, \ldots \) be i.i.d. standard Gaussian random variables and set \( S_n = X_1 + \cdots + X_n, \ n \geq 1 \). Then

\[
\limsup_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = 1 \quad \text{a.s.}
\]

and

\[
\liminf_{n \to \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -1 \quad \text{a.s.}
\]

**Proof.** Let

\[
h(x) = \sqrt{2x \log \log x}, \quad x > e.
\]

The statement about \( \lim inf \) follows from the one about \( \lim sup \) by symmetry (\(-S_n\) has the same distribution as \( S_n \)). To prove the latter, we split the argument into two parts.

*Upper bound.* First we exploit Doob’s maximal inequality, to get an exponential bound on tail probabilities for the maximum. Fix \( \lambda > 0 \). We have,

\[
\mathbb{E} e^{\lambda S_n} = e^{\frac{1}{2} \lambda^2 n}
\]

(because \( S_n \sim N(0, n) \)). Moreover, \((e^{\lambda S_n})_{n \geq 1}\) is a submartingale because \( x \mapsto e^x \) is convex. By Doob’s maximal inequality 14.5, for every \( t \), we have

\[
P \left( \max_{k \leq n} S_k \geq t \right) = P \left( \max_{k \leq n} e^{\lambda S_k} \geq e^{\lambda t} \right) \leq e^{-\lambda t} \mathbb{E} e^{\lambda S_n} = e^{-\lambda t + \frac{1}{2} \lambda^2 n}
\]

and optimising over \( \lambda \) yields

\[
P \left( \max_{k \leq n} S_k \geq t \right) \leq e^{-\frac{t^2}{2n}}.
\]

Fix \( \alpha > 1 \) and let \( a_n = \alpha h(\alpha^{n-1}) \). Since

\[
P \left( \max_{k \leq \alpha^n} S_k \geq a_n \right) \leq e^{-\frac{a^2}{2n}} = \exp \left\{ -\frac{\alpha^2 \cdot 2^{\alpha n-1} \log \log \alpha^{\alpha - 1}}{2\alpha n} \right\} = \exp \left\{ -\alpha \log \log \alpha^{\alpha - 1} \right\} = (n-1)^{-\alpha} e^{-\alpha \log \log \alpha},
\]

the series \( \sum P(\max_{k \leq \alpha^n} S_k \geq a_n) \) converges and by the first Borel-Cantelli lemma, a.s., \( \max_{k \leq \alpha^n} S_k \geq a_n \) holds only for finitely many \( n \)’s. Consequently, the event

\[
\{ \max_{k \leq \alpha^n} S_k < a_n \quad \text{for all but finitely many } n \text{’s} \}
\]

170
has probability 1. This event is contained in the event
\[ U_\alpha = \{ \exists n_0 \, \forall n \geq n_0 \, \forall \alpha^{n-1} \leq k \leq \alpha^n \, S_k \leq \alpha h(k) \} \]
because of the monotonicity of \( h(x) \), so \( U_\alpha \) has probability 1. On the event \( U_\alpha \),
\[ \limsup_{n \to \infty} \frac{S_k}{h(k)} \leq \alpha, \]
thus on the event \( \bigcap_{j=2}^{\infty} U_{1+1/j} \), which also has probability 1,
\[ \limsup_{n \to \infty} \frac{S_k}{h(k)} \leq 1, \]

hence
\[ \limsup_{n \to \infty} \frac{S_k}{h(k)} \leq 1 \text{ a.s.} \]

**Lower bound.** We shall need the following elementary estimate on Gaussian tails (cf. Lemma 11.4).

**Claim.** If \( g \) is a standard Gaussian random variable, then for \( t > 0 \),
\[ \mathbb{P}(g > t) \geq \frac{1}{\sqrt{2\pi}} t e^{-t^2/2}. \]

Indeed,
\[
\mathbb{P}(g > t) = \int_t^{\infty} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \geq \int_t^{\infty} \frac{1 + x^{-2}}{1 + t^{-2}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}
= \frac{1}{1 + t^{-2}} \int_t^{\infty} \left( -\frac{1}{x} e^{-x^2/2} \right)' \frac{dx}{\sqrt{2\pi}}
= \frac{1}{\sqrt{2\pi}} t e^{-t^2/2}.
\]

We fix \( \varepsilon \in (0, 1) \) and an integer \( N > 1 \). We consider the events
\[ A_\alpha = \{ S_{N^{n+1}} - S_{N^n} > (1 - \varepsilon) h(N^{n+1} - N^n) \}. \]

Since \( S_{N^{n+1}} - S_{N^n} \) has the same distribution as \( \sqrt{N^{n+1} - N^n} g \), where \( g \) is standard Gaussian, by the claim, we get
\[
\mathbb{P}(A_\alpha) = \mathbb{P}
\left( g > (1 - \varepsilon) \sqrt{2 \log \log(N^{n+1} - N^n)} \right)
\geq \frac{1}{\sqrt{2\pi}} \frac{(1 - \varepsilon) \sqrt{2 \log \log(N^{n+1} - N^n)}}{1 + 2(1 - \varepsilon)^2 \log \log(N^{n+1} - N^n)} (\log N^n (N - 1))^{-(1 - \varepsilon)^2}
= \Omega(n^{-1})
\]
provided \( N \) is large enough. This gives \( \sum \mathbb{P}(A_\alpha) = \infty \) and of course, the events \( A_\alpha \) are independent, so by the second Borel-Cantelli lemma, infinitely many \( A_\alpha \) occur with probability 1. In other words,
\[ \mathbb{P}(S_{N^{n+1}} > (1 - \varepsilon) h(N^{n+1} - N^n) + S_{N^n}, \text{ for infinitely many } n \text{'s}) = 1. \]
By the upper bound, a.s., \( S_{N_n} > -2h(N^n) \) eventually, so the events

\[ V_{N,\varepsilon} = \{ S_{N_{n+1}} > (1-\varepsilon)h(N^{n+1} - N^n) - 2h(N^n), \text{ for infinitely many } n's \} \]

have probability 1. On \( V_{N,\varepsilon} \), we have

\[
\limsup \frac{S_n}{h(n)} \geq \limsup \frac{S_{N_{n+1}}}{h(N_{n+1})} \geq \limsup \left( 1 - \varepsilon \right) \frac{h(N^{n+1} - N^n)}{h(N^{n+1})} - 2 \frac{h(N^n)}{h(N^{n+1})} \\
= \limsup \left( 1 - \varepsilon \right) \sqrt{\frac{2N^n(N - 1) \log \log N^n(N - 1)}{2N^{n+1} \log \log N^{n+1}}} - 2 \sqrt{\frac{2N^n \log \log N^n}{2N^{n+1} \log \log N^{n+1}}} \\
= (1 - \varepsilon) \sqrt{\frac{N - 1}{N}} - \frac{2}{\sqrt{N}}.
\]

Therefore, on \( \bigcap_{N=2}^{\infty} \bigcap_{l=2}^{\infty} V_{N,1/l} \), we have

\[
\limsup \frac{S_n}{h(n)} \geq 1.
\]

\[
\square
\]

15.8 Remark. We have,

\[
\frac{S_n}{\sqrt{2n \log \log n}} \xrightarrow{P} 0.
\]

This is very simple: since \( S_n \) has the same distribution as \( \sqrt{n}g \), where \( g \) is standard Gaussian, we have

\[
P \left( \left| \frac{S_n}{\sqrt{2n \log \log n}} \right| > \varepsilon \right) = P \left( |g| > \varepsilon \sqrt{\log \log n} \right) \to 0.
\]

15.9 Remark. Theorem 15.7 can be substantially generalised: it holds for an arbitrary sequence of i.i.d. random variables with mean 0 and variance 1 (the Hartman-Wintner law of the iterated logarithm). There are several proofs (see e.g. [1] for an elementary proof, or [3] for a modern proof using Brownian motion and Donsker’s theorem).

15.5 The Radon-Nikodym theorem

Let \( \mu \) and \( \nu \) be finite measures on \((\Omega, \mathcal{F})\). We say that \( \nu \) is absolutely continuous with respect to \( \mu \) if for every \( A \in \mathcal{F} \) with \( \mu(A) = 0 \), we also have \( \nu(A) = 0 \). This is sometimes denoted \( \nu \ll \mu \). The Radon-Nikodym theorem implies that then \( \nu \) has a density with respect to \( \mu \), that is there is a measurable function \( g : \Omega \to [0, +\infty) \) such that for every measurable set \( A \),

\[
\nu(A) = \int_A g \, d\mu.
\]

This function is sometimes called the Radon-Nikodym derivative, denoted \( \frac{d\nu}{d\mu} \). Clearly, the converse holds as well. We shall present a martingale proof of the Radon-Nikodym theorem.
15.10 Theorem (Radon-Nikodym). Let $\mu$ and $\nu$ be finite measures on $(\Omega, F)$. There is an $F$-measurable function $g: \Omega \to [0, +\infty)$ and a set $S \in F$ such that $\mu(S) = 0$ and

$$\nu(A) = \nu(A \cap S) + \int_A gd\mu, \quad A \in F.$$ 

Moreover, $g$ is unique up to sets of $\mu$-measure 0 and $S$ is unique up to sets of $(\mu + \nu)$-measure 0.

15.11 Remark. In particular, if $\nu \ll \mu$, then $\nu(S) = 0$, so

$$\nu(A) = \nu(A \cap S) + \int_A gd\mu, \quad A \in F.$$ 

15.12 Remark. Considering $\mu$ and $\nu$ on pieces where they are finite, the theorem extends to the case when $\mu$ and $\nu$ are $\sigma$-finite.

To “construct” $g$, we will work with sequences in $L_1(\Omega, F, \mu + \nu)$ indexed by finite sub-$\sigma$-algebras of $F$, so we need to extend a bit notions of convergence to such sequences. The completeness of $L_1$ will play a crucial role.

Let $(E, d)$ be a metric space. Let $T$ be a directed set, that is a partially ordered set by a relation $\preceq$ (reflexive, antisymmetric and transitive) with the property that every two elements of $T$ have an upper bound, that is for every $s, t \in T$, there is $u \in T$ with $s \preceq u$ and $t \preceq u$. We say that a sequence $(a_t)_{t \in T}$ in $E$ indexed by $T$ converges to $a \in E$ if

$$\forall \varepsilon > 0 \ \exists t_0 \in T \ \forall t \in T \ \ t \succeq t_0 \Rightarrow d(a_t, a) < \varepsilon.$$ 

(15.1)

We say that the sequence $(a_t)_{t \in T}$ satisfies the Cauchy condition (or, simply, is Cauchy) if

$$\forall \varepsilon > 0 \ \exists t_0 \in T \ \forall t \in T \ \ t \succeq t_0 \Rightarrow d(a_t, a_{t_0}) < \varepsilon.$$ 

(15.2)

15.13 Lemma. Let $(E, d)$ be a complete metric space and let $(T, \preceq)$ be a directed set.

(i) If $(a_t)_{t \in T}$ is a sequence in $E$ such that for every nondecreasing sequence of indices $t_1 \preceq t_2 \preceq \ldots$, the sequence $(a_{t_n})_{n \geq 1}$ converges, then $(a_t)_{t \in T}$ is Cauchy.

(ii) If $(a_t)_{t \in T}$ is a Cauchy sequence in $E$, then it converges to some $a \in E$ and there exists a nondecreasing sequence of indices $t_1 \preceq t_2 \preceq \ldots$ such that $a_{t_n} \to a$ in $E$.

Proof. (i): If $(a_t)$ does not satisfy the Cauchy condition, then there is $\varepsilon > 0$ such that for every $t_0 \in T$, there is $t \in T$ with $t \succeq t_0$ and $d(a_t, a_{t_0}) \geq \varepsilon$. Choose $t_1 \in T$ arbitrarily. Given $t_n$, define $t_{n+1}$ as the index $t$ given by the previous condition applied to $t_n = t_{n-1}$. We obtain the sequence $t_1 \preceq t_2 \preceq \ldots$ with $d(a_{t_n}, a_{t_{n+1}}) \geq \varepsilon$, so $(a_{t_n})_n$ does not converge in $E$, a contradiction.

(ii): For $n = 1, 2, \ldots$, we apply the Cauchy condition with $\varepsilon = \frac{1}{n}$ and set $t'_n$ to be the index $t_0$ provided in (15.2). Then we define the sequence $(t_n)$ recursively, $t_1 = t'_1$ and
given \( t_n \), we choose \( t_{n+1} \) as any common upper bound of \( t_n \) and \( t'_n+1 \). This way we obtain the sequence of indices \( t_1 \leq t_2 \leq \ldots \) such that for every \( n \geq 1 \) and every \( t \in T \) with \( t \geq t_n \), we have

\[
d(a_{t_n}, a_t) \leq d(a_{t_n}, a_{t'_n}) + d(a_{t'_n}, a_t) < \frac{2}{n}.
\]

This shows that the sequence \((a_{t_n})_n\) is Cauchy. Since \( E \) is complete, it converges to, say \( a \in E \). It remains to show that the whole sequence \((a_t)_{t \in T}\) also converges to \( a \) in the sense of (15.1). Fix \( \varepsilon > 0 \). Let \( n \) be such that \( \varepsilon > \frac{1}{n} \) and \( d(a_{t_n}, a) < \varepsilon \). Choose \( t_0 = t_n \).

