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Extra questions, first year maths students, Term 1 2014/2015

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Question 1. Suppose you take a friend of yours, go to a bar and order 8 pints of beer. The bartender brings you one 8-pint jug full of beer, and two empty jugs: one 3-pint and one 5-pint jug. How will you divide the 8 pints in half using the jugs provided?

Question 2. Prove that for every positive integer n we have

$$\frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq 4^n.$$

Question 3. Prove that for every real number x the following inequality holds

$$|x+1| + |x+2| + \ldots + |x+2014| \geq 1007^2.$$

When does the equality hold?

Question 4. Prove by induction that for every positive reals $a_1, a_2, \ldots, a_n$ satisfying $a_1 a_2 \cdot \ldots \cdot a_n = 1$ we have $a_1 + a_2 + \ldots + a_n \geq n$.

Question 5. Let $a_1, \ldots, a_n$ be positive real numbers and let $a_1', \ldots, a_n'$ be the same numbers but possibly reordered. Prove that

$$\frac{a_1}{a_1'} + \ldots + \frac{a_n}{a_n'} \geq n.$$

Question 6. Prove the following inequalities

(a) If a, b, b are positive real numbers, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

(b) If $0 < a_1 \leq a_2 \leq \ldots \leq a_n$, then

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_1} + \ldots + \frac{a_{n-2}}{a_{n-1}+a_n} + \frac{a_{n-1}}{a_n+a_1} + \frac{a_n}{a_1+a_2} \geq \frac{n}{2}.$$

(c) If $a_1, a_2, \ldots, a_n$ are arbitrary positive real numbers, then

$$\frac{a_1}{a_2+a_3} + \frac{a_2}{a_3+a_1} + \ldots + \frac{a_{n-2}}{a_{n-1}+a_n} + \frac{a_{n-1}}{a_n+a_1} + \frac{a_n}{a_1+a_2} \geq \frac{n}{4}.$$

Question 7. Prove that the limit of the sequence $a_n = \sin(n)$ when $n \to \infty$ does not exist.

Question 8. Examine the convergence of the sequence

$$a_n = \sum_{k=1}^{n} \frac{k}{n^2+k} = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \dots \frac{n}{n^2+n}.$$

Question 9. Does the series $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$ converge?

Question 10. Given an irrational number α prove that the set $\{\{k\alpha\} : k \in \mathbb{Z}\}$ is a dense subset of the interval $[0, 1]$, i.e. prove that for any numbers $0 < a < b < 1$ there exists an integer k such that $\{k\alpha\} \in (a, b)$.

Remark. The fractional part, denoted by $\{x\}$ for real x, is defined by the formula

$$\{x\} = x - \lfloor x \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the usual floor function.

Question 11. Does there exist a positive integer n such that the number $\sqrt[n]{\sqrt{2}+1} + \sqrt[n]{\sqrt{2}-1}$ is rational?

Question 12. Let φ be Euler's totien function, i.e. for a positive integer n we define $\varphi(n)$ to be the number of positive integers less than or equal to n that are relatively prime to n. Prove that for any positive integer n

$$\sum_{d|n} \varphi(d) = n,$$

where the sum is over all positive divisors of n.

Question 13. Nonzero integers a, b, c are chosen so that the number $a/b + b/c + c/a$ is an integer. Prove that abc is the cube of an integer.

Question 14. Prove that the cube of any integer can be written as the difference of two squares.

Question 15. Does there exist a non-abelian group with less than 6 elements?

Question 16. Show that if n is odd, it is not possible for a knight to visit all the squares of an $n \times n$ chessboard exactly once and return to its starting point.