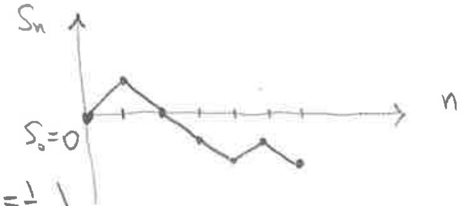
- SIMPLE RANDOM WALK -

Let $$S_n = S_0 + X_1 + \dots + X_n$$

S_0: starting point; indep; $X_1, \dots, X_n$ i.i.d

$\mathbb{P}(X_i = +1) = p$
$\mathbb{P}(X_i = -1) = 1-p$



The process $(S_n)_{n \geq 0}$ is called

<u>simple random walk</u> $\begin{pmatrix} \text{symmetric if } p = \frac{1}{2} \\ \text{asymmetric o/w} \end{pmatrix}$

In other words, $$S_{n+1} = \begin{cases} S_n + 1 & \text{with prob } p \\ S_n - 1 & \text{with prob } 1-p \end{cases}$$

(future depends only on the present, not past $\longrightarrow$ Markov process)

E.g. $$\mathbb{P}(S_n = S_0) = \mathbb{P}(X_1 + \dots + X_n = 0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{2m}{m} p^m (1-p)^m & \text{if } n = 2m \end{cases}$$

<u>Q</u> What is the chance that the walk revisits its starting point,

$$\beta = \mathbb{P}(\exists n \geq 1 \; S_n = S_0) \quad ?$$

Intuition:

- $p \neq q$ $\qquad \dfrac{S_n}{n} \xrightarrow[\text{a.s.}]{\text{LLN}} \mathbb{E}X_1 = p - q \rightsquigarrow \beta \overset{?}{<} 1$

- $p = q$ $\qquad \dfrac{S_n}{n} \xrightarrow{\text{LLN}} 0 \rightsquigarrow \beta \overset{?}{=} 1$

A walk is called recurrent if $\beta = 1$, <u>transient</u> if $\beta < 1$

The walk is recurrent iff it is sym

Thm. For a simple random walk $(S_n)_{n\geq 0}$,

$$\mathbb{P}(\ S_n = 0 \text{ for some } n\geq 1 \mid S_0=0) = 1-|p-q|.$$
$$= 1-|1-2p|.$$

Cor. For a sym. simple RW $(S_n)_{n\geq 0}$,

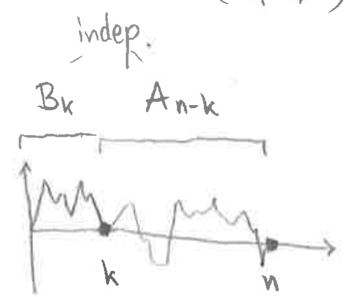
$$\mathbb{P}(\ S_n \text{ revisits } 0 \ \infty\text{-many times} \mid S_0=0) = 1.$$

Proof Idea:
$$\mathbb{P}(\ S_{n+m}=0 \text{ for some } m\geq 1 \mid S_n=0)$$
$$= \mathbb{P}(S_n=0 \text{ for some } n\geq 1 \mid S_0=0)$$
$$= 1 \quad \square$$

Proof of Thm We can assume $S_0=0$, let

revisit at n $\quad A_n = \{ S_n = 0 \}$ $\quad , n\geq 1$

first revisit at n $\quad B_n = \{ S_n=0, \quad S_k\neq 0, \ 1\leq k\leq n-1 \}$ $\quad , n\geq 1$
$(B_1=\emptyset)$



$$\mathbb{P}(A_n) = \mathbb{P}\left(A_n \cap \bigcup_{k=1}^{n} B_k \right)$$
$$= \sum_{k=1}^{n} \mathbb{P}(A_n\cap B_k)$$
$$\overset{\text{Markov}}{=} \sum_{k=1}^{n} \mathbb{P}(B_k)\mathbb{P}(A_{n-k})$$

Let $\quad u_k = \mathbb{P}(A_k)$, $\quad f_k = \mathbb{P}(B_k)$ $\quad (u_0=1)$

$$\boxed{u_n = \sum_{k=1}^{n} f_k u_{n-k}} \qquad n\geq 1$$

Goal: Compute $\beta = \mathbb{P}\left(\bigcup_n \{S_n = 0\}\right) = \mathbb{P}\left(\bigcup_n B_n\right)$

$$= \sum \mathbb{P}(B_n) = f_1 + f_2 + \dots$$

Gen. functions: $U(s) = \sum_{n=0}^{\infty} u_n s^n$, $F(s) = \sum_{n=0}^{\infty} f_n s^n$ $(f_0 = 0)$

(well def. for $|s|<1$)

$$U(s) - 1 = \sum_{n=1}^{\infty} u_n s^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} f_k u_{n-k} \right) s^n$$

$$= \sum_{k=1}^{\infty} f_k s^k \sum_{n=k}^{\infty} u_{n-k} s^{n-k}$$

$$= F(s) U(s), \qquad |s| < 1$$

$$F(s) = 1 - \frac{1}{U(s)}$$

$$\beta = \sum_{n=1}^{\infty} f_n = \lim_{s \nearrow 1-} F(s) = \lim_{s \nearrow 1-} \left(1 - \frac{1}{U(s)}\right)$$

⚠ so $\beta = 1$ iff $\sum_{n=1}^{\infty} u_n = \infty$.