Then for every \( t \in T \) with \( t \geq t_0 \), we have

\[
d(a_t, a) \leq d(a_t, a_{t_n}) + d(a_{t_n}, a) < \frac{2}{n} + \varepsilon < 3\varepsilon.
\]

Thus (15.1) holds and the proof is finished.

\[\square\]

Proof of Theorem 15.10. Let \( c = \mu(\Omega) + \nu(\Omega) \). If \( c = 0 \), there is nothing to prove, so we assume \( c > 0 \). Let \( P = \frac{1}{c}(\mu + \nu) \), so that \((\Omega, F, P)\) is a probability space. We shall denote the integral against \( P \) by \( \mathbb{E} \), that is

\[
\mathbb{E} f = \int_{\Omega} f \frac{d(\mu + \nu)}{c}, \quad f: \Omega \to \mathbb{R}, \ f \text{ is measurable}.
\]

Step I (martingale argument). We define

\[
T = \{ \mathcal{G} \subset F : \mathcal{G} \text{ is a finite sub-}\sigma\text{-algebra, i.e.} \mathcal{G} = \sigma(A_1, \ldots, A_n) \text{ for some } A_1, \ldots, A_n \in F \}
\]

Equipped with the inclusion relation \( \subset \), this is a directed set (a common upper bound for \( \mathcal{G}_1, \mathcal{G}_2 \in T \) is simply \( \sigma(\mathcal{G}_1, \mathcal{G}_2) \)). We set

\[
E = L_1(\Omega, F, P)
\]

which is a complete metric space (see Theorem 6.10). For \( \mathcal{G} \in T \) generated by atoms \( A_1, \ldots, A_n \) (meaning that \( \Omega = A_1 \cup \cdots \cup A_n \) is a disjoint partition and every set in \( \mathcal{G} \) is of the form \( \bigcup_{i \in I} A_i \) for a subset \( I \) of \( \{1, \ldots, n\} \)), we define

\[
X_{\mathcal{G}}(\omega) = \begin{cases} 
\frac{\nu(A_j)}{P(A_j)}, & \text{for } \omega \in A_j, \text{ if } P(A_j) > 0 \\
0, & \text{for } \omega \in A_j, \text{ if } P(A_j) = 0
\end{cases}
\]

Note that

(a) \( 0 \leq X_{\mathcal{G}} \leq c \),

(b) \( X_{\mathcal{G}} \) is the density of \( \nu \) with respect to \( P \) on \( \mathcal{G} \), that is

\[
\nu(A) = \mathbb{E} X_{\mathcal{G}} 1_A, \quad \text{for every } A \in \mathcal{G},
\]

174
(c) For every sequence \( G_1 \subset G_2 \subset \ldots, \ G_n \in T, \ n = 1, 2, \ldots, \) letting \( X_n = X_{G_n}, \) we have \((X_n)_{n \geq 1}\) is a martingale on \((\Omega, \mathcal{F}, \{G_n\}_{n \geq 1}, \mathbb{P})\).

This is because for every \( n \geq 1 \) and every \( A \in G_n, \) thanks to (a), we have

\[
E X_{n+1} 1_A = \nu(A) = E X_n 1_A,
\]

or, equivalently, \( E(X_{n+1} | G_n) = X_n. \)

Moreover, since \((X_n)\) is bounded (by virtue of (a)), \((X_n)\) is a uniformly integrable martingale (Lemma I.1), so it converges a.s. and in \( L_1. \)

By (c) and Lemma 15.13, the sequence \((X_G)_{G \in T}\) converges in \( L_1 \) (in the sense of (15.1)) to a random variable \( X \in L_1. \) Moreover, there is a sequence \( G_1 \subset G_2 \subset \ldots \) in \( T \) such that \( X_{G_n} \to X \) in \( L_1. \) As a uniformly integrable martingale, this sequence \((X_{G_n})\) also converges a.s. and in \( L_1 \) to some \( L_1 \) random variable, say \( X_\infty. \) By the uniqueness of limits in \( L_1, \) \( X_\infty = X \) and by the a.s. convergence, \( X \in [0, c] \) a.s.

**Step II (limit argument).** The idea is of course that because each \( X_G \) is the density of \( \nu \) with respect to \( \mathbb{P} \) on \( G, \) the random variable \( X \) constructed in the previous step as the limit of \( X_G, \) should be the density of \( \nu \) with respect to \( \mathbb{P} \) on the whole \( \mathcal{F}. \)

Formally, fix \( \varepsilon > 0. \) Since \( X_G \to X \) (in the sense of (15.1)), there is \( K \in T \) such that for every \( G \supset K, \) we have

\[
E|X_G - X| < \varepsilon.
\]

Fix \( A \in \mathcal{F} \) and let \( G = \sigma(K, A). \) Since \( G \supset K, \) we have

\[
|E X_G 1_A - E X 1_A | \leq E|X_G - X| 1_A < \varepsilon.
\]

By (b), \( E X_G 1_A = \nu(A), \) so we obtain

\[
|\nu(A) - E X 1_A | < \varepsilon,
\]

hence

\[
\nu(A) = E X 1_A
\]

(because \( \varepsilon \) is arbitrary). In view of \( E X 1_A = \frac{1}{c} \int_A X d\mu + \frac{1}{c} \int_A X d\nu, \) we equivalently have

\[
\int_A (c - X) d\nu = \int_A X d\mu,
\]

for every \( A \in \mathcal{F}. \) By a standard argument of complicating measurable functions, we also obtain from this that

\[
\int_A (c - X) f d\nu = \int_A X f d\mu,
\]

for every \( A \in \mathcal{F} \) and every \( \mathcal{F} \)-measurable function \( f: \Omega \to \mathbb{R}. \)
**Step III (derivation of density).** It remains to define $S$ and $g$. We set

$$S = \{X = c\}.$$  

Choosing $A = S$, $f = 1$ in (15.3) yields

$$0 = \int_S (c - X) d\nu = \int_S X d\mu = \int_S c d\mu = c \mu(S),$$

hence $\mu(S) = 0$. Recall that $X \in [0, c]$ P-a.s., so in particular we have, $\mu(\{X > c\}) = \nu(\{X > c\}) = 0$. Applying now (15.3) with

$$f(\omega) = \begin{cases} 1, & \omega \in S_c, \\ 0, & \omega \in S, \end{cases}$$

we get

$$\nu(A \cap S^c) = \int_A (c - X) f d\nu = \int_A X f d\mu = \int_{A \cap S_c} \frac{X}{c - X} d\mu.$$

Therefore, we define

$$g = \begin{cases} \frac{X}{c - X}, & \text{on } S_c, \\ 0, & \text{on } S, \end{cases}$$

and the previous identity becomes

$$\nu(A \cap S^c) = \int_A g d\mu.$$  

Finally,

$$\nu(A) = \nu(A \cap S) + \nu(A \cap S^c) = \nu(A \cap S) + \int_A g d\mu,$$

as desired.

**Step IV (uniqueness).** Suppose we have another set $\tilde{S}$ and function $\tilde{g}$ satisfying the required properties. Then, for every $A \in \mathcal{F},$

$$\nu(A \cap S) + \int_A \tilde{g} d\mu = \nu(A \cap \tilde{S}) + \int_A \tilde{g} d\mu.$$  

Taking $A = S$ gives $\nu(S) = \nu(S \cap \tilde{S})$ (because $\int_S \tilde{g} d\mu = 0 = \int_S \tilde{g} d\mu$, as $\mu(S) = 0$).

By symmetry, $\nu(\tilde{S}) = \nu(S \cap \tilde{S})$, hence $\nu(S \Delta \tilde{S}) = 0$. Thus $S$ is unique up to sets of $(\mu + \nu)$-measure 0. In particular, now we know that $\nu(A \cap S) = \nu(A \cap \tilde{S})$, so

$$\int_A g d\mu = \int_A \tilde{g} d\mu,$$

for every $A \in \mathcal{F}$. As a result, $g = \tilde{g}$ $\mu$-a.e. \[\square\]
15.6 Exercises

1. Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs, that is $P(\varepsilon_k = 1) = \frac{1}{2} = P(\varepsilon_k = -1)$ for every $k$. Then $\sum_{n=1}^{\infty} a_n \varepsilon_n$ converges a.s. if and only if $\sum_{n=1}^{\infty} a_n^2 < \infty$.

2. Let $X_1, X_2, \ldots$ be i.i.d. exponential random variables with parameter 1. Let $(a_n)_{n \geq 1}$ be a sequence of nonnegative numbers. Show that $\sum_{n=1}^{\infty} a_n X_n$ converges a.s. if and only if $\sum a_n < \infty$.

3. If $X_1, X_2, \ldots$ are i.i.d. random variables with $E X_1^2 < \infty$ and $E X_1 = 0$, then for every $\varepsilon > 0$, we have
\[
\frac{X_1 + \cdots + X_n}{n^{1/2} \log^{1/2+\varepsilon} n} \xrightarrow{a.s.} 0, \quad n \to \infty.
\]
In other words, $\frac{X_1 + \cdots + X_n}{n} = o \left( \frac{\log^{1/2+\varepsilon} n}{n^{1/2}} \right)$ a.s., giving the rate of convergence in the strong law large numbers under the assumption that $E X_1^2 < \infty$.

4. Prove Marcinkiewicz’s theorem: if $p \in (0, 2)$ and $X_1, X_2, \ldots$ are i.i.d. with $E|X_1|^p < \infty$, then
\[
\frac{X_1 + \cdots + X_n - n \mu}{n^{1/p}} \xrightarrow{a.s.} 0, \quad n \to \infty,
\]
where $\mu = 0$ for $p \in (0, 1)$ and $\mu = E X_1$ for $p \in [1, 2)$.
In other words, for $p \in (1, 2)$, $\frac{X_1 + \cdots + X_n}{n} - E X_1 = o(n^{1/p-1})$ a.s. which is the strong law of large numbers with the rate of convergence.

5. Let $\mu, \nu$ be two finite measures on $(\Omega, F)$. Show that $\nu$ is absolutely continuous with respect to $\mu$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $A \in F$ with $\mu(A) < \delta$, we have $\nu(A) < \varepsilon$. 

177
16 Large deviations

The weak law of large numbers tells us that for every \( \epsilon > 0 \),
\[
\mathbb{P}\left( \frac{S_n}{n} - \mu > \epsilon \right) \xrightarrow{n \to \infty} 0,
\]
where \( S_n = X_1 + \ldots + X_n \), \( X_1, X_2, \ldots \) are i.i.d. integrable random variables and \( \mu = \mathbb{E}X_1 \).
How fast do these probabilities converge to 0?

Large deviations theory answers this question and establishes, under additional assumptions, precise exponential rates of convergence. Cramér’s theorem provides a convex function \( I: (\mu, +\infty) \to (0, +\infty) \) (determined by the distribution of \( X_1 \)) such that
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \frac{S_n}{n} > a \right) = -I(a), \quad a > \mu,
\]
so that, roughly, \( \mathbb{P}\left( \frac{S_n}{n} > a \right) \approx e^{-nI(a)} \).

We stress out that as opposed to the central limit theorem, which identifies the limiting behaviour of the probabilities for the bulk (centre), that is a narrow window of width \( O(1/\sqrt{n}) \) around the mean,
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( S_n \geq na \right) = \int_a^b e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}, \quad a < b,
\]
large deviations treat the limiting behaviour of the probabilities for the tail, a constant away from the mean,
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( \frac{S_n}{n} - \mu > \epsilon \right) = -I(\mu + \epsilon), \quad \epsilon > 0.
\]

We end this introduction with a simple lemma showing that as a consequence of independence, such limit always exists. The ultimate goal would be to determine its value.

16.1 Lemma. Let \( X_1, X_2, \ldots \) be i.i.d. integrable random variables. For every \( a \in \mathbb{R} \), the limit
\[
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}\left( S_n \geq na \right) \in [-\infty, 0]
\]
extists. It equals \(-\infty\) if and only \( \mathbb{P}(X_1 \geq a) = 0 \).

Proof. Fix \( a \). Let \( b_n = \log \mathbb{P}(S_n \geq na) \in [-\infty, 0] \), \( n \geq 1 \). Note that for \( n \geq m \geq 1 \),
\[
b_{n+m} \geq \log \mathbb{P}(S_n - S_m \geq (n-m)a, S_m \geq ma),
\]
so, thanks to independence,
\[
b_{n+m} \geq b_n + b_m,
\]
that is the sequence \( (b_n) \) is subadditive and hence \( \lim_{n \to \infty} \frac{b_n}{n} \) exists and it is equal to \( \sup_{m \geq 1} \frac{b_m}{m} \in [-\infty, 0] \), as explained in the next basic lemma. The last part is left as an exercises. \( \square \)
16.2 Lemma. Let \((b_n)_{n \geq 1}\) be a sequence of real numbers which is subadditive, that is for every \(m, n \geq 1\),
\[
b_{m+n} \geq b_m + b_n.
\]
Then \(\lim_{n \to \infty} \frac{b_n}{n}\) exists and equals \(\sup_{n \geq 1} \frac{b_n}{n} \in (-\infty, +\infty]\).

Proof. Let \(a = \sup_{n \geq 1} \frac{b_n}{n}\). Plainly, \(\limsup b_n \leq a\). To show that \(\liminf b_n \geq a\), it is enough to prove that for every \(m\), we have \(\liminf \frac{b_n}{n} \geq b_m (b_0 = 0)\), thus, dividing by \(n = km + r\),
\[
\frac{b_n}{n} \geq \frac{km}{km + r} \frac{b_m}{m} + \frac{b_r}{km + r}.
\]
As \(n \to \infty\), also \(k \to \infty\), so taking \(\liminf\) and using that \(|b_r| \leq \max_{1 \leq j < m} |b_j|\) is bounded gives the desired claim. \(\square\)

16.1 Moment generating functions

For a random variable \(X\), we define its moment generating function \(\psi: \mathbb{R} \to (0, +\infty)\) as
\[
\psi(\lambda) = \mathbb{E} e^{\lambda X}.
\]
Note that
\[
\psi(0) = 1.
\]

16.3 Lemma. Function \(\log \psi\) is convex (in other words, \(\psi\) is log-convex).

Proof. It is evident from the fact that sums of log-convex functions are log-convex (by Hölder’s inequality) and \(\lambda \mapsto e^{\lambda X}\) is log-affine. Alternatively, it can be seen by applying Hölder’s inequality directly (with weights \(1/p = t, 1/q = 1 - t\),
\[
\log \psi(t\lambda_1 + (1 - t)\lambda_2) = \log \mathbb{E} e^{t\lambda_1 X} e^{(1 - t)\lambda_2 X} \leq \log(\mathbb{E} e^{\lambda_1 X})^t (\mathbb{E} e^{\lambda_2 X})^{1 - t} = t \log \psi(\lambda_1) + (1 - t) \log \psi(\lambda_2),
\]
for every \(\lambda_1, \lambda_2 \in \mathbb{R}\) and \(t \in (0, 1)\). \(\square\)

16.4 Corollary. If for some \(\lambda_1 < \lambda_2\), \(\psi(\lambda_1), \psi(\lambda_2) < \infty\), then \(\psi(\lambda) < \infty\) for all \(\lambda \in [\lambda_1, \lambda_2]\).

In view of this corollary, it makes sense to define
\[
\lambda_- = \inf\{\lambda \in \mathbb{R} : \mathbb{E} e^{\lambda X} < \infty\}, \quad \lambda_+ = \sup\{\lambda \in \mathbb{R} : \mathbb{E} e^{\lambda X} < \infty\}
\]
and then \((\lambda_-, \lambda_+)\) is the largest open interval where \(\psi < \infty\). Since \(\psi(0) = 1\), of course
\[
\lambda_- \leq 0 \leq \lambda_+.
\]
16.5 Lemma. Suppose $\psi(\lambda) = \mathbb{E} e^{\lambda X}$ is the moment generating function of a random variable $X$.

