– DISTRIBUTION FUNCTIONS and DENSITY –

Pick a point from $[0,1]$ uniformly at random. This gives rise to a r.v. $X : \Omega \to [0,1]$, $X(\omega)$ = point we pick, which is not discrete.

Recall **Def** $X : \Omega \to \mathbb{R}$ is a (real valued) r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$ if

$$\forall x \in \mathbb{R} \qquad \underbrace{\{\omega \in \Omega,\ X(\omega) \leq x\}}_{\{X \leq x\}} \in \mathcal{F}$$

⚠ If X is a discrete r.v. taking values say in $\mathbb{Z}$, then

$$\{X \leq x\} = \dots \cup \{X=-2\} \cup \{X=-1\} \cup \{X=0\} \cup \dots \cup \{X = \lfloor x \rfloor\}$$

each in $\mathcal{F}$, so their countable union in $\mathcal{F}$

so every discrete r.v. is a r.v.

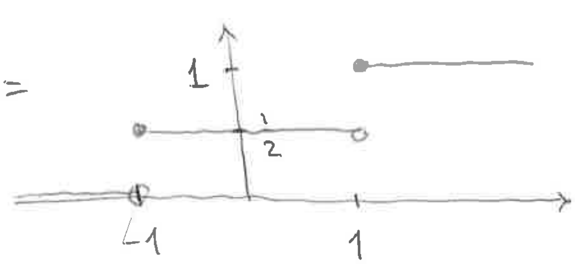
⚠ $\forall x \ \{X \leq x\} \in \mathcal{F} \underset{\text{(Check)}}{\Longrightarrow} \begin{matrix} \{X < x\} \\ \{x < X \leq y\} \\ \{X \geq x\} \\ \vdots \end{matrix} \in \mathcal{F}$

To describe the distribution of a r.v. X we define its

<u>(cummulative) distribution function</u> (cdf)

$$F_X(t) = \mathbb{P}(X \leq t), \quad t \in \mathbb{R}$$

E.g. X = random sign $\qquad F_X =$



Properties of the cdf F_X of a r.v. X

(1) nondecreasing: if $s \le t$, then $F_X(s) \le F_X(t)$

(2) $\lim_{t\to-\infty} F_X(t) = 0$, $\lim_{t\to+\infty} F_X(t) = 1$

(3) continuity from the right $\forall t_0 \in \mathbb{R}$ $\lim_{t\to t_0^+} F_X(t) = F_X(t_0)$.

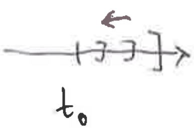
Proof (1) $\{X \le s\} \subset \{X \le t\} \Rightarrow \mathbb{P}(X \le s) \le \mathbb{P}(X \le t)$.

(2) If $t_n \nearrow \infty$, $\{X \le t_n\}$ is an increasing family,

The same for $t_n \searrow -\infty$

$\bigcup_{n=1}^{\infty} \{X \le t_n\} = \Omega$, so $\lim F_X(t_n) = \lim \mathbb{P}(X \le t_n) \underset{\text{continuity of } \mathbb{P}}{=} \mathbb{P}(\Omega) = 1$.

(3) If $t_n \searrow t_0$, then $\{X \le t_n\}$ is a decreasing family,

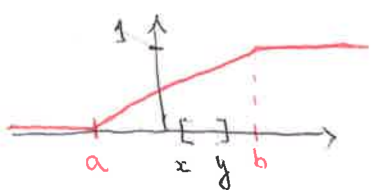


$\bigcap_{n=1}^{\infty} \{X \le t_n\} = \{X \le t_0\}$, so $\lim F_X(t_n) = F_X(t_0)$. $\square$

Measure theory

Thm If a function $F: \mathbb{R} \to \mathbb{R}$ satisfies (1) - (3), then it is the cdf of some r.v.

E.g.

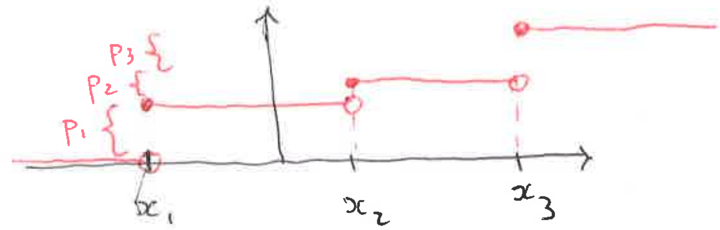


This is the dist fun of $X \sim \mathrm{Unif}([a,b])$ because

$$\mathbb{P}(x < X \le y) = \mathbb{P}(X \le y) - \mathbb{P}(X \le x) = \frac{y-a}{b-a} - \frac{x-a}{b-a} = \frac{y-x}{b-a}.$$

