1. Let X_1, X_2, X_3 be i.i.d. standard Gaussian random variables. Find the mean and variance of $Y = 3X_1 - X_2 + 2X_3$. Find its density.

2. Show that a Gaussian random vector in $\mathbb{R}^n$ has independent components if and only if they are uncorrelated.

3. Let (X, Y) be a standard Gaussian random vector in $\mathbb{R}^2$. Let $\rho \in (-1, 1)$ and define

$$(U, V) = \left(\frac{\sqrt{1+\rho} + \sqrt{1-\rho}}{2} X + \frac{\sqrt{1+\rho} - \sqrt{1-\rho}}{2} Y, \frac{\sqrt{1+\rho} + \sqrt{1-\rho}}{2} Y + \frac{\sqrt{1+\rho} - \sqrt{1-\rho}}{2} X \right).$$

Find the density of (U, V). Is this a Gaussian random vector? What is its covariance matrix? What is the distribution of U and V? Determine the values of ρ for which U and V are independent.

4. Let $\rho \in (-1, 1)$ and let (U, V) be a random vector in $\mathbb{R}^2$ with density

$$f(u, v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)}(u^2 - 2\rho uv + v^2) \right\}, \qquad (u, v) \in \mathbb{R}^2.$$

Is it a Gaussian random vector? Find the covariance matrix of (U, V). Find the distributions of the marginals U and V. Find the conditional density of V given $U = u$ and the conditional expectation $\mathbb{E}(V|U = u)$. Determine the values of ρ for which U and V are independent.

5. Suppose (X, Y) is a centred (i.e., $\mathbb{E}X = \mathbb{E}Y = 0$) Gaussian random vector in $\mathbb{R}^2$ with $\mathrm{Cov}(\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$. Find, a) the density of (X, Y), b) the density of $X + 3Y$, c) all $\alpha \in \mathbb{R}$ for which $X + Y$ and $X + \alpha Y$ are independent.

6. Let G be a standard Gaussian vector in $\mathbb{R}^n$ and let U be an $n \times n$ orthogonal matrix. Find the density of UG. Are the components of this vector independent?

7. Let g be a standard Gaussian random variable. Show that $\mathbb{E}g^{2m} = 1 \cdot 3 \cdot \ldots \cdot (2m-1)$, $m = 1, 2, \ldots$.

8. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each with mean zero and finite fourth moment. Show that

$$\mathbb{E}\left(\sum_{i=1}^n X_i\right)^4 = \sum_{i=1}^n \mathbb{E}X_i^4 + 6 \sum_{1 \le i < j \le n} \mathbb{E}X_i^2 \mathbb{E}X_j^2.$$

9.* Let $0 < p < q < 1$. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{Ber}(p)$ random variables and let $Y_1, \ldots, Y_n$ be i.i.d. $\mathrm{Ber}(q)$ random variables. Show that for any $t \leq n$,

$$\mathbb{P}\left(X_1 + \ldots + X_n \geq t\right) \leq \mathbb{P}\left(Y_1 + \ldots + Y_n \geq t\right)$$

(intuitively, probability of getting at least t heads when tossing a biased coin showing heads with probability p does not decrease as we increase p).