(i) If $\psi < \infty$ on $(-\delta, \delta)$ for some $\delta > 0$, then $\mathbb{E}|X|^k < \infty$ and $\psi^{(k)}(0) = \mathbb{E}X^k$, for every integer $k \geq 0$.

(ii) If $\psi < \infty$ on $(\lambda_0 - \delta, \lambda_0 + \delta)$ for some $\lambda_0 \in \mathbb{R}$ and $\delta > 0$, then $\mathbb{E}|X|^k e^{\lambda_0 X} < \infty$ and $\psi^{(k)}(\lambda_0) = \mathbb{E}X^k e^{\lambda_0 X}$, for every integer $k \geq 0$.

Proof. (i): Note that $e^{x} \leq e^x + e^{-x}$, $x \in \mathbb{R}$. In particular, for $-\delta < h < \delta$, $k \geq 0$,
\[
\frac{|h|^k |X|^k}{k!} \leq e^{|hX|} \leq e^{-hX} + e^{hX}
\]
and the right hand side is integrable because $\psi(-h), \psi(h) < \infty$. Thus $\mathbb{E}|X|^k < \infty$.
Moreover,
\[
\sum_{k=0}^{\infty} \frac{(hX)^k}{k!} \leq \sum_{k=0}^{\infty} \frac{|h|^k |X|^k}{k!} = e^{|hX|},
\]
so $\sum_{k=0}^{\infty} \frac{(hX)^k}{k!}$ is integrable and by Fubini’s theorem,
\[
\psi(h) = \mathbb{E} \sum_{k=0}^{\infty} \frac{(hX)^k}{k!} = \sum_{k=0}^{\infty} h^k \mathbb{E}X^k / k!.
\]
Consequently, $\psi$ is $C^\infty$ on $(-\delta, \delta)$ (as a convergent power series) and $\psi^{(k)}(0) = \mathbb{E}X^k$.

(ii): We shall deduce this part from (i) using the so-called “exponential tilting” of measure, one of the key ideas of large deviations. Since $\psi(\lambda_0) < \infty$, we can define a new random variable $Y$ which is absolutely continuous with respect to $X$ with density $\frac{e^{\lambda_0 X}}{\psi(\lambda_0)}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, that is
\[
\mathbb{P}(Y \in A) = \mathbb{E} \frac{e^{\lambda_0 X}}{\psi(\lambda_0)} 1_{X \in A},
\]
or, equivalently,
\[
d\mu_Y(x) = \frac{e^{\lambda_0 x}}{\psi(\lambda_0)} d\mu_X(x).
\]
Then,
\[
\mathbb{E}f(Y) = \mathbb{E}f(X) \frac{e^{\lambda_0 X}}{\psi(\lambda_0)},
\]
for every Borel function $f$ (for which the right hand side exists). In particular, for the moment generating function $\psi_Y$ of $Y$, we get
\[
\psi_Y(\lambda) = \mathbb{E} e^{\lambda Y} = \mathbb{E} \frac{e^{\lambda_0 X}}{\psi(\lambda_0)} e^{\lambda X} = \frac{\psi(\lambda + \lambda_0)}{\psi(\lambda_0)},
\]
which is finite on $(-\delta, \delta)$. Applying (i) to $Y$, we thus get
\[
\mathbb{E}|Y|^k = \mathbb{E}|X|^k e^{\lambda_0 X} / \psi(\lambda_0) < \infty
\]
and
\[
\frac{\psi^{(k)}(\lambda_0)}{\psi(\lambda_0)} = \frac{\psi^{(k)}(0)}{\psi(\lambda_0)} = \mathbb{E}Y^k = \mathbb{E}X^k \frac{e^{\lambda_0 X}}{\psi(\lambda_0)},
\]
which gives $\psi^{(k)}(\lambda_0) = \mathbb{E}X^k e^{\lambda_0 X}$, as desired.\qed
16.6 Corollary. Function $\psi$ is $C^\infty$ on $(\lambda_-, \lambda_+)$.  

When $\lambda_- = \lambda_+ = 0$, we will not be able to establish exponential rates of convergence to 0 for probabilities $P \left( \frac{S_n}{n} > \mu + \varepsilon \right)$ (for a reason – the rates are slower – see exercises). We shall thus make the minimal assumption that at least one of $\lambda_-, \lambda_+$ is nonzero. Since we can always consider $-X$ instead of $X$, we shall focus on the case when

$$\lambda_+ = \sup\{ \lambda \in \mathbb{R} : E e^{\lambda X} < \infty \} > 0.$$  

(16.2)

As it will turn out, this assumption allows to control upper tails: $P(S_n > \mu + \varepsilon)$ for a reason – the rates are slower – see exercises.

16.7 Remark. If (16.2) holds, then $E X_+^k < \infty$, for every $k \geq 1$. In particular,

$E X \in [-\infty, +\infty)$.

Proof. Fix $0 < \lambda_0 < \lambda_+$. Since $\left(\frac{\lambda_0 X}{k!}\right)^k \leq e^{\lambda_0 X_+}$, we have $E X_+^k < \infty$.  

16.8 Lemma. Suppose $\psi$ is the moment generating function of a random variable $X$ and (16.2) holds. Then

(i) $\psi$ is continuous at 0,

(ii) $\lim_{\lambda \to 0^+} \psi^{(k)}(\lambda) = E X^k$, for every $k \geq 1$.

Proof. (i): We need to show that $\lim_{\lambda \to 0^+} \psi(\lambda) = \psi(0)$, that is $\lim_{\lambda \to 0^+} E e^{\lambda X} = 1$, which will of course follow if we can change the order of taking the limit and expectation. Fix $0 < \lambda_0 < \lambda_+$ and note that

$$e^{\lambda X} \leq 1 + e^{\lambda_0 X}, \quad 0 < \lambda < \lambda_0$$

(this holds because if $X < 0$, then $e^{\lambda X} \leq 1$ and if $X \geq 0$, then $e^{\lambda X} \leq e^{\lambda_0 X}$, by monotonicity). Lebesgue’s dominated convergence theorem finishes the argument.

(ii): Following the same argument, we want to dominate $X^k e^{\lambda X}$ for all $0 < \lambda < \lambda_0$ by an integrable random variable. We write $X^k = X_+^k - X_-^k$. For small enough $\varepsilon > 0$,

$$\frac{e^{kX_+}}{k!} \leq e^{\varepsilon X_+} \leq 1 + e^{\varepsilon X},$$

so that $X_+^k e^{\lambda X}$ is dominated by $(1 + e^{\varepsilon X})(1 + e^{\lambda_0 X})$ which is integrable provided that $\lambda_0 + \varepsilon < \lambda_+$. Hence,

$$\lim_{\lambda \to 0^+} E X_+^k e^{\lambda X} = E X_+^k.$$  

181
Now we analyse $X_k e^{\lambda X}$. If $E X_k = +\infty$, Fatou’s lemma gives

$$\liminf_{\lambda \to 0^+} E X_k e^{\lambda X} \geq \liminf_{\lambda \to 0^+} X_k e^{\lambda X} = E X_k = +\infty.$$ 

As a result, in this case,

$$\lim_{\lambda \to 0^+} \psi^{(k)}(\lambda) = \lim_{\lambda \to 0^+} E X_k e^{\lambda X} = -\infty = E X_k.$$ 

If $E X_k < \infty$, then since $X_k e^{\lambda X} \leq X_k e^{\lambda_0 X} \leq \frac{k!}{\varepsilon^k} e^{(X_+ + (\lambda_0 - \varepsilon)X)} = \frac{k!}{\varepsilon^k} e^{X_+ + (\lambda_0 - \varepsilon)X}$ and $e^{\varepsilon X} \leq 1 + e^{\varepsilon X}$, we can conclude by Lebesgue’s dominated convergence theorem. 

16.9 Remark. A similar argument shows that $\psi(\lambda) \to \psi(\lambda_+)$ as $\lambda \to \lambda_+$: we write $e^{\lambda X} = e^{\lambda X_+} 1_{\{X \geq 0\}} + e^{\lambda X_-} 1_{\{X < 0\}}$ and use Lebesgue’s monotone convergence theorem for the first term and Lebesgue’s dominated convergence theorem for the second one.

We close this section with a relationship between moment generating functions and tilted measures.

16.10 Lemma. Let $X$ be a random variable with moment generating function $\psi$ and let $\lambda_\pm$ be given by (16.1). For $\lambda \in (\lambda_-, \lambda_+)$, let $\mu_\lambda$ be the probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\mu_\lambda(A) = \int_A \frac{e^{\lambda x}}{\psi(\lambda)} d\mu_X(x), \quad A \in \mathcal{B}(\mathbb{R}),$$

where $\mu_X$ is the distribution of $X$. Let $Y_\lambda$ be a random variable with distribution $\mu_\lambda$. Then

$$(\log \psi)'(\lambda) = E Y_\lambda,$$

$$(\log \psi)''(\lambda) = \text{Var}(Y_\lambda).$$

Proof. By the definition of $Y_\lambda$,

$$E f(Y_\lambda) = E f(X) \frac{e^{\lambda X}}{\psi(\lambda)},$$

for every measurable function $f$ for which the right hand side exists. Thus, by Lemma 16.5,

$$(\log \psi)'(\lambda) = \frac{\psi'(\lambda)}{\psi(\lambda)} = \frac{E X e^{\lambda X}}{\psi(\lambda)} = E Y_\lambda$$

and

$$(\log \psi)''(\lambda) = \frac{\psi''(\lambda)}{\psi(\lambda)} - \left( \frac{\psi'(\lambda)}{\psi(\lambda)} \right)^2 = \frac{E X^2 e^{\lambda X}}{\psi(\lambda)} - (E Y_\lambda)^2 = \text{Var}(Y_\lambda).$$
16.2 Upper bounds: Chernoff’s inequality

16.11 Lemma (Chernoff’s bound). Let $X$ be a random variable with moment generating function $\psi$ and let $a \in \mathbb{R}$. Then

$$P(X \geq a) \leq e^{-(a \lambda - \log \psi(\lambda))}, \quad \lambda \geq 0.$$  

Proof. The assertion follows from exponential Chebyshev’s inequality, since for $\lambda > 0$,

$$P(X \geq a) = P(e^{\lambda X} \geq e^{\lambda a}) \leq e^{-\lambda a} E e^{\lambda X} = e^{-(\lambda a - \log \psi(\lambda))}.$$  

16.12 Corollary. Let $X_1, X_2, \ldots$ be i.i.d. random variables with moment generating function $\psi$, $S_n = X_1 + \ldots + X_n$. For $a \in \mathbb{R}$,

$$P(S_n \geq an) \leq \exp\left\{-n \sup_{\lambda > 0} \{\lambda a - \log \psi(\lambda)\}\right\}.$$  

If (16.2) holds, then for every $a > \mu = \mathbb{E}X_1 \in [-\infty, +\infty)$, we have

$$\sup_{\lambda > 0} \{\lambda a - \log \psi(\lambda)\} > 0,$$

that is the above upper bound is meaningful (and is exponentially small in $n$).

Proof. The upper bound on $P(S_n \geq an)$ follows from Chernoff’s bound applied to $X = S_n$. Independence yields, $\psi_{S_n}(\lambda) = (\psi(\lambda))^n$ and the supremum appears because the bound holds for all $\lambda > 0$.

If (16.2) holds, then by the intermediate value theorem applied to $\log \psi$ on $(0, \lambda)$, we have

$$a \lambda - \log \psi(\lambda) = a \lambda - (\log \psi(\lambda) - \log \psi(0)) = \lambda \cdot [a - (\log \psi)'(\theta)],$$

for some $\theta \in (0, \lambda)$. Since $(\log \psi)'(\theta) = \frac{\psi'(\theta)}{\psi(\theta)} \to \mathbb{E}X = \mu$ as $\lambda \to 0$ (Lemma 16.8), for small $\lambda$, the expression in the square bracket is close to $a - \mu > 0$, which shows that the supremum $\sup_{\lambda > 0} \{\lambda a - \log \psi(\lambda)\}$ is positive, as desired.

The above upper bound motivates the following definition: the rate function $I: \mathbb{R} \to [0, +\infty]$ of a random variable $X$ with moment generating function $\psi$ is defined as

$$I(a) = \sup_{\lambda \in \mathbb{R}} \{\lambda a - \log \psi(\lambda)\}, \quad a \in \mathbb{R}.$$  

(It is the Legendre transform of the log-moment generating function $\log \psi$.) As a pointwise supremum of linear functions, $I$ is a convex function.

16.13 Example. For the common distributions, in some cases, the rate function can be written down explicitly (see exercises).
1) Standard Gaussian distribution, \( X \sim N(0, 1) \). We have
\[
\psi(\lambda) = E e^{\lambda X} = e^{\lambda^2/2}
\]
Then, given \( a \in \mathbb{R} \), \( \lambda a - \log \psi(\lambda) \) is maximised over \( \lambda \in \mathbb{R} \) when
\[
a = (\log \psi)'(\lambda) = \lambda,
\]
which gives
\[
I(a) = a^2 - \log \psi(a) = \frac{a^2}{2}.
\]
2) Standard exponential distribution, \( X \sim \text{Exp}(1) \). We have
\[
\psi(\lambda) = \begin{cases} 
1 - \frac{1}{\lambda}, & \lambda < 1 \\
+\infty, & \lambda \geq 1
\end{cases}
\]
and
\[
I(a) = \begin{cases} 
\log a, & a > 0 \\
\infty, & a \leq 0
\end{cases}
\]
3) Bernoulli distribution, \( X \sim \text{Ber}(p) \). We have
\[
\psi(\lambda) = 1 - p + pe^{\lambda}
\]
and
\[
I(a) = \begin{cases} 
\log \frac{a}{p} + (1 - a) \frac{1}{1-p}, & 0 < a < 1 \\
- \log(1 - p), & a = 0 \\
- \log p, & a = 1 \\
\infty, & a < 0 \text{ or } a > 1.
\end{cases}
\]
4) Poisson distribution, \( X \sim \text{Poiss}(\mu) \). We have
\[
\psi(\lambda) = \exp\{-\mu + \mu e^{\lambda}\}
\]
and
\[
I(a) = \begin{cases} 
\infty, & a < 0 \\
\mu, & a = 0 \\
\mu \frac{a}{e^\mu} + \mu, & a > 0.
\end{cases}
\]
16.14 Example. Let \( X \) be a random variable with density \( g(x) = Cx^{-3}e^{-x}1_{[1, +\infty)}(x) \), where \( C \) is a normalising constant. We have
\[
\psi(\lambda) = C \int_1^\infty x^{-3}e^{\lambda x - x}dx < \infty
\]
if and only if $\lambda \leq 1$, so $\lambda - 1 = -\infty$, $\lambda + 1 = 1$. Note that as $\lambda \to 1^-$,

$$\frac{\psi'(\lambda)}{\psi(\lambda)} \sim \frac{\psi'(1)}{\psi(1)} = \frac{\int_1^{\infty} x^{-2}dx}{\int_1^{\infty} x^{-3}dx} = 2.$$ 

Thus, the supremum in the definition of $I(a)$ is attained if and only if $a \leq 2$, the case when the equation $a = \frac{\psi'(\lambda)}{\psi(\lambda)}$ has a solution.