E.g. $\quad p_1 + p_2 + p_3 = 1, \quad p_1, p_2, p_3 > 0, \quad \mathbb{P}(X = x_i) = p_i$

F_X



⚠ $F_X(t_0-) = \lim_{t \to t_0-} F_X(t) = \mathbb{P}(X < t)$

Jumps: $\quad F_X(x_2) - F_X(x_2-) = \mathbb{P}(X \le x_2) - \mathbb{P}(X < x_2)$

$$= (p_1 + p_2) - p_1 = p_2 .$$

A r.v. X is called <u>continuous</u> if its dist. fun. F_X,

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\, dx$$

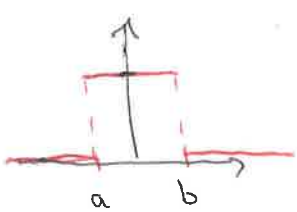
for some nonneg function $f_X : \mathbb{R} \to [0, \infty)$. We say then X has <u>(probability) density function</u> f_X (pdf).

⚠ X continuous $\Rightarrow$ F_X continuous

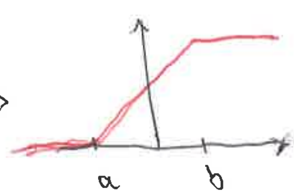
$\not\Leftarrow$

that is, no jumps, because $\lim_{t \to t_0-} F_X(t) = \lim_{t \to t_0-} \int_{-\infty}^{t} f_X = \int_{-\infty}^{t_0} f_X .$

E.g. $\quad X \sim \mathrm{Unif}([a,b]) \qquad F_X(t) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a,b] \\ 1 & x > b \end{cases}$



$\int \rightsquigarrow$



$$= \int_{-\infty}^{t} \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)\, dx$$

For a continuous r.v.

$$\frac{d}{dt} F_X(t) = f_X(t) \quad \text{for almost all } t$$

so if we are given F_X, we can find f_X by

$$f_X(x) = \begin{cases} F_X'(x) & \text{if this derivative exists} \\ 0 & \text{o/w} \end{cases}$$

<u>Properties</u> of the density f_X of a cts r.v. X

(1) $f_X \geq 0$

(2) $\int_{-\infty}^{\infty} f_X = 1$

Prob mass fun.

$p_X(x) \geq 0$

$\sum p_X(x) = 1$

~~Thm~~ (3) $\forall x \in \mathbb{R}$ $\quad \mathbb{P}(X=x)=0, \quad \mathbb{P}(X \leq x) = \mathbb{P}(X<x)$

(4) $\forall a<b \quad \mathbb{P}(a \underset{\leq}{<} X \underset{\leq}{<} b) = \int_a^b f_X(x)\,dx$

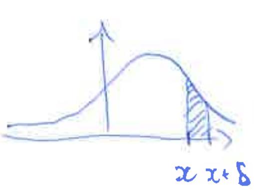
(5) $\forall A \in \underbrace{\mathcal{B}(\mathbb{R})}_{\text{Borel sets}} \quad \mathbb{P}(X \in A) = \int_A f_X$

<u>Proof</u> (2) $\int_{-\infty}^{\infty} f_X = \lim_{t \to \infty} \int_{-\infty}^{t} f_X = \lim_{t\to\infty} F_X(t) = 1.$

(3) already done: F_X is cts

(4) $\mathbb{P}(a<X<b) = F_X(b) - F_X(a) = \int_{-\infty}^{b} f_X - \int_{-\infty}^{a} f_X$

$= \int_a^b f_X.$



Meaning of $f_X(x)$: for small δ

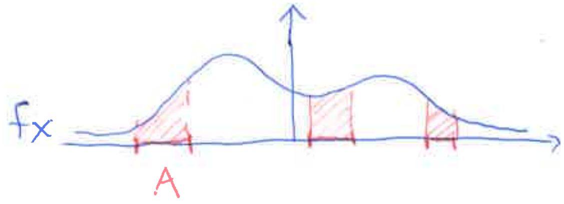
$$\mathbb{P}(x \leq X \leq x+\delta) = \int_x^{x+\delta} f_X \approx \delta \cdot f_X(x)$$

$f_X(x)$ can be >1

so it is not prob. that $X=x$, but

$f_X(x)\cdot dx = \mathbb{P}(X \in [x, x+dx])$

$\mathbb{P}(X \in A) =$ surface area under f_X over A

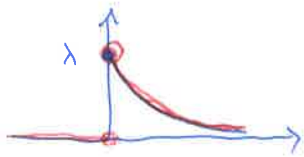


Important examples

- uniform distribution $X \sim \mathcal{U}nif(K)$, $K \subset \mathbb{R}$

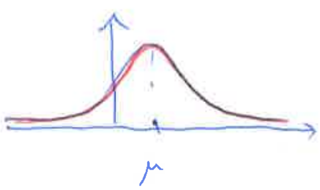
$$f_X(x) = \frac{1}{|K|} \mathbb{1}_K(x)$$

one-sided



- exponential distribution $X \sim \mathcal{E}xp(\lambda)$, $\lambda > 0$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , x \leq 0 \end{cases} = \lambda e^{-\lambda x} \mathbb{1}_{(0,\infty)}(x).$$

