Probability 21-325 **Final exam** **TT**

Instructions

Time: **180 minutes**

Books, notes, calculators, or any electronic devices **are not allowed**

Please write down your solutions for each question on an **individual** sheet

Please write down your name on each sheet in **capital** letters

Each question is worth 5 points. Question 7 is for extra credit and may be harder

Questions

1. You are dealt a poker hand of five cards from a regular deck of 52. What is the chance that you get three of a kind, e.g., three 7s or three aces (but not four of a kind or a full house)?

2. Find the probability generating function of a Poisson random variable with parameter λ. Let $X_1, \ldots, X_n$ be independent random variables with the Poisson distribution, each with parameter 1. Find the probability generating function of $S_n = X_1 + \ldots + X_n$. What is the distribution of S_n? What is the mean and variance of S_n? Prove that for positive t, $\mathbb{P}\left(S_n \geq (1+t)n\right) \leq \frac{1}{t^2 n}$. Show that $\lim_{n\to\infty} e^{-n} \sum_{k \geq 1.1n} \frac{n^k}{k!} = 0$.

3. Let X be a random variable with density $f(x) = \frac{1}{2}e^{-|x|}$. Find $\mathbb{E}X$ and $\mathbb{E}|X|$. Find the distribution function of X^2.

4. Let S_n be the number of heads after throwing n times a biased coin showing heads with probability 1/3. What is the mean and variance of S_n? Show that

$$\lim_{n\to\infty} \mathbb{P}\left(S_n > n/3 + \sqrt{n}\right) = \int_{\frac{3}{\sqrt{2}}}^{\infty} e^{-x^2/2} \frac{\mathrm{d}x}{\sqrt{2\pi}}.$$

5. What is the density of a standard Gaussian random variable, that is a Gaussian random variable with mean zero and variance one? Let X and Y be independent standard Gaussian random variables. What is the distribution of $\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$? Are the variables X and $X+Y$ independent? Are the variables $\frac{1}{2}X - \frac{\sqrt{3}}{2}Y$ and $\frac{\sqrt{3}}{2}X + \frac{1}{2}Y$ independent? Find the density of $\sqrt{X^2+Y^2}$.

6. Let f be a continuous function on $[0,1]$. Find $\lim_{n\to\infty} \int_0^1 \ldots \int_0^1 f\left(\frac{x_1+\ldots+x_n}{n}\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$ (or show it does not exist).

7.* Let $\varepsilon_1, \varepsilon_2, \ldots$ be independent random signs. Let $X_n = \frac{2}{n}\sum_{1\leq i<j\leq n} \varepsilon_i\varepsilon_j$. Does the sequence X_n converge in distribution? If yes, find its limit.