### 16.3 Cramér’s theorem

**16.15 Theorem.** Let $X_1, X_2, \ldots$ be i.i.d. random variables with moment generating function $\psi$ satisfying (16.2) and rate function $I$. Let $\mu = \mathbb{E}X_1$ and $S_n = X_1 + \ldots + X_n$, $n \geq 1$. For every $a > \mu$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(S_n \geq na) = -I(a).$$

**Proof.** We begin with two remarks which will help us understand better the claimed value of the limit $-I(a)$. By its definition, the rate function $I(a)$ is $\sup_{\lambda \in \mathbb{R}} f(\lambda)$ with

$$f(\lambda) = \lambda a - \log \mathbb{E}e^{\lambda X_1}.$$

First, we remark that under our assumptions, in fact we have

$$I(a) = \sup_{0 < \lambda < \lambda_+} f(\lambda). \quad (16.3)$$

Plainly $\lambda > \lambda_+$ results in $f(\lambda) = -\infty$, so those $\lambda$ do not count in the supremum. Moreover, note that by Jensen’s inequality,

$$f(\lambda) = \lambda a - \log \mathbb{E}e^{\lambda X_1} \leq \lambda a - \log e^{\lambda \mathbb{E}X_1} = \lambda(a - \mu).$$

For $a > \mu$ and $\lambda \leq 0$, the above is thus nonpositive. If (16.2) holds, we know that the supremum over $\lambda > 0$ is positive (Corollary 16.12). Thus $\lambda \leq 0$ can also be neglected in the supremum defining $I(a)$.

Second, thanks to Lemma 16.10,

$$f''(\lambda) = -(\log \psi)'(\lambda) = -\text{Var}(Y_\lambda)$$

which is strictly negative (unless $Y_\lambda$, equivalently $X$ is a point mass, in which case there is nothing to do). Thus, $f'$ is strictly increasing and $f$ is strictly concave on $(0, \lambda_+)$. In particular, if $f$ attains its supremum, it is unique, attained at $\lambda = \lambda_a \in (0, \lambda_+)$ which is a unique solution of the equation

$$f'(\lambda) = 0,$$

that is $a = (\log \psi)'(\lambda) = \frac{\psi'(\lambda)}{\psi(\lambda)}$.

We now break the proof into two parts.
**Upper bound.** By Chernoff’s bound,
\[
\limsup_n \frac{1}{n} \log P(S_n \geq na) \leq -\sup_{\lambda>0} \{\lambda a - \log \psi(\lambda)\} = -I(a),
\]
where the last equality is justified by (16.3).

**Lower bound.** Fix \(a > \mu\). It remains to show that
\[
\liminf_n \frac{1}{n} \log P(S_n \geq na) \geq -I(a) \tag{16.4}
\]
with the right hand side given by (16.3).

**Claim.** If there is \(\lambda \in (0, \lambda_+)\) such that \(a = \frac{\psi'(\lambda a)}{\psi(\lambda a)}\), then lower bound (16.4) holds.

To prove the claim, fix \(\lambda \in (0, \lambda_+)\). Let \(Y_1, \ldots, Y_n\) be i.i.d. copies of the tilted random variable \(Y_\lambda\) from Lemma 16.10. Note that then, thanks to independence, the vector \((Y_1, \ldots, Y_n)\) has density \(e^{\lambda (X_1 + \cdots + X_n)} / \psi(\lambda)^n\) with respect to \((X_1, \ldots, X_n)\) and we have
\[
\mathbb{E} f(Y_1, \ldots, Y_n) = \mathbb{E} f(X_1, \ldots, X_n) \cdot \frac{e^{\lambda S_n}}{\psi(\lambda)^n},
\]
for every measurable function \(f : \mathbb{R}^n \to \mathbb{R}\) for which the right hand side exists. For \(a' > a\), we thus have
\[
P(S_n \geq na) \geq P(S_n \in (na, na')) \geq \mathbb{E} e^{\lambda S_n} e^{-\lambda na'} 1_{\{S_n \in (na, na')\}}
\]
\[
= \psi(\lambda)^n e^{-\lambda na'} \mathbb{E} 1_{\{Y_1 + \cdots + Y_n \in (na, na')\}}
\]
\[
= \psi(\lambda)^n e^{-\lambda na'} P(Y_1 + \cdots + Y_n \in (na, na')).
\]

By the weak law of large numbers,
\[
\frac{Y_1 + \cdots + Y_n - n \mathbb{E} Y_1}{n} \xrightarrow{p} 0,
\]
so as long as \(a' > \mathbb{E} Y_1 > a\), we get
\[
P(Y_1 + \cdots + Y_n \in (na, na'))
\]
\[
= P \left( \frac{Y_1 + \cdots + Y_n - n \mathbb{E} Y_1}{n} \in (a - \mathbb{E} Y_1, a' - \mathbb{E} Y_1) \right) \xrightarrow{n \to \infty} 1
\]
Since \(\mathbb{E} Y_1 = \frac{\psi'(\lambda)}{\psi(\lambda)} > \frac{\psi'(\lambda a)}{\psi(\lambda a)} = a\), given \(\lambda > \lambda_a\), we thus fix \(a'\) such that \(a' > \frac{\psi'(\lambda)}{\psi(\lambda)}\) and get
\[
\liminf_{n \to \infty} \frac{1}{n} \log P(S_n \geq na) \geq -(\lambda a' - \log \psi(\lambda)).
\]
Letting \(\lambda \searrow \lambda_a\) and then \(a' \searrow a\), we get
\[
\liminf_{n \to \infty} \frac{1}{n} \log P(S_n \geq na) \geq -(\lambda_a - \log \psi(\lambda_a)) = -I(a),
\]
as desired. This finishes the proof of the claim.
Depending on the behaviour of $\psi$, the claim may or may not be applicable. To decide when it is, we consider

$$A = \text{esssup } X \in (-\infty, +\infty].$$

**Case 1:** $A < +\infty$. Then $\lambda_+ = +\infty$. First we claim that

$$\frac{\psi'(\lambda)}{\psi(\lambda)} = \frac{EXe^{\lambda X}}{Ee^{\lambda X}} \xrightarrow{\lambda \to \infty} A. \quad (16.5)$$

Indeed, since $X \leq A$, clearly $EXe^{\lambda X} \leq A$. On the other hand, for $\varepsilon > 0$, we have

$$EXe^{\lambda X} \geq (A - \varepsilon)EXe^{(A - \varepsilon)}1_{\{X > A - \varepsilon\}} + EXe^{\lambda X}1_{\{X \leq A - \varepsilon\}}$$

$$= (A - \varepsilon)EXe^{\lambda X} + EX(X - (A - \varepsilon))e^{\lambda X}1_{\{X \leq A - \varepsilon\}},$$

since $|ye^{\lambda e^{\lambda X}}| \leq e^{-1}$ and

$$\frac{e^{\lambda X}}{EXe^{\lambda X}} \leq \frac{e^{\lambda X}}{EXe^{\lambda X}1_{\{X > A - \varepsilon/2\}}} \leq P(X > A - \varepsilon/2),$$

we obtain

$$EXe^{\lambda X} \geq A - \varepsilon - \frac{1}{\lambda} e^{-\lambda \varepsilon/2} \frac{1}{P(X > A - \varepsilon/2)},$$

thus

$$\liminf_{\lambda \to \infty} \frac{EXe^{\lambda X}}{EXe^{\lambda X}} \geq A - \varepsilon.$$ 

Consequently, (16.5) holds.

In view of (16.5), if $a < A$, then there is $\lambda_+ = a$ with $\psi'(\lambda) = \psi(\lambda_+)$ and the claim finishes the proof in this case. If $a > A$, then trivially $P(S_n \geq an) = 0$ for every $n$, so it remains to argue that $I(a) = +\infty$. This holds because $f'(\lambda) > a - A > 0$ for every $\lambda > 0$ (as $f'$ is strictly decreasing), so $f(\lambda) \to \infty$ as $\lambda \to \infty$. Finally, if $a = A$, then $P(S_n \geq an) = P(X_1 = A)^n$, so it remains to argue that $I(A) = -\log P(X_1 = A)$. On one hand, for every $\lambda > 0$, $\psi(\lambda) = EXe^{\lambda X_1} \geq e^{\lambda A}P(X_1 = A)$, so

$$I(A) = \sup_{\lambda > 0} \{\lambda A - \log \psi(\lambda)\} \leq -\log P(X_1 = A).$$

On the other hand, this upper bound is attained in the limit as $\lambda \to \infty$ because by Lebesgue’s dominated convergence theorem,

$$\lambda A - \psi(\lambda) = -\log EXe^{\lambda X - A} \xrightarrow{\lambda \to \infty} -\log P(X_1 = A),$$

which finishes the whole argument in this case.

**Case 2:** $A = +\infty$. If $\lambda_+ = \infty$, then the proof of the lower bound in (16.5) shows that

$$\frac{\psi'(\lambda)}{\psi(\lambda)} \to \infty \quad \text{as } \lambda \to \infty,$$
so regardless of the value of $a$, the claim is applicable. Suppose now that $\lambda_+ < \infty$. Let

$$\alpha = \lim_{\lambda \to \lambda_+} \frac{\psi'(\lambda)}{\psi(\lambda)}.$$ 

Note that then, thanks to the monotonicity of $(\log \psi)' = \frac{\psi'}{\psi}$,

$$\log \psi(\lambda_+) = \int_0^{\lambda_+} (\log \psi)' \leq \lambda_+ \alpha < \infty$$

Writing $e^{\lambda X} = \epsilon_\lambda X 1_{X > 0} + e^{\lambda X} 1_{X \leq 0}$, by Lebesgue’s monotone and dominated convergence theorems we see that in fact

$$\psi(\lambda) = E e^{\lambda X} \to E e^{\lambda_+ X} = \psi(\lambda_+)$$

and

$$\psi'(\lambda) = E X e^{\lambda X} \to E X e^{\lambda_+ X} = \psi'(\lambda_+)$$

as $\lambda \to \lambda_+$. It remains to consider the case when $\alpha < \infty$ and $a \geq \alpha$ (otherwise, again, the claim is applicable). We have,

$$I(a) = a \lambda_+ - \log \psi(\lambda_+), \quad a \geq \alpha,$$

($I(a)$ is linear). Indeed, $f'(\lambda) = a - (\log \psi)'(\lambda) > a - \alpha \geq 0$, for every $\lambda < \lambda_+$, so $f$ is strictly increasing, hence $I(a) = \sup_{0 < \lambda < \lambda_+} f(\lambda) = f(\lambda_+) = a \lambda_+ - \log \psi(\lambda_+)$. We fix $a \geq \alpha$ and our goal is to show that

$$\liminf_{n \to \infty} \frac{1}{n} \log P(S_n \geq an) \geq -(a \lambda_+ - \log \psi(\lambda_+)).$$

Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with the law given by the tilted measure $\mu_\lambda$ from Corollary 16.6 with $\lambda = \lambda_+$, so that $E Y_1 = \alpha$. We proceed as in the proof of the claim: for $a' > a$, we have

$$P(S_n \geq an) \geq \psi(\lambda_+) \alpha e^{-n \lambda_+} P \left( \sum_{k=1}^{n} Y_k \in (an, an') \right).$$

Using independence, for $\varepsilon > 0$,

$$P \left( \sum_{k=1}^{n} Y_k \in (an, an') \right) \geq P \left( \sum_{k=1}^{n-1} Y_k \in ((a - \varepsilon)n, (a + \varepsilon)n) \right) P(Y_n \in (a - \alpha + \varepsilon)n, (a' - \alpha - \varepsilon)n)).$$

By the weak law of large numbers, the first term is at least, say $\frac{1}{2}$ for large $n$. Choosing $a' = a + 3\varepsilon$ and using that $(a' - \alpha - \varepsilon)n = (a - \alpha + 2\varepsilon)n > (a - \alpha + \varepsilon)(n + 1)$ for large $n$, we thus get

$$\frac{1}{n} \log P(S_n \geq an) \geq -((a + 3\varepsilon)\lambda_+ - \log \psi(\lambda_+))$$

$$+ \frac{1}{n} \log \frac{1}{2} + \frac{1}{n} \log P(Y_1 \in (a - \alpha + \varepsilon)n, (a - \alpha + \varepsilon)(n + 1)).$$
for large \( n \). Finally, if we had

\[
\limsup \frac{1}{n} \log P(Y_1 \in (a - \alpha + \varepsilon)n, (a - \alpha + \varepsilon)(n + 1)) < 0,
\]

then for some small enough \( \delta > 0 \), the series

\[
\sum_{n=1}^{\infty} e^{\delta(a - \alpha + \varepsilon)(n + 1)} P(Y_1 \in (a - \alpha + \varepsilon)n, (a - \alpha + \varepsilon)(n + 1))
\]

would converge, so

\[
E e^{\delta Y_1} < \infty
\]

and, equivalently, \( E e^{(\delta + \lambda_+)X} < \infty \) contradicting the definition of \( \lambda_+ \). Therefore the \( \limsup \) above is zero and thus

\[
\limsup \frac{1}{n} \log P(S_n \geq an) \geq -(a + 3\varepsilon)\lambda_+ - \log \psi(\lambda_+)
\]

for every \( \varepsilon > 0 \). By virtue of Lemma 16.1, this \( \limsup \) is the same as \( \liminf \) because the limit in fact exists, so we get the desired lower bound after letting \( \varepsilon \to 0 \).

16.4 Quantitative bounds

A careful inspection of the exponential tilting argument used in the lower bound of Cramér’s reveals that if instead of crudely applying the weak law of large numbers, we use a quantitative central limit theorem-type bound, we get a quantitative large deviation-type bound. For simplicity, we only consider the case when the supremum defining the rate function is attained.