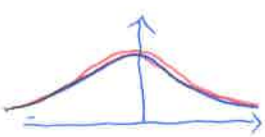


- Gaussian (aka normal) distribution $X \sim N(\underset{\text{mean}}{\mu}, \underset{\text{var}}{\sigma^2})$

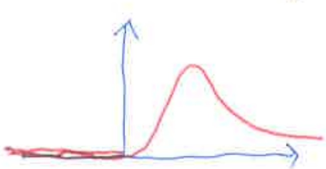
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\boxed{\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-x^2/2} = 1}$$

$N(0,1)$ = standard Gaussian $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.



- Cauchy distribution $f_X(x) = \frac{1}{\pi(1+x^2)}$.



- Gamma distribution $X \sim Gamma(\beta)$, $\beta > 0$

$$f_X(x) = \frac{1}{\Gamma(\beta)} x^{\beta-1} e^{-x} \mathbb{1}_{(0,\infty)}(x) \qquad \Gamma(\beta) = \int_0^\infty x^{\beta-1} e^{-x} dx$$

special case of Gamma

- $\chi^2(n)$ (chi squared with $n = 1, 2, \ldots$ degs of freedom)

$$f_X(x) = \frac{1}{2\Gamma(n/2)} \left(\frac{x}{2}\right)^{n/2-1} e^{-x/2} \mathbb{1}_{(0,\infty)}(x).$$

E.g. X cts r.v. with density f_X

$Y = aX + b$, $a > 0$, what density f_Y ?

- $F_Y(t) = \mathbb{P}(Y \le t) = \mathbb{P}(aX + b \le t) \underset{a>0}{=} \mathbb{P}\left(X \le \frac{t-b}{a}\right)$

$$= F_X\left(\frac{t-b}{a}\right)$$

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \frac{d}{dt} F_X\left(\frac{t-b}{a}\right) = \frac{1}{a} F_X'\left(\frac{t-b}{a}\right) = \frac{1}{a} f_X\left(\frac{t-b}{a}\right)$$

- another method: for $A \subset \mathbb{R}$

want to find

$$\mathbb{P}(Y \in A) = \int_A \boxed{f_Y}$$

$$= \mathbb{P}(aX + b \in A) = \int_{x:\, ax+b \in A} f_X(x)\, dx$$

change of variables $y = ax + b$, $dx = \frac{dy}{|a|}$

$$= \int_{y \in A} f_X\left(\frac{y-b}{a}\right) \frac{dy}{|a|}$$

$$= \int_A \underbrace{\boxed{\frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}}_{f_Y(y)} dy$$

E.g. $Y = X^2$

$$\mathbb{P}(Y \le t) = \mathbb{P}(X^2 \le t) = \begin{cases} 0 & , t < 0 \\ \underbrace{\mathbb{P}(-\sqrt{t} \le X \le \sqrt{t})}_{\mathbb{P}(X \le \sqrt{t}) - \mathbb{P}(X \le -\sqrt{t})} & , t \ge 0 \end{cases}$$

$$f_Y(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{d}{dt}\left(F_X(\sqrt{t}) - F_X(-\sqrt{t})\right), & t \ge 0 \end{cases}$$

$$= \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2\sqrt{t}}\left(f_X(\sqrt{t}) + f_X(-\sqrt{t})\right), & t > 0. \end{cases}$$

The expectation of a cts r.v. X with density f_X is

$$\mathbb{E}X = \int_{\mathbb{R}} x f_X(x)\, dx$$

(provided this integral converges absolutely, $\int_{\mathbb{R}} |x| f_X(x)\, dx < \infty$)

⚠ $\mathbb{E}$ has the usual properties of the integral

- $\mathbb{E}(aX + bY) = a\mathbb{E}X + b\mathbb{E}Y$
- $\underbrace{X \le Y \text{ a.s.}}_{\substack{\text{almost surely i.e.} \\ \mathbb{P}(X \le Y) = 1}} \Rightarrow \mathbb{E}X \le \mathbb{E}Y$

