Probability 21-325 Homework 10 (due 23rd April) TT

1. Show that for positive t, $\int_t^\infty e^{-x^2/2}\mathrm{d}x \le \frac{1}{t}e^{-t^2/2}$ and $\int_t^\infty e^{-x^2/2}\mathrm{d}x \le \sqrt{\frac{\pi}{2}}e^{-t^2/2}$. Conclude that for a standard Gaussian random variable Z and positive t,

$$\mathbb{P}\left(Z > t\right) \le \frac{1}{\sqrt{2\pi}} \min\left\{\frac{1}{t}, \sqrt{\frac{\pi}{2}}\right\} e^{-t^2/2}.$$

2. Find the characteristic functions of random variables with distribution $\mathrm{Ber}(p)$, $\mathrm{Bin}(n,p)$, $\mathrm{Poiss}(\lambda)$, $\mathrm{Unif}([-1,1])$.

3. Let $X_1, X_2, \ldots$ be random variables such that $\mathbb{P}\left(X_n = \frac{k}{n}\right) = \frac{1}{n}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots$. Does the sequence (X_n) converge in distribution? If yes, find the limiting distribution.

4. Let $U_1, U_2, \ldots$ be i.i.d. random variables uniformly distributed on $[0,1]$. Let $X_n = \min\{U_1, \ldots, U_n\}$. Show that $\mathbb{E}X_n = \frac{1}{n+1}$. Show that nX_n converges in distribution to an exponential random variable with parameter one.

5. Let S be the number of ones when throwing a fair die 18000 times. Find a good approximation to $\mathbb{P}\left(2950 < S < 3050\right)$. How can you bound the error you make?

6. Let G be a standard Gaussian random vector in $\mathbb{R}^n$. Let $\|G\| = \sqrt{G_1^2 + \ldots + G_n^2}$ be its magnitude. Let $a_n = \mathbb{P}\left(\sqrt{n} - 1 \le \|G\| \le \sqrt{n} + 1\right)$. Find $a = \lim_{n\to\infty} a_n$ and show that $|a_n - a| \le \frac{8}{\sqrt{n}}$ for all $n \ge 1$.

7. Show that $e^{-n}\sum_{k=1}^n \frac{n^k}{k!} \xrightarrow[n\to\infty]{} \frac{1}{2}$.

Hint: Poiss(n) random variable is a sum of n i.i.d. Poiss(1) random variables.

8. Suppose that $X, X_1, X_2, \ldots$ are nonnegative integer-valued random variables. Show that $X_n \xrightarrow[n\to\infty]{d} X$, if and only if $\mathbb{P}\left(X_n = k\right) \xrightarrow[n\to\infty]{} \mathbb{P}\left(X = k\right)$, for every $k = 0, 1, 2, \ldots$.

9. Let $X_1, X_2, \ldots$ be i.i.d. standard Cauchy random variables. Show that for any reals $a_1, \ldots, a_n$, the sum $a_1X_1 + \ldots + a_nX_n$ has the same distribution as $(|a_1| + \ldots + |a_n|)X_1$.

10. Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. random signs. Show that $X_n = \sum_{k=1}^n \frac{\varepsilon_k}{2^k}$ converges in distribution to a random variable uniformly distributed on $(-1,1)$.

11. Suppose that a random variable X with variance one has the following property: $\frac{X+X'}{\sqrt{2}}$ has the same distribution as X, where X' is an independent copy of X. Show that $X \sim N(0,1)$.