16.16 Theorem. Let \( X_1, X_2, \ldots \) be i.i.d. random variables with moment generating function \( \psi \) such that \( \psi < \infty \) on \( (-\delta, \delta) \) for some \( \delta > 0 \). Let \( I \) be the rate function of \( X_1 \). Let \( \mu = EX_1 \). If \( a > \mu \) and

\[
a = \frac{\psi'(\lambda)}{\psi(\lambda)},
\]

has a solution, say \( \lambda = \lambda_a \), then

\[
P(S_n \geq an) = e^{-nI(a)} \left( e^{\sigma^2 \lambda_a^2 n/2} \int_{\sigma \lambda_a \sqrt{n}}^{\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}} + O \left( \frac{\rho}{\sqrt{n}} \right) \right),
\]

where

\[
\sigma = \sqrt{(\log \psi)''(\lambda_a)}
\]

and

\[
\rho = \frac{1}{\sigma^2 \psi'(\lambda_a)} \mathbb{E} |X_1 - a|^3 e^{\lambda_a X_1}.
\]

Proof. We fix \( \lambda = \lambda_a \) to be the unique solution to the equation \( a = \frac{\psi'(\lambda)}{\psi(\lambda)} \), that is the value of \( \lambda \) attaining the supremum in the definition of \( I(a) \). We let \( Y_1, \ldots, Y_n \) be i.i.d.
copies of a random variable with the tilted law \( \mu \) from Lemma 16.10. We have \( \mathbb{E}Y_1 = a \) and \( \text{Var}(Y_1) = \sigma^2 \). Moreover,

\[
\mathbb{P}(S_n \geq an) = \psi(\lambda)'^n \mathbb{E} \left[ \mathbb{1}_{\{\sum_{k=1}^n Y_k \geq an\}} e^{-\lambda \sum_{k=1}^n Y_k} \right].
\]

Denote \( V = \sum_{k=1}^n Y_k \). Observe that

\[
\mathbb{E} \mathbb{1}_{\{V \geq an\}} e^{-\lambda V} = \mathbb{E} \left[ \mathbb{1}_{\{V \geq an\}} \int_V^\infty \lambda e^{-\lambda x} \, dx \right] = \int_{an}^\infty \lambda e^{-\lambda x} \mathbb{P}(an \leq V \leq x) \, dx.
\]

A change of variables \( x = na + \sigma \sqrt{n} y \) yields

\[
\mathbb{E} \mathbb{1}_{\{V \geq an\}} e^{-\lambda V} = e^{-na\lambda} \int_0^\infty \sigma \sqrt{n} \lambda e^{-\sigma \sqrt{n} \lambda y} \mathbb{P} \left( 0 \leq \frac{V - na}{\sigma \sqrt{n}} \leq y \right) \, dy.
\]

By the Berry-Esseen theorem (Theorem 11.1) applied to the \( Y_k \),

\[
\mathbb{P} \left( 0 \leq \frac{V - na}{\sigma \sqrt{n}} \leq y \right) = \int_0^y e^{-u^2/2} \frac{du}{\sqrt{2\pi}} + O \left( \frac{\rho}{\sqrt{n}} \right),
\]

where the error term is quantified by \( \rho = \mathbb{E} \left| \frac{Y_1 - a}{\sigma} \right|^3 \leq \frac{1}{\sigma^3} \mathbb{E} |Y_1 - a|^3 e^{\lambda Y_1} \), as stated. Since \( \int_0^\infty \sigma \sqrt{n} \lambda e^{-\sigma \sqrt{n} \lambda y} \, dy = 1 \), by putting these together, we obtain

\[
\mathbb{P}(S_n \geq an) = e^{-n(a\lambda - \log \psi(\lambda))} \left( \int_0^\infty \sigma \sqrt{n} \lambda e^{-\sigma \sqrt{n} \lambda y} \left( \int_0^y e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right) \, dy + O \left( \frac{\rho}{\sqrt{n}} \right) \right).
\]

By the choice of \( \lambda \), \( a\lambda - \log \psi(\lambda) = I(a) \). By Fubini’s theorem,

\[
\int_0^\infty \sigma \sqrt{n} \lambda e^{-\sigma \sqrt{n} \lambda y} \left( \int_0^y e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \right) \, dy = \int_0^\infty e^{-u^2/2} \left( \int_u^\infty \sigma \sqrt{n} \lambda e^{-\sigma \sqrt{n} \lambda y} \, dy \right) \frac{du}{\sqrt{2\pi}}
\]

\[
= \int_0^\infty e^{-\sigma \sqrt{n} \lambda u - u^2/2} \frac{du}{\sqrt{2\pi}}
\]

\[
= e^{(\sigma \sqrt{n} \lambda)^2/2} \int_0^\infty e^{-(u+\sigma \sqrt{n} \lambda)^2/2} \frac{du}{\sqrt{2\pi}}
\]

\[
= e^{\sigma^2 \lambda^2 n/2} \int_{\sigma \lambda n}^\infty e^{-u^2/2} \frac{du}{\sqrt{2\pi}},
\]

which finishes the proof. \( \square \)

**16.17 Remark.** By the asymptotics of Gaussian tail (Lemma 11.4 and the claim from the proof of Theorem 15.7),

\[
e^{\sigma^2 \lambda^2 n/2} \int_{\sigma \lambda n}^\infty e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \approx \frac{1}{\sqrt{2\pi \sigma^2 \lambda^2 n}}
\]

for large \( n \), thus, in fact this is of the same order as the error term \( O(\frac{\rho}{\sqrt{n}}) \). Consequently, under the assumptions of Theorem 16.16, we obtain

\[
\frac{1}{n} \log \mathbb{P}(S_n \geq an) = -I(a) + O(n^{-1} \log n^{-1})
\]

as \( n \to \infty \) (the implicit constants depending on \( a \) and the distribution of \( X_1 \)). This can be viewed as a quantitative version of Cramér’s theorem under these special assumptions.
The next result shows that for sums of i.i.d. random variables, with probabilities exponentially close to 1 (in $n$), $S_k^k$ is $\varepsilon$-close to its mean for all $k \geq n$. It relies on maximal inequalities combined with the basic idea used in Chernoff’s bounds. It can be thought of as a quantitative version of the strong law of large numbers (cf. Exercises 15.3 and 15.4).

16.18 Theorem. Let $X_1, X_2, \ldots$ be i.i.d. random variables with moment generating function $\psi$ such that $\psi < \infty$ on $(-\delta, \delta)$ for some $\delta > 0$. Let $I$ be the rate function of $X_1$. Let $\mu = \mathbb{E}X_1$. Then for every $\varepsilon > 0$, we have

$$
P \left( \sup_{k \geq n} \left| \frac{S_k}{k} - \mu \right| > \varepsilon \right) \leq 2e^{-n \min\{I(\mu - \varepsilon), I(\mu + \varepsilon)\}}.
$$

Proof. Fix $\lambda > 0$ and $a$ such that $\lambda a - \log \psi(\lambda) \geq 0$. Observe that

$$
P \left( \sup_{k \geq n} \left| \frac{S_k}{k} - \mu \right| > \varepsilon \right) = P \left( \exists k \geq n : S_k > ak \right)
$$

$$
= P \left( \exists k \geq n : e^{\lambda S_k - k \log \psi(\lambda)} > e^{k(\lambda a - \log \psi(\lambda))} \right)
$$

$$
\leq P \left( \exists k \geq n : e^{\lambda S_k - k \log \psi(\lambda)} > e^{a(\lambda a - \log \psi(\lambda))} \right)
$$

$$
= P \left( \sup_{k \geq n} e^{\lambda S_k - k \log \psi(\lambda)} > e^{n(\lambda a - \log \psi(\lambda))} \right).
$$

Since $(e^{\lambda S_k - k \log \psi(\lambda)})_{k \geq n}$ is a martingale (as the product of independent random variables with mean 1), by Doob’s maximal inequality (14.5),

$$
P \left( \sup_{k \geq n} \frac{S_k}{k} > a \right) \leq e^{-n(\lambda a - \log \psi(\lambda))}.
$$

Using this with $a = \mu + \varepsilon$ and optimising over $\lambda > 0$ with $a\lambda - \log \psi(\lambda) > 0$ (recall (16.3)) gives

$$
P \left( \sup_{k \geq n} \frac{S_k}{k} > \mu + \varepsilon \right) \leq e^{-nI(\mu + \varepsilon)}.
$$

To finish the proof, note that

$$
\left\{ \sup_{k \geq n} \left| \frac{S_k}{k} - \mu \right| > \varepsilon \right\} = \left\{ \sup_{k \geq n} \left( \frac{S_k}{k} - \mu \right) > \varepsilon \right\} \cup \left\{ \sup_{k \geq n} \left( \mu - \frac{S_k}{k} \right) > \varepsilon \right\}
$$

so it remains to apply the above inequality to $-X_1, -X_2, \ldots$, to get

$$
P \left( \sup_{k \geq n} \left| \frac{S_k}{k} - \mu \right| > \varepsilon \right) \leq e^{-nI(\mu + \varepsilon)} + e^{-nI(\mu - \varepsilon)} \leq 2e^{-n \min\{I(\mu - \varepsilon), I(\mu + \varepsilon)\}}.
$$

\[\square\]
16.5 Exercises

1. Under the hypothesis of Lemma 16.1, we have that \( \lim \frac{1}{n} \log P(S_n \geq an) = 0 \) if and only if \( P(S_n \geq an) = 0 \) for all \( n \), if and only if \( P(X_1 \geq a) = 0 \).

2. Find the rate functions claimed in Example 16.13.
A Appendix: Carathéodory’s theorem

Our goal here is to give a proof of Carathéodory’s theorem about extensions of measures.

A.1 Theorem (Carathéodory). Let $\Omega$ be a set and let $\mathcal{A}$ be an algebra on $\Omega$. Suppose a function $P: \mathcal{A} \rightarrow [0, +\infty)$ satisfies

(i) $P(\Omega) = 1$,

(ii) $P$ is finitely additive, that is for every $A_1, \ldots, A_n \in \mathcal{A}$ which are pairwise disjoint, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i),$$

(iii) for every $A_1, A_2, \ldots \in \mathcal{A}$ with $A_1 \subset A_2 \subset \ldots$ such that $A = \bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{A}$, we have

$$\lim_{n \to \infty} P(A_n) = P(A).$$

Then $P$ can be uniquely extended to a probability measure on the $\sigma$-algebra $\mathcal{F} = \sigma(\mathcal{A})$ generated by $\mathcal{A}$.

Proof. We break the proof into 3 steps.

I. We define a nonnegative function $P^*$ on all subsets of $\Omega$ satisfying: $P^*(\Omega) = 1$, $P^*$ is monotone and subadditive (the so-called exterior or outer measure).

II. We define a family of subsets $\mathcal{M}$ of $\Omega$ which is a $\sigma$-algebra and $P^*$ is countably-additive on $\mathcal{M}$.

III. We show that $P^*$ agrees with $P$ on $\mathcal{M}$ and that $\mathcal{M}$ contains $\mathcal{A}$.

We proceed with proving the steps I, II, III. Then we argue about the uniqueness.

I. For a subset $A$ of $\Omega$, we define

$$P^*(A) = \inf \sum_{n} P(A_n),$$

where the infimum is taken over all sets $A_1, A_2, \ldots \in \mathcal{A}$ such that $\bigcup_{n} A_n \supset A$.

Clearly, $P^*$ is nonnegative. Since $\emptyset \in \mathcal{A}$, we have $P^*(\emptyset) = 0$. It is also clear that $P^*$ is monotone, that is if $A \subset B$, then $P^*(A) \leq P^*(B)$. Finally, we show that $P^*$ is subadditive, that is for every sets $A_1, A_2, \ldots$, we have

$$P^*\left(\bigcup_{n} A_n\right) \leq \sum_{n} P^*(A_n).$$

Indeed, by the definition of $P^*$, for $\varepsilon > 0$, there are sets $B_{n,k} \in \mathcal{A}$ such that $A_n \subset \bigcup_{k} B_{n,k}$ and $\sum_{k} P(B_{n,k}) < P^*(A_n) + \varepsilon 2^{-n}$. Then $\bigcup_{n} A_n \subset \bigcup_{n,k} B_{n,k}$ and consequently,

$$P^*\left(\bigcup_{n} A_n\right) \leq \sum_{n,k} P(B_{n,k}) < \sum_{n} P^*(A_n) + \varepsilon.$$
Since $\epsilon > 0$ is arbitrary, we get the desired inequality.

II. We define the following class of subsets of $\Omega$,

$$M = \{ A \subset \Omega, \forall E \subset \Omega \ P^*(A \cap E) + P^*(A^c \cap E) = P^*(E) \}.$$ 

Since $P^*$ is subadditive, $A \in M$ is equivalent to the so-called Carathéodory’s condition: for all $E \subset \Omega$,

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E). \quad (A.1)$$

First we show that $M$ is an algebra and $P^*$ is finitely additive on $M$. Clearly, $\Omega \in M$ and if $A \in M$, then $A^c \in M$. Let $A, B \in M$. Then for an arbitrary subset $E$ of $\Omega$, we have

$$P^*(E) = P^*(B \cap E) + P^*(B^c \cap E)$$

$$= P^*(A \cap B \cap E) + P^*(A^c \cap B \cap E) + P^*(A \cap B^c \cap E) + P^*(A^c \cap B^c \cap E)$$

$$\geq P^*(A \cap B \cap E) + P^* \left( (A^c \cap B \cap E) \cup (A \cap B^c \cap E) \cup (A^c \cap B^c \cap E) \right)$$

$$= P^* \left( (A \cap B) \cap E \right) + P^* \left( (A \cap B)^c \cap E \right).$$

Thus $A \cap B \in M$ and consequently, $M$ is an algebra.

To prove the finite additivity of $P^*$ on $M$, take $A, B \in M$ with $A \cap B = \emptyset$ and note that since $A \in M$, we have

$$P^*(A \cup B) = P^* \left( A \cap (A \cup B) \right) + P^* \left( A^c \cap (A \cup B) \right) = P^*(A) + P^*(B).$$

By induction, we easily get the desired finite additivity.

Now we argue that $P^*$ is in fact countably additive on $M$. If $A_1, A_2, \ldots \in M$ are pairwise disjoint and we let $A = \bigcup_{k=1}^{\infty} A_k$, then

$$\sum_{k=1}^{n} P^*(A_k) = P^* \left( \bigcup_{k=1}^{n} A_k \right)$$

$$= P^* \left( A \cap \bigcup_{k=1}^{n} A_k \right)$$

$$\leq P^*(A)$$

because $P^*$ is monotone (see I.). Taking the limit $n \to \infty$, we get $\sum_{k=1}^{\infty} P^*(A_k) \leq P^*(A)$. By the subadditivity of $P^*$, we also have the reverse inequality, hence we have equality and the countable additivity of $P^*$ follows.