We can even compute β,

$$U(s) = \sum_{n=0}^{\infty} u_n s^n = \sum_{m=0}^{\infty} \binom{2m}{m} p^m q^m (s^2)^m$$

$$\underset{\text{neg. binom}}{=} \left[1 - 4pq s^2\right]^{-1/2}$$

$$U(s) \xrightarrow[s \nearrow 1-]{} (1 - 4pq)^{-1/2}.$$

$$\beta = 1 - \sqrt{1 - 4pq} = 1 - \sqrt{1 - 4p(1-p)} = 1 - |1 - 2p| \quad \square$$

We just showed $\mathbb{P}(T<\infty)=1$

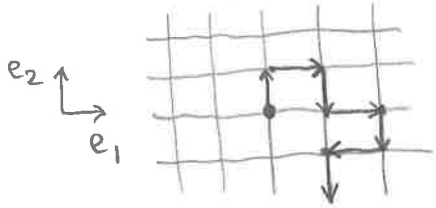
Thm Let $T = \min\{n \geq 1,\ S_n = 0\}$ (waiting time to return) for a sym. simple RW (S_n).

Then $\mathbb{E}T = \infty$.

Proof $$\mathbb{E}T = \sum_{n=1}^{\infty} n f_n \underset{\text{Abel}}{=} \lim_{s \nearrow 1-} \sum_{n=1}^{\infty} n f_n s^{n-1} = \lim_{s\nearrow 1-} F'(s)$$

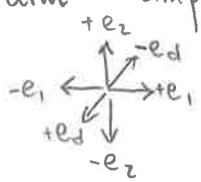
$$F(s) = 1-\sqrt{1-s^2}\,,\quad F'(s) = \frac{s}{\sqrt{1-s^2}} \longrightarrow = \infty. \qquad \square$$

2D Simple RW



$$S_{n+1} = \begin{cases} S_n + e_1 & 1/4 \\ S_n - e_1 & 1/4 \\ S_n + e_2 & 1/4 \\ S_n - e_2 & 1/4 \end{cases} \quad \text{with prob}$$

d-dim Simple RW



$$S_{n+1} = \left\{ S_n \pm e_i \right. \qquad \text{with prob. } \frac{1}{2d}$$

Thm
- $d=1$ (S_n) is recurrent
- $d=2$ (S_n) is recurrent
- $d\geq 3$ (S_n) is transient

Proof $\cdot\ d=1$ done

$\cdot\ d=2$
$$u_n = \mathbb{P}(S_n = 0) \overset{n=2m}{=} \sum_{k=0}^{2m} \binom{2m}{k}\binom{2m-k}{k}\binom{2m-2k}{m-k}\left(\frac{1}{4}\right)^{2m}$$
$$\Big(\underbrace{+\dots+}_{k}\underbrace{-\dots-}_{k},\ \underbrace{+\dots+}_{m-k}\underbrace{-\dots-}_{m-k}\Big)$$
$$= \binom{2m}{m}^2\left(\frac{1}{4}\right)^{2m} \underset{\text{Stirling}}{\approx} \frac{C}{m}$$

so $\sum u_n = \infty$, so, by $\triangle$!, $\underline{\beta = 1}$.

• $d \geq 3$

$$u_n \overset{n=2m}{=} \sum_{k_1+\dots+k_d=m} \binom{2m}{k_1}\binom{2m-k_1}{k_1}\dots\binom{2m-2k_1-\dots-2k_{d-1}}{k_d}\left(\frac{1}{2d}\right)^{2m}$$

$$= \left(\frac{1}{2d}\right)^{2m} \sum_{k_1+\dots+k_d=m} \frac{(2m)!}{k_1!^2 \cdot \dots \cdot k_d!^2}$$

$$= \underbrace{\left(\frac{1}{2}\right)^{2m}\binom{2m}{m}}_{\sim \frac{c}{\sqrt{m}}} \underbrace{\sum_{k_1+\dots+k_d=m}\left(\frac{m!}{d^m\, k_1! \cdot \dots \cdot k_d!}\right)^2}_{\leq \frac{c_d}{\sqrt{m}^{d-1}}}$$

• $\frac{1}{d^m}\frac{m!}{k_1! \cdot \dots \cdot k_d!} \overset{\text{equal}}{\leq} \frac{1}{d^m}\frac{m!}{\left(\frac{m}{d}\right)!^d} \approx \frac{c_d}{\sqrt{m}^{d-1}}$

$(\ )^2 = \underbrace{(\ )}_{\leq \max \leq \frac{c_d}{\sqrt{m}^{d-1}}} \cdot \underbrace{(\ )}$ add up to 1

• $\sum_{k_1+\dots+k_d=m} \frac{1}{d^m}\frac{m!}{k_1! \cdot \dots \cdot k_d!} = \frac{1}{d^m}(1+\dots+1)^m = 1$

$$\leq \frac{c_d}{\sqrt{m}^d}$$

For $d \geq 3$ $\sum u_n \leq \sum \frac{c_d}{m^{3/2}} < \infty$, so $\underline{\beta < 1}$. □