Probability 21-325 **Homework 3 (due 14th Feb)** **TT**

1. Find the expected value and variance of the distributions: Ber(p), Bin(n, p), Poiss(λ), Geom(p), for $p \in (0,1)$, $\lambda \in (0, \infty)$, $n \in \{1, 2, \ldots\}$.

2. Let X be a Poisson random variable with parameter λ. Show that for any nonnegative integer k we have

$$\mathbb{E}X(X-1)\cdot \ldots \cdot (X-k) = \lambda^{k+1}.$$

3. Let X be a discrete random variable taking values in the set of nonnegative integers. Show that

$$\mathbb{E}X = \sum_{k=0}^{\infty} \mathbb{P}\left(X > k\right).$$

Show the following generalisation: for an increasing function $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$,

$$\mathbb{E}f(X) = \sum_{k=0}^{\infty} [f(k+1) - f(k)] \cdot \mathbb{P}\left(X > k\right).$$

4. We throw repeatedly a fair die which has two faces coloured red, two yellow and two green. Let X be the random variable which takes the value n if all three colours occur in the first n throws but only two of the colours occur in the first $n-1$ throws. Find the expected value of X. *Hint: You may first want to compute the probability that not all colours occur in the first k throws and use the formula from Q3.*

5. We toss repeatedly a biased coin showing heads with probability $p \in (0,1)$. Let X be the number of tosses until n heads have occurred in a row. Find the expectation of X.

6. Let X and Y be discrete random variables, each taking two distinct values. Show that X and Y are independent if and only if $\mathbb{E}XY = \mathbb{E}X \cdot \mathbb{E}Y$.

7. Let X and Y be independent Poisson random variables with parameters λ and μ. Show that $X + Y$ is a Poisson random variable with parameter $\lambda + \mu$.

8. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be independent random signs. Show that the vectors $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1})$ and $(\varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_3, \ldots, \varepsilon_{n-1}\varepsilon_n)$ have the same distribution. In particular, the variables $\varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_3, \ldots, \varepsilon_{n-1}\varepsilon_n$ are independent. Are the variables $\varepsilon_1\varepsilon_2, \varepsilon_2\varepsilon_3, \varepsilon_3\varepsilon_1$ independent?

9. Let X be a discrete random variable. Show that X is zero with probability one if and only if $\mathbb{E}X^2 = 0$. Suppose that the expectation of X exists. Show that X has variance zero if and only if $X = \mathbb{E}X$.

10.* Show that the number of σ-fields on the n-element set equals $\frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$.