It remains to show that $M$ is a $\sigma$-algebra. It is enough to consider pairwise disjoint sets $A_1, A_2, \ldots \in M$ and argue that $A = \bigcup_{n=1}^{\infty} A_n \in M$ (if they are not disjoint, we consider $B_n = A_n \cap A_{n+1}^c \cap \ldots \cap A_1^c$ which are pairwise disjoint, which are in $M$.

194
and $\bigcup_n B_n = A$). To this end, we want to verify (A.1) for $A$. Fix $E \subset \Omega$ and let $K_n = \bigcup_{k=1}^n A_k$. By induction, we show that

$$P^*(K_n \cap E) = \sum_{k=1}^n P^*(A_k \cap E).$$

The base case $n = 1$ is clear. Further,

$$P^*(K_{n+1} \cap E) = P^*(K_n \cap K_{n+1} \cap E) + P^*(K_n^c \cap K_{n+1} \cap E)$$

$$= P^*(K_n \cap E) + P^*(A_{n+1} \cap E)$$

$$= \sum_{k=1}^n P^*(A_k \cap E) + P^*(A_{n+1} \cap E),$$

where in the last equality we used the inductive hypothesis. This finishes the inductive argument. Since $K_n \in M$, we obtain

$$P^*(E) = P^*(E \cap K_n) + P^*(E \cap K_n^c) \geq \sum_{k=1}^n P^*(A_k \cap E) + P^*(E \cap K_n^c)$$

$$\geq \sum_{k=1}^n P^*(A_k \cap E) + P^*(E \cap A^c),$$

where the last inequality holds because $P^*$ is monotone (see I.) and $K_n \subset A$. Letting $n \to \infty$ and using subadditivity, we get

$$P^*(E) \geq \sum_{k=1}^\infty P^*(A_k \cap E) + P^*(E \cap A^c) \geq P^*(E \cap A) + P^*(E \cap A^c),$$

so $A$ satisfies (A.1).

**III.** We show 1) $A \subset M$ which also gives $\sigma(A) \subset M$ because $M$ is a $\sigma$-algebra. Moreover, we show 2) $P^* = P$ on $A$, so $P^*$ is the desired extension of $P$ on $\sigma(A)$. The uniqueness follows immediately from Dynkin’s theorem on $\pi$-$\lambda$ systems (see Appendix B and Remark 2.11).

To prove 1), take $A \in A$ and an arbitrary subset $E$ of $\Omega$. Fix $\varepsilon > 0$. By the definition of $P^*$, there are sets $B_1, B_2, \ldots \in A$ such that $E \subset \bigcup_n B_n$ and $\sum_{n=1}^\infty P(B_n) \leq P^*(E) + \varepsilon$. Since $E \cap A \subset \bigcup(B_n \cap A)$ and $E \cap A^c \subset \bigcup(B_n \cap A^c)$ and $B_n \cap A, B_n \cap A^c \in A$, by the definition of $P^*$,

$$P^*(E \cap A) \leq \sum_n P(B_n \cap A)$$

and similarly

$$P^*(E \cap A^c) \leq \sum_n P(B_n \cap A^c).$$

Adding these up and using the additivity of $P$ on $A$, we get

$$P^*(E \cap A) + P^*(E \cap A^c) \leq \sum_n (P(B_n \cap A) + P(B_n \cap A^c)) = \sum_n P(B_n) \leq P^*(E) + \varepsilon,$$
so (A.1) holds, so $A \in \mathcal{M}$.

To prove 2), take $A \in \mathcal{A}$. By the definition of $\mathbb{P}^*$, clearly, $\mathbb{P}^*(A) \leq \mathbb{P}(A)$. To argue for the opposite inequality, suppose $A \subset_n A_n$ for some $A_1, A_2, \ldots \in \mathcal{A}$. Let 

$$C_n = \bigcup_{k=1}^{n}(A \cap A_k).$$

We have that $C_1, C_2, \ldots$ are all in $\mathcal{A}$, $C_1 \subset C_2 \subset \ldots$ and 

$$\bigcup_n C_n = \bigcup_{k=1}^{\infty}(A \cap A_k) = A \cap \bigcup_{k=1}^{\infty} A_k = A$$

is also in $\mathcal{A}$. Using finite subadditivity of $\mathbb{P}$ on $\mathcal{A}$ and its monotonicity, we have

$$\mathbb{P}(C_n) \leq \sum_{k=1}^{n} \mathbb{P}(A \cap A_k) \leq \sum_{k=1}^{n} \mathbb{P}(A_k).$$

Letting $n \to \infty$, by assumption (ii) (finally used for the first and last time!), we obtain

$$\mathbb{P}(A) = \lim_{n \to \infty} \mathbb{P}(C_n) \leq \sum_{k=1}^{\infty} \mathbb{P}(A_k).$$

After taking the infimum over the $A_k$, this gives $\mathbb{P}(A) \leq \mathbb{P}(A^*)$, hence $\mathbb{P}(A) = \mathbb{P}(A^*)$. This finishes the whole proof. \qed
Appendix: Dynkin’s theorem

Recall that a family $\mathcal{A}$ of subsets of a set $\Omega$ is a $\pi$-system if it is closed under finite intersections, that is for every $A, B \in \mathcal{A}$, we have $A \cap B \in \mathcal{A}$. A family $\mathcal{L}$ of subsets of a set $\Omega$ is a $\lambda$-system if $\Omega \in \mathcal{L}$, for every $A, B \in \mathcal{L}$ with $A \subset B$, we have $B \setminus A \in \mathcal{L}$ and for every $A_1, A_2, \ldots \in \mathcal{L}$ such that $A_1 \subset A_2 \subset \ldots$, we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$.

B.1 Remark. If a family is a $\pi$-system and a $\lambda$-system, then it is a $\sigma$-algebra.

B.2 Theorem (Dynkin). Let $\Omega$ be a set. If a $\lambda$-system $\mathcal{L}$ on $\Omega$ contains a $\pi$-system $\mathcal{A}$ on $\Omega$, then $\mathcal{L}$ contains $\sigma(\mathcal{A})$.

Proof. Let $\mathcal{L}_0$ be the smallest $\lambda$-system containing $\mathcal{A}$. By Remark B.1, it suffices to show that $\mathcal{L}_0$ is a $\pi$-system. To this end, we first consider the family

$$\mathcal{C} = \{ A \subset \Omega, \ A \cap B \in \mathcal{L}_0 \ for \ every \ B \in \mathcal{A} \}.$$ 

Clearly, $\mathcal{C}$ contains $\mathcal{A}$. Moreover, $\mathcal{C}$ is a $\lambda$-system. Indeed,

(i) $\Omega \in \mathcal{C}$ because $\mathcal{A} \subset \mathcal{L}_0$,

(ii) let $U, V \in \mathcal{C}$ with $U \subset V$, then for $B \in \mathcal{A}$,

$$ (V \setminus U) \cap B = (U \cap B) \setminus (V \cap B)$$

which is in $\mathcal{L}_0$ because $U \cap B \subset V \cap B$ and $\mathcal{L}_0$ is a $\lambda$-system

(iii) let $A_1, A_2, \ldots \in \mathcal{C}$ with $A_1 \subset A_2 \subset \ldots$, then for $B \in \mathcal{A}$, we have $A_1 \cap B \subset A_2 \cap B \subset \ldots$ and

$$ \left( \bigcup_{i=1}^{n} A_i \right) \cap B = \bigcup_{i=1}^{n} (A_i \cap B)$$

which is in $\mathcal{L}_0$ because $A_i \cap B$ are in $\mathcal{L}_0$ and it is a $\lambda$-system.

We thus get that $\mathcal{C}$, as a $\lambda$-system containing $\mathcal{A}$, contains the smallest $\lambda$-system containing $\mathcal{A}$, that is $\mathcal{L}_0$. This means that $A \cap B \in \mathcal{L}_0$ whenever $A \in \mathcal{L}_0$ and $B \in \mathcal{A}$.

The rest of the proof is a repetition of the same argument. We consider the family

$$\tilde{\mathcal{C}} = \{ A \subset \Omega, \ A \cap B \in \mathcal{L}_0 \ for \ every \ B \in \mathcal{L}_0 \}.$$ 

By the previous step, we know that $\tilde{\mathcal{C}} \supset \mathcal{A}$. We show that $\tilde{\mathcal{C}}$ is a $\lambda$-system, hence, as above, it contains $\mathcal{L}_0$. Therefore, for every $A, B \in \mathcal{L}_0$, $A \cap B \in \mathcal{L}_0$, that is $\mathcal{L}_0$ is a $\pi$-system, as required. \qed
C Appendix: Fubini’s theorem

Let $(\Omega_i, F_i, \mathbb{P}_i), i = 1, 2$ be two probability measures. Let

$$\Omega = \Omega_1 \times \Omega_2.$$  

Define the product $\sigma$-algebra

$$F = \sigma(A_1 \times A_2, A_1 \in F_1, A_2 \in F_2),$$

denoted $F_1 \otimes F_2$.

For $A \subset \Omega$, define the sections of $A$,

$$A_{\omega_1} = \{\omega_2 \in \Omega_2, (\omega_1, \omega_2) \in A\}, \quad \omega_1 \in \Omega_1,$$

$$A^{\omega_2} = \{\omega_1 \in \Omega_1, (\omega_1, \omega_2) \in A\}, \quad \omega_2 \in \Omega_2.$$

Similarly, for a function $X : \Omega \rightarrow \mathbb{R}$, define its section functions

$$X_{\omega_1} : \Omega_2 \rightarrow \mathbb{R}, \quad X_{\omega_1}(\omega_2) = X(\omega_1, \omega_2), \quad \omega_1 \in \Omega_1,$$

$$X^{\omega_2} : \Omega_1 \rightarrow \mathbb{R}, \quad X^{\omega_2}(\omega_1) = X(\omega_1, \omega_2), \quad \omega_2 \in \Omega_2.$$

We have the following lemma about $F$-measurability.

**C.1 Lemma.** For every $A \in F$, every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, we have

$$A_{\omega_1} \in F_2, \quad A^{\omega_2} \in F_1.$$

For every $F$-measurable function $X : \Omega \rightarrow \mathbb{R}$, every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, we have

$X_{\omega_1}$ is $F_2$-measurable, $X^{\omega_2}$ is $F_1$-measurable.

If moreover $X$ is nonnegative, we have that

$$\omega_1 \mapsto \int_{\Omega_2} X_{\omega_1}(\omega_2) d\mathbb{P}_2(\omega_2) \quad \text{is } F_1 \text{-measurable}$$

and

$$\omega_2 \mapsto \int_{\Omega_1} X^{\omega_2}(\omega_1) d\mathbb{P}_1(\omega_1) \quad \text{is } F_2 \text{-measurable}$$

**Proof.** Let $\mathcal{M}$ be the class of all subsets $A$ of $\Omega$ such that for every $\omega_1$, $A_{\omega_1}$ is $F_2$-measurable. Clearly $\mathcal{M}$ contains product sets $B_1 \times B_2, B_i \in F_i, i = 1, 2$ which form a $\pi$-system generating $F$. Moreover, it is easy to check that $\mathcal{M}$ is a $\sigma$-algebra. Thus $\mathcal{M} \supset F$. We argue similarly about $A^{\omega_2}$.

To prove the $F_2$-measurability of $X_{\omega_1}$, note that for $B \in \mathcal{B}(\mathbb{R})$,

$$X_{\omega_1}^{-1}(B) = \{\omega_2 \in \Omega_2, X(\omega_1, \omega_2) \in B\} = X^{-1}(B)_{\omega_1}.$$
which is in $\mathcal{F}_2$ by the previous part because $X^{-1}(B) \in \mathcal{F}$. The $\mathcal{F}_1$-measurability of $X^{\omega_2}$, we proceed in the same way.

Finally, if $X = 1_{B_1 \times B_2}$ for some $B_i \in \mathcal{F}_i$, $i = 1, 2$, we have

$$\int_{\Omega_2} X_{\omega_1}(\omega_2) dP_2(\omega_2) = 1_{B_1}(\omega_1) \int_{\Omega_2} 1_{B_2}(\omega_2) dP_2(\omega_2),$$

which is clearly $\mathcal{F}_1$-measurable. Thus, by the standard arguments (see the proof of Theorem E.6), the same holds when $X$ is a simple function and consequently, thanks to Lebesgue’s monotone convergence theorem, when $X$ is nonnegative. □

We define $P: \mathcal{F} \to [0, 1]$ as follows: for $A \in \mathcal{F}$, let $X = 1_A$ and

$$P(A) = \int_{\Omega_1} \left( \int_{\Omega_2} X_{\omega_1}(\omega_2) dP_2(\omega_2) \right) dP_1(\omega_1).$$

We have the following important result saying that $P$ is the so-called product measure on $\Omega$.

C.2 Theorem (The uniqueness of product measures). The set function $P$ is a unique probability measure on $(\Omega, \mathcal{F})$ such that for every $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$, we have

$$P(A_1 \times A_2) = P_1(A_1)P_2(A_2).$$

Moreover,

$$P(A) = \int_{\Omega_2} \left( \int_{\Omega_1} X^{\omega_1}(\omega_1) dP_1(\omega_1) \right) dP_2(\omega_2).$$

Proof. By Lemma C.1, the inner integral in the definition of $P$ is an $\mathcal{F}_2$-measurable function, thus $P$ is well defined on $\mathcal{F}$. Clearly, $P(A_1 \times A_2) = P_1(A_1)P_2(A_2)$, so in particular $P(\Omega) = 1$. If $B_1, B_2, \ldots \in \mathcal{F}$ are disjoint, then so are their sections, that is we have $1_{\bigcup_n (B_n)_{\omega_1}} = \sum_n 1_{(B_n)_{\omega_1}}$, consequently, by the linearity of integrals, we get that $P$ is countably-additive. The uniqueness follows from the fact that the product sets $A_1 \times A_2$, $A_i \in \mathcal{F}_i$, form a $\pi$-system generating $\mathcal{F}$, combined with Remark 2.11. The formula with the integrals over $\Omega_1$ and $\Omega_2$ swapped follows by considering

$$\tilde{P}(A) = \int_{\Omega_2} \left( \int_{\Omega_1} X^{\omega_2}(\omega_1) dP_1(\omega_1) \right) dP_2(\omega_2),$$

checking that $\tilde{P}$ satisfies the same defining property, $\tilde{P}(A_1 \times A_2) = P_1(A_1)P_2(A_2)$ and using the uniqueness. □

We say that $P$ is the product of $P_1$ and $P_2$, denoted

$$P = P_1 \otimes P_2.$$ 

C.3 Theorem (Fubini). Let $X: \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable.
(i) If $X \geq 0$, then
\[
\int_{\Omega_1 \times \Omega_2} X \, dP = \int_{\Omega_1} \left( \int_{\Omega_2} X_{\omega_1}(\omega_2) \, dP_2(\omega_2) \right) \, dP_1(\omega_1)
= \int_{\Omega_2} \left( \int_{\Omega_1} X_{\omega_2}(\omega_1) \, dP_1(\omega_1) \right) \, dP_2(\omega_2).
\]

(ii) If
\[
\int_{\Omega_1} \left( \int_{\Omega_2} |X_{\omega_1}(\omega_2)| \, dP_2(\omega_2) \right) \, dP_1(\omega_1) < \infty,
\]
or
\[
\int_{\Omega_2} \left( \int_{\Omega_1} |X_{\omega_2}(\omega_1)| \, dP_1(\omega_1) \right) \, dP_2(\omega_2) < \infty,
\]
then
\[
\int_{\Omega_1 \times \Omega_2} |X| \, dP < \infty,
\]
that is $X$ is $(\Omega, F, P)$-integrable.