Thm If X is with density f_X, $\underset{\text{measurable}}{g: \mathbb{R} \to \mathbb{R}}$, then

$$\mathbb{E}g(X) = \int_{\mathbb{R}} g(x) f_X(x)\, dx$$

Proof Standard argument from measure theory

- OK for indicator functions $g(x) = 1_A(x)$

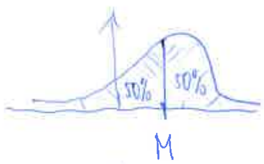
$$\text{LHS} = \mathbb{E}g(x) = \mathbb{E}\underbrace{\boxed{1_A(X)}}_{\substack{\text{discrete r.v.} \\ \text{values } 0,1}} = \mathbb{P}(1_A(X) = 1) = \mathbb{P}(X \in A)$$

$$\text{RHS} = \int_{\mathbb{R}} 1_A(x) f_X(x)\, dx = \int_A f_X(x)\, dx = \mathbb{P}(X \in A).$$

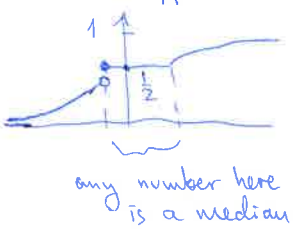
- OK for "simple" functions $g = \sum a_i 1_{A_i}$
- OK for all functions. □

Important quantities

- mean : $\mathbb{E}X$
- variance : $\operatorname{Var} X = \mathbb{E}(X-\mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$
- p^{th} moments : $\|X\|_p = (\mathbb{E}|X|^p)^{1/p}$, $p > 0$.
- median of X : a number M s.t. $\mathbb{P}(X \ge M), \mathbb{P}(X \le M) \ge \frac{1}{2}$.



e.g. for a random sign, its median is any number in $[-1,1]$.



E.g. • $U \sim \text{Unif}[0,1]$ $\quad f_U(x) = 1_{[0,1]}(x)$

$$\mathbb{E}U = \int_0^1 x\,dx = \frac{1}{2}, \qquad \mathbb{E}|U|^p = \int_0^1 x^p\,dx = \frac{1}{1+p}$$

$$\|U\|_p = \frac{1}{(1+p)^{1/p}}$$

$$\operatorname{Var} U = \mathbb{E}U^2 - (\mathbb{E}U)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\operatorname{Med}(U) = \frac{1}{2}$$

• $V \sim \text{Unif}[a,b]$, $\quad \underset{[a,b]}{V} = a + (b-a)\underset{[0,1]}{U}$

$$\mathbb{E}V = \mathbb{E}(a+(b-a)U) = a + (b-a)\mathbb{E}U = a + \frac{b-a}{2} = \frac{a+b}{2}$$

$\operatorname{Var}(X+b) = \operatorname{Var} X$

$\operatorname{Var}(aX) = a^2 \operatorname{Var} X$

$$\operatorname{Var} V = \operatorname{Var}(a+(b-a)U) = (b-a)^2 \operatorname{Var} U = \frac{(b-a)^2}{12}.$$

Med (X):

$\mathbb{P}(X \le M) = \frac{1}{2} = \mathbb{P}(X>M)$

$\int_M^\infty e^{-x} = \frac{1}{2}$

$e^{-M} = \frac{1}{2}$

$M = \log 2$

$\mathrm{Med}(X) < \mathbb{E}X$

- $X \sim \mathcal{E}xp(1)$, $f_X(x) = e^{-x} \mathbb{1}_{(0,\infty)}(x)$

$$\mathbb{E}X = \int_0^\infty x e^{-x} dx = \int_0^\infty x(-e^{-x})' dx \underset{\text{by parts}}{=} -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}$$

$$= 1$$

$$\mathbb{E}X^2 = \int_0^\infty x^2 e^{-x} dx = \dots = 2$$

$$\mathbb{E}X^n = n! \quad {}_{n=1,2,\dots} \qquad \mathbb{E}|X|^p = \int_0 x^p e^{-x} = \Gamma(p+1).$$

$$\mathrm{Var}\, X = 2 - 1^2 = 1.$$

- $Y \sim \mathcal{E}xp(\lambda)$ $\quad Y = \frac{1}{\lambda} X$, $\quad \mathbb{E}Y = \frac{1}{\lambda}$, $\mathrm{Var}\, Y = \frac{1}{\lambda^2}$.

- $X \sim N(0,1)$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$\mathbb{E}X = \int_{-\infty}^\infty \underbrace{x}_{\text{odd}} \frac{1}{\sqrt{2\pi}} \underbrace{e^{-x^2/2}}_{\text{even}} dx = 0$$

$$\mathbb{E}X^2 = \int_{\mathbb{R}} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \int_{\mathbb{R}} x (-e^{-x^2/2})' \frac{dx}{\sqrt{2\pi}}$$

$$= -xe^{-x^2/2}\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} = 1$$

$Y \sim N(\cancel{0}\mu, \sigma^2)$, $\quad Y = \mu + \sigma X$

$\mathbb{E}Y = \mu$, $\quad \mathrm{Var}\, Y = \sigma^2$

$\mathrm{Med}(X) = 0$, $\quad \mathrm{Med}(Y) = \mu$ $\quad$ (symmetry)