(iii) If $X$ is $(\Omega, F, P)$-integrable, then
\[
P_1 \left\{ \omega_1 \in \Omega_1, \, \int_{\Omega_2} |X_{\omega_1}(\omega_2)| \, dP_2(\omega_2) < \infty \right\} = 1,
\]
\[
P_2 \left\{ \omega_2 \in \Omega_2, \, \int_{\Omega_1} |X_{\omega_2}(\omega_1)| \, dP_1(\omega_1) < \infty \right\} = 1
\]
and (i) holds.

Proof. (i) By Theorem C.2, the formula holds for $X = 1_A$, $A \in F$. Thus it holds for simple functions and by Lebesgue’s monotone convergence theorem, it holds for nonnegative functions.

(ii) Follows from (i) applied to $|X|$.

(iii) By the construction of Lebesgue integrals, $|X|$ being integrable gives
\[
\int_{\Omega} X^+ \, dP < \infty \quad \text{and} \quad \int_{\Omega} X^- \, dP < \infty.
\]
Thus from (a) applied to $X^+$,
\[
\int_{\Omega} X^+ \, dP = \int_{\Omega_1} \left( \int_{\Omega_2} X^+_{\omega_1}(\omega_2) \, dP_2(\omega_2) \right) \, dP_1(\omega_1),
\]
which by basic properties of Lebesgue integrals means that
\[
\int_{\Omega_2} X^+_{\omega_1}(\omega_2) \, dP_2(\omega_2) < \infty
\]
for $P_1$-a.e. $\omega_1$. Similarly for $X^-$. Therefore,
\[
P_1 \left\{ \omega_1 \in \Omega_1, \, \int_{\Omega_2} |X_{\omega_1}(\omega_2)| \, dP_2(\omega_2) < \infty \right\} = 1.
\]
In particular, for every $\omega_1$ in this event,
\[
\int_{\Omega_2} X_{\omega_1}(\omega_2) dP_2(\omega_2) = \int_{\Omega_2} X_{\omega_1}^+(\omega_2) dP_2(\omega_2) - \int_{\Omega_2} X_{\omega_1}^-(\omega_2) dP_2(\omega_2).
\]
For the remaining $\omega_1$, we can set all these integrals to be 0 and then we get
\[
\int_{\Omega_1} \left( \int_{\Omega_2} X_{\omega_1}(\omega_2) dP_2(\omega_2) \right) dP_1(\omega_1) = \int_{\Omega_1} \left( \int_{\Omega_2} X_{\omega_1}^+(\omega_2) dP_2(\omega_2) \right) dP_1(\omega_1)
- \int_{\Omega_1} \left( \int_{\Omega_2} X_{\omega_1}^-(\omega_2) dP_2(\omega_2) \right) dP_1(\omega_1)
= \int_{\Omega_1} X^+ dP - \int_{\Omega_1} X^- dP
= \int_{\Omega_1 \times \Omega_2} X dP.
\]
We proceed in the same way for the swapped order of taking the integrals over $\Omega_1$ and $\Omega_2$.

Fubini’s theorem generalises to $\sigma$-finite measures as well as products of more than two but finitely many measures. Extensions to products of infinitely many measures are more delicate and are handled in the next appendix.
D Appendix: Infinite products of measures
E Appendix: Construction of expectation

The goal of this section is to define expectation of random variables and establish its basic properties. We shall only consider real-valued random variables. Recall that a function $X : \Omega \to \mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a random variable if for every $x \in \mathbb{R}$, the preimage $\{X \leq x\} = \{\omega \in \Omega, X(\omega) \leq x\} = X^{-1}((-\infty, x])$ is an event (belongs to the sigma-field $\mathcal{F}$).

A random variable $X$ is called simple if its image $X(\Omega)$ is a finite set, that is

$$X = \sum_{k=1}^{n} x_k 1_{A_k},$$

for some distinct $x_1, \ldots, x_n \in \mathbb{R}$ (values) and events $A_1, \ldots, A_n$ which form a partition of $\Omega$ (we have, $A_k = \{X = x_k\}$).

The expectation of the simple random variable $X$, denoted $\mathbb{E}X$, is defined as

$$\mathbb{E}X = \sum_{k=1}^{n} x_k \mathbb{P}(A_k).$$

The expectation of a nonnegative random variable $X$ is defined as

$$\mathbb{E}X = \sup\{\mathbb{E}Z, Z \text{ is simple and } Z \leq X\}.$$

Note that $\mathbb{E}X \geq 0$ because we can always take $Z = 0$. We can have $\mathbb{E}X = +\infty$ (for instance, for a discrete random variable $X$ with $\mathbb{P}(X = k) = \frac{1}{k(k-1)}, k = 2, 3, \ldots$). For an arbitrary random variable $X$, we write

$$X = X^+ - X^-,$$

where

$$X^+ = \max\{X, 0\} = X 1_{\{X \geq 0\}}$$

is the positive part of $X$ and

$$X^- = -\min\{X, 0\} = -X 1_{\{X \leq 0\}}$$

is the negative part of $X$. These are nonnegative random variables and the expectation of $X$ is defined as

$$\mathbb{E}X = \mathbb{E}X^+ - \mathbb{E}X^-$$

provided that at least one of the quantities $\mathbb{E}X^+$, $\mathbb{E}X^-$ is finite (to avoid $\infty - \infty$). We say that $X$ is integrable if $\mathbb{E}|X| < \infty$. Since $|X| = X^+ + X^-$, we have that $X$ is integrable if and only if $\mathbb{E}X^+ < \infty$ and $\mathbb{E}X^- < \infty$.

One of the desired properties of expectation is linearity. It of course holds for simple random variables.
E.1 Theorem. Let $X$ and $Y$ be simple random variables. Then $\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$.

Proof. Let $X = \sum_{k=1}^{m} x_k \mathbf{1}_{A_k}$ and $Y = \sum_{l=1}^{n} y_l \mathbf{1}_{B_l}$ for some reals $x_k, y_l$ and events $A_k$ and $B_l$ are such that the $A_k$ partition $\Omega$ and the $B_l$ partition $\Omega$. Then the events $A_k \cap B_l$, $k \leq m, l \leq n$ partition $\Omega$ and

$$X + Y = \sum_{k \leq m, l \leq n} (x_k + y_l) \mathbf{1}_{A_k \cap B_l}.$$

This is a simple random variable with

$$\mathbb{E}(X + Y) = \sum_{k \leq m, l \leq n} (x_k + y_l) \mathbb{P}(A_k \cap B_l)$$

$$= \sum_{k \leq m, l \leq n} x_k \mathbb{P}(A_k \cap B_l) + \sum_{k \leq m, l \leq n} y_l \mathbb{P}(A_k \cap B_l)$$

$$= \sum_{k \leq m} x_k \sum_{l \leq n} \mathbb{P}(A_k \cap B_l) + \sum_{l \leq n} y_l \sum_{k \leq m} \mathbb{P}(A_k \cap B_l)$$

$$= \sum_{k \leq m} x_k \mathbb{P} \left( \bigcup_{l \leq n} B_l \right) + \sum_{l \leq n} y_l \mathbb{P} \left( \bigcup_{k \leq m} A_k \cap B_l \right)$$

$$= \sum_{k \leq m} x_k \mathbb{P}(A_k) + \sum_{l \leq n} y_l \mathbb{P}(B_l),$$

which is $\mathbb{E}X + \mathbb{E}Y$ and this finishes the proof. \qed

E.1 Nonnegative random variables

Our main goal is to prove linearity of expectation. We first establish a few basic properties of expectation for nonnegative random variables.

E.2 Theorem. Let $X$ and $Y$ be nonnegative random variables. We have

(a) if $X \leq Y$, then $\mathbb{E}X \leq \mathbb{E}Y$,

(b) for $a \geq 0$, $\mathbb{E}(a + X) = a + \mathbb{E}X$ and $\mathbb{E}(aX) = a\mathbb{E}X$,

(c) if $\mathbb{E}X = 0$, then $X = 0$ a.s. (i.e. $\mathbb{P}(X = 0) = 1$)

(d) if $A$ and $B$ are events such that $A \subset B$, then $\mathbb{E}X \mathbf{1}_A \leq \mathbb{E}X \mathbf{1}_B$.

Proof. (a) Let $\varepsilon > 0$. By definition, there is a simple random variable $Z$ such that $Z \leq X$ and $\mathbb{E}Z > \mathbb{E}X - \varepsilon$. Then also $Z \leq Y$, so by the definition of $\mathbb{E}Y$, we have $\mathbb{E}Z \leq \mathbb{E}Y$. Thus $\mathbb{E}X - \varepsilon < \mathbb{E}Y$. Sending $\varepsilon$ to 0 finishes the argument.

(b) For a simple random variable $Z$, clearly $\mathbb{E}(a + Z) = a + \mathbb{E}Z$ and $\mathbb{E}(aZ) = a\mathbb{E}Z$. It remains to follow the proof of (a).
(c) For $n \geq 1$, we have $X \geq X_{1/n} \geq \frac{1}{n} \mathbf{1}_{\{X \geq 1/n\}}$, so by (a) we get

$$0 = \mathbb{E}X \geq \mathbb{E}\frac{1}{n} \mathbf{1}_{\{X \geq 1/n\}} = \frac{1}{n} \mathbb{P}(X \geq 1/n),$$

thus $\mathbb{P}(X \geq 1/n) = 0$, so

$$\mathbb{P}(X > 0) = \mathbb{P}\left( \bigcap_{n \geq 1} \{X \geq 1/n\} \right) = \lim_{n \to \infty} \mathbb{P}(X \geq 1/n) = 0.$$

(d) follows immediately from (a).

The following lemma gives a way to approximate nonnegative random variables with monotone sequences of simple ones.

**E.3 Lemma.** If $X$ is a nonnegative random variable, then there is a sequence $(Z_n)$ of nonnegative simple random variables such that for every $\omega \in \Omega$, $Z_n(\omega) \leq Z_{n+1}(\omega)$ and $Z_n(\omega) \xrightarrow{n \to \infty} X(\omega)$.

**Proof.** Define

$$Z_n = \sum_{k=1}^{n/2^n} \left( \frac{k-1}{2^n} \mathbf{1}_{\{k-1/2^n \leq X < k/2^n\}} + n \mathbf{1}_{\{X \geq n\}} \right).$$

Fix $\omega \in \Omega$. Then $Z_n(\omega)$ is a nondecreasing sequence (check!). Since $n > X(\omega)$ for large enough $n$, we have for such $n$ that $0 \leq X(\omega) - Z_n(\omega) \leq 2^{-n}$.

The following is a very important and useful tool allowing to exchange the order of taking the limit and expectation for monotone sequences.

**E.4 Theorem** (Lebesgue’s monotone convergence theorem). If $X_n$ is a sequence of nonnegative random variables such that $X_n \leq X_{n+1}$ and $X_n \xrightarrow{n \to \infty} X$, then

$$\mathbb{E}X_n \xrightarrow{n \to \infty} \mathbb{E}X.$$

**Proof.** By E.2 (a), $\mathbb{E}X_n \leq \mathbb{E}X_{n+1}$ and $\mathbb{E}X_n \leq \mathbb{E}X$, so $\lim_n \mathbb{E}X_n$ exists and is less than or equal to $\mathbb{E}X$. It remains to show that $\mathbb{E}X \leq \lim_n \mathbb{E}X_n$. Take a simple random variable $Z$ such that $0 \leq Z \leq X$, with the largest value say $K$. Observe that for every $n \geq 1$ and $\varepsilon > 0$,

$$Z \leq (X_n + \varepsilon) \mathbf{1}_{\{Z < X_n + \varepsilon\}} + K \mathbf{1}_{\{Z \geq X_n + \varepsilon\}}.$$  \hspace{1cm} (E.1)

**Claim.** For nonnegative random variables $X$, $Y$ and an event $A$, we have

$$\mathbb{E}(X \mathbf{1}_A + Y \mathbf{1}_A^c) \leq \mathbb{E}X \mathbf{1}_A + \mathbb{E}Y \mathbf{1}_A^c.$$

**Proof of the claim.** Fix $\varepsilon > 0$. Take a simple random variable $Z$ such that $Z \leq X \mathbf{1}_A + Y \mathbf{1}_A^c$ and $\mathbb{E}Z > \mathbb{E}(X \mathbf{1}_A + Y \mathbf{1}_A^c) - \varepsilon$. Note that

$$Z \mathbf{1}_A \leq X \mathbf{1}_A \quad \text{and} \quad Z \mathbf{1}_A^c \leq Y \mathbf{1}_A^c.$$
Thus by E.2 (a),
\[ \mathbb{E}Z1_A \leq \mathbb{E}X1_A \quad \text{and} \quad \mathbb{E}Z1_{A^c} \leq \mathbb{E}Y1_{A^c}. \]

Adding these two inequalities together and using that \( \mathbb{E}Z1_A + \mathbb{E}Z1_{A^c} = \mathbb{E}Z \), which follows from linearity of expectation for simple random variables (Theorem E.1), we get
\[ \mathbb{E}(X1_A + Y1_{A^c}) - \varepsilon < \mathbb{E}Z \leq \mathbb{E}X1_A + \mathbb{E}Y1_{A^c}. \]

Sending \( \varepsilon \to 0 \) finishes the argument. \( \square \)

Applying the claim to (E.1), we obtain
\[ \mathbb{E}Z \leq \mathbb{E}X_n + \varepsilon + K\mathbb{P}(Z \geq X_n + \varepsilon). \]

The events \( \{ Z \geq X_n + \varepsilon \} \) form a decreasing family (because \( X_n \leq X_{n+1} \) and their intersection is \( \{ Z \geq X + \varepsilon \} = \emptyset \) (because \( X_n \to X \) and \( Z \leq X \)). Therefore taking \( n \to \infty \) in the last inequality gives
\[ \mathbb{E}Z \leq \lim_n \mathbb{E}X_n + \varepsilon. \]

Taking the supremum over simple random variables \( Z \leq X \) gives
\[ \mathbb{E}X \leq \lim_n \mathbb{E}X_n + \varepsilon. \]

Letting \( \varepsilon \to 0 \), we finish the proof. \( \square \)

As a corollary we obtain a result about the limit inferior of nonnegative random variables and its expectation.

**E.5 Theorem** (Fatou’s lemma). If \( X_1, X_2, \ldots \) are nonnegative random variables, then
\[ \mathbb{E}\liminf_{n \to \infty} X_n \leq \liminf_{n \to \infty} \mathbb{E}X_n. \]

**Proof.** Let \( Y_n = \inf_{k \geq n} X_k \). Then this is a nondecreasing sequence which converges to \( \liminf_{n \to \infty} X_n \) and \( Y_n \leq X_n \). Note that
\[ \liminf_{n \to \infty} \mathbb{E}X_n \geq \liminf_{n \to \infty} \mathbb{E}Y_n = \lim_{n \to \infty} \mathbb{E}Y_n, \]
where the last equality holds because the sequence \( \mathbb{E}Y_n \), as nondecreasing, is convergent.

By Lebesgue’s monotone convergence theorem,
\[ \lim_{n \to \infty} \mathbb{E}Y_n = \mathbb{E}\left( \lim_{n \to \infty} Y_n \right) = \mathbb{E}\liminf_{n \to \infty} X_n, \]
which in view of the previous inequality finishes the proof. \( \square \)

We are ready to prove linearity of expectation for nonnegative random variables.
**E.6 Theorem.** Let \( X \) and \( Y \) be nonnegative random variables. Then

\[
E(X + Y) = EX + EY.
\]

**Proof.** By Lemma E.3, there are nondecreasing sequences \((X_n)\) and \((Y_n)\) of nonnegative simple random variables such that \( X_n \to X \) and \( Y_n \to Y \). Then the sequence \((X_n + Y_n)\) is also monotone and \( X_n + Y_n \to X + Y \). By Theorem E.1,

\[
E(X_n + Y_n) = EX_n + EY_n.
\]

Letting \( n \to \infty \), by the virtue of Lebesgue’s monotone convergence theorem, we get in the limit 

\[
E(X + Y) = EX + EY.
\]

\( \Box \)

**E.2 General random variables**

Key properties of expectation for general random variables are contained in our next theorem.

**E.7 Theorem.** If \( X \) and \( Y \) are integrable random variables, then

(a) \( X + Y \) is integrable and \( E(X + Y) = EX + EY \),

(b) \( E(aX) = aXE \) for every \( a \in \mathbb{R} \),

(c) if \( X \leq Y \), then \( EX \leq EY \),

(d) \( |EX| \leq E|X| \).

**Proof.** (a) By the triangle inequality Theorem E.2 (a) and Theorem E.6,

\[
E|X + Y| \leq E(|X| + |Y|) = E|X| + E|Y|
\]

and the right hand side is finite by the assumption, thus \( X + Y \) is integrable.

To show the linearity, write \( X + Y \) in two different ways

\[(X + Y)^+ - (X + Y)^- = X + Y = X^+ - X^- + Y^+ - Y^-,
\]

rearrange

\[(X + Y)^+ + X^- + Y^- = (X + Y)^- + X^+ + Y^+,
\]

to be able to use the linearity of expectation for nonnegative random variables (Theorem E.6) and get

\[
E(X + Y)^+ + EX^- + EY^- = E(X + Y)^- + EX^+ + EY^+,
\]

which rearranged again gives \( E(X + Y) = EX + EY \).

(b) We leave this as an exercise.
(c) Note that $X \leq Y$ is equivalent to saying that $X^+ \leq Y^+$ and $X^- \geq Y^-$. (if $X = X^+$, then $X \leq Y$ implies that $Y = Y^+$, hence $X^+ \leq Y^+$; similarly, if $Y = -Y^-$, then $X \leq Y$ implies that $X = -X^-$, hence $X^- \geq Y^-$). It remains to use Theorem E.2 (a).

(d) Since $-|X| \leq X \leq |X|$, by (c) we get $-\mathbb{E}|X| \leq \mathbb{E}X \leq \mathbb{E}|X|$, that is $|\mathbb{E}X| \leq \mathbb{E}|X|$. □

E.3 Lebesgue’s dominated convergence theorem

We finish with one more limit theorem, quite useful in various applications; we also show one of them.

**E.8 Theorem** (Lebesgue’s dominated convergence theorem). If $(X_n)$ is a sequence of random variables and $X$ is a random variable such that for every $\omega \in \Omega$, we have $X_n(\omega) \xrightarrow{n \to \infty} X(\omega)$ and there is an integrable random variable $Y$ such that $|X_n| \leq Y$, then

$$\mathbb{E}|X_n - X| \xrightarrow{n \to \infty} 0.$$  

In particular,

$$\mathbb{E}X_n \xrightarrow{n \to \infty} \mathbb{E}X.$$

**Proof.** Since $|X_n| \leq Y$, taking $n \to \infty$ yields $|X| \leq Y$. In particular, $X$ is integrable as well. By the triangle inequality,

$$|X_n - X| \leq 2Y$$

and Fatou’s lemma (Theorem E.5) gives

$$\mathbb{E}(2Y) = \mathbb{E}\lim\inf(2Y - |X_n - X|) \leq \lim\inf \mathbb{E}(2Y - |X_n - X|) = 2\mathbb{E}Y - \lim\sup \mathbb{E}|X_n - X|.$$  

As a result, $\lim\sup \mathbb{E}|X_n - X| \leq 0$, so

$$\mathbb{E}|X_n - X| \xrightarrow{n \to \infty} 0.$$  

In particular, since by Theorem E.7 (d),

$$|\mathbb{E}(X_n - X)| \leq \mathbb{E}|X_n - X|,$$

we get that the left hand side goes to 0, that is $\mathbb{E}X_n \to \mathbb{E}X$. □
Appendix: Lindeberg’s swapping argument
Appendix: The moment method
Appendix: Feller’s converse to the central limit theorem
I Appendix: Uniform integrability

We recall the definition: a family of random variables \( \{X_t\}_{t \in T} \) is uniformly integrable if for every \( \varepsilon > 0 \), there is \( K > 0 \) such that for all \( t \in T \), we have \( \mathbb{E}|X_t|1_{\{|X_t| > K\}} \leq \varepsilon \).

We start with two easy criteria guaranteeing uniform integrability.

I.1 Lemma. Let \( \{X_t\}_{t \in T} \) be a family of random variables such that there is a nonnegative random variable \( Y \) with \( \mathbb{E}Y < \infty \) and \( |X_t| \leq Y \) for every \( t \in T \). Then the family \( \{X_t\}_{t \in T} \) is uniformly integrable.

I.2 Remark. In particular, if \( \{X_t\}_{t \in T} \) is a finite family of integrable random variables, then it is uniformly integrable (we simply take \( Y = \sum_{t \in T} |X_t| \)).

Proof of Lemma I.1. We have, \( \mathbb{E}|X_t|1_{\{|X_t| > K\}} \leq \mathbb{E}1_{\{Y > K\}} \) and the right hand side goes to 0 as \( K \to \infty \) (by Lebesgue’s dominated convergence theorem).

I.3 Lemma. Let \( \{X_t\}_{t \in T} \) be a family of random variables bounded in \( L_p \) for some \( p > 1 \) (that is, \( M = \sup_{t \in T} \mathbb{E}|X_t|^p < \infty \)). Then the family \( \{X_t\}_{t \in T} \) is uniformly integrable.

Proof. Write \( p = 1 + \delta \) with \( \delta > 0 \). We have,
\[
\mathbb{E}|X_t|1_{\{|X_t| > K\}} = \mathbb{E}|X_t|^{1+\delta}|X_t|^{-\delta}1_{\{|X_t| > K\}} \leq K^{-\delta}\mathbb{E}|X_t|^{1+\delta} \leq K^{-\delta}M,
\]
where \( M = \sup_{t \in T} \mathbb{E}|X_t|^p \), so that the right hand side goes to 0 as \( K \to \infty \).

There is an equivalent definition of uniform integrability which perhaps explains the name better and is often useful.

I.4 Theorem. Let \( \{X_t\}_{t \in T} \) be a family of random variables. It is uniformly integrable if and only if the following two conditions hold

(i) it is bounded in \( L_1 \), that is \( \sup_{t \in T} \mathbb{E}|X_t| < \infty \),
(ii) for every \( \varepsilon > 0 \), there is \( \delta > 0 \) such that for every event \( A \) with \( \mathbb{P}(A) < \delta \) and every \( t \in T \), we have \( \mathbb{E}|X_t|1_A < \varepsilon \).

Proof. “\( \Rightarrow \)”: Fix \( \varepsilon > 0 \) and choose \( K > 0 \) such that \( \mathbb{E}|X_t|1_{\{|X_t| > K\}} \leq \varepsilon \) for every \( t \in T \).
For an event \( A \), we have
\[
\mathbb{E}|X_t|1_A = \mathbb{E}|X_t|1_A1_{\{|X_t| \leq K\}} + \mathbb{E}|X_t|1_A1_{\{|X_t| > K\}} \leq K\mathbb{P}(A) + \varepsilon < 2\varepsilon
\]
if \( \mathbb{P}(A) < \delta = \frac{\varepsilon}{K} \). This shows (ii). Taking \( A = \Omega \), the first inequality shows (i).

“\( \Leftarrow \)”: Fix \( \varepsilon > 0 \) and take \( \delta \) provided by (ii). We would like to choose \( K \) such that \( \mathbb{E}|X_t|1_{\{|X_t| > K\}} < \varepsilon \) for every \( t \in T \). Let \( A = \{|X_t| > K\} \). Let \( M = \sup_{t \in T} \mathbb{E}|X_t| \) which is finite by (i). By Chebyshev’s inequality,
\[
\mathbb{P}(A) \leq \frac{1}{K} \mathbb{E}|X_t| \leq \frac{M}{K} < \delta
\]
provided that \( K > \frac{M}{\delta} \). Thus, for this choice of \( K \), what we want follows from (ii) with
\[
A = \{|X_n| > K\}. \tag{\ref{I.5}} \]

\section*{I.5 Remark.}
Let \( X \) be an integrable random variable. Combining Remark I.2 (applied to the one-element family \( \{X\} \)) with (ii) of Theorem I.4, we get a “uniform continuity” property of expectation:
\[
\forall \varepsilon > 0 \exists \delta > 0 \forall A : \mathbb{P}(A) < \infty \quad \mathbb{E}|X|\mathbf{1}_A < \varepsilon. \tag{\ref{I.6}}
\]

The usefulness of uniform integrability lies in the fact that it captures \( L_p \) convergence.

\section*{I.6 Theorem.}
Let \( p > 0 \). For random variables \( X, X_1, X_2, \ldots \) in \( L_p \), we have that \( X_n \to X \) in \( L_p \) if and only if the following two conditions hold
\begin{enumerate}[(i)]
  \item \( X_n \to X \) in probability,
  \item \( \{|X_n|^p\}_{n \geq 1} \) is a uniformly integrable family.
\end{enumerate}

\begin{proof}
“\( \Rightarrow \)” : Clearly (i) holds (see Theorem 6.14). To see (ii), we shall use Theorem I.4. Fix \( \varepsilon > 0 \). Let \( A \) be an event. First we use \(|a + b|^p \leq C(|a|^p + |b|^p), a, b \in \mathbb{R}\) \((C = \max\{2^{p-1}, 1\} \) is good\), to bound
\[
\mathbb{E}|X_n|^p\mathbf{1}_A \leq C(\mathbb{E}|X|^p\mathbf{1}_A + \mathbb{E}|X_n - X|^p\mathbf{1}_A).
\]

For large \( n \), say \( n > N \), we have \( \mathbb{E}|X_n - X|^p\mathbf{1}_A \leq \mathbb{E}|X_n - X|^p < \varepsilon \). For \( n \leq N \), we can choose \( \delta \) from (ii) of Theorem I.4 applied to the finite family \( \{|X|, |X_n - X|, n \leq N\} \) to bound each term by \( \varepsilon \). This finishes the argument.

“\( \Leftarrow \)” : Fix \( \varepsilon > 0 \). Let \( \delta \) be chosen from (ii) of Theorem I.4 for the uniformly integrable family \( \{|X_n - X|^p\}_{n \geq 1} \) (it is uniformly integrable because \(|X_n - X|^p \leq C(|X_n|^p + |X|^p))\). We have,
\[
\mathbb{E}|X_n - X|^p = \mathbb{E}|X_n - X|^p\mathbf{1}_{\{|X_n - X|\leq \varepsilon\}} + \mathbb{E}|X_n - X|^p\mathbf{1}_{\{|X_n - X| > \varepsilon\}}
\leq \varepsilon^p + \mathbb{E}|X_n - X|^p\mathbf{1}_{\{|X_n - X| > \varepsilon\}}.
\]

Let \( A = \{|X_n - X| > \varepsilon\} \). For \( n \) large enough, by (i), \( \mathbb{P}(A) < \delta \), thus the second term for all such \( n \) is bounded by \( \varepsilon \). This shows that \( X_n \to X \) in \( L_p \).
\end{proof}

\section*{I.7 Example.}
Consider the probability space \((\{0, 1\}, \mathcal{B}(\{0, 1\}), \text{Leb})\) and the random variables \( X_n = n\mathbf{1}_{[0, 1/n]}, n \geq 1 \). Then, \( \mathbb{E}|X_n| = 1 \), so \(|X_n|\) is bounded in \( L_1 \), but the family is not uniformly integrable: for every \( K > 0 \), we have \( \mathbb{E}|X_n|\mathbf{1}_{\{|X_n| > K\}} = \mathbb{E}|X_n| = 1 \) for all \( n \geq K \). In this example, \( X_n \to 0 \) a.s. and in probability, but not in \( L_1 \) (see Example 6.15).
References


