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Combinatorics, Support Classes, Term 1 2014/2015

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Week 2

Class	1, 4, 6 (HW01)
Wishes	—

Extra Question. Prove that for every positive integer N we have

$$\sum_{k=1}^{N} \frac{(-1)^{k-1}}{k}\binom{N}{k} = \sum_{k=1}^{N} \frac{1}{k}.$$

Week 3

Class	2, 9 (HW01)
Wishes	?

Extra Question. Prove that for every positive integer N we have

$$1 + \sum_{k=1}^{N} \frac{(-1)^{k-1}}{k}\binom{N}{k}\left(1-\frac{k}{N}\right)^{N} = \sum_{k=1}^{N} \frac{1}{k}.$$

Week 4

Class	[3, 5, 7, 8 (HW01) optional] 2, 3 (HW02)
Wishes	?

Extra Question. Prove the identity

$$\sum_{k=0}^{m} \binom{n-k}{m-k} = \binom{n+1}{m}.$$

Extra Question. Let $\varphi(n)$ be the number of integers between 1 and n coprime to n. Prove that if n has the prime factorisation $p_1^{\alpha_1} \dots p_r^{\alpha_r}$, then

$$\begin{aligned}\varphi(n) &= \sum_{I \subset \{1,\dots,r\}} (-1)^{|I|} \frac{n}{\prod_{i \in I} p_i} \\ &= n \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).\end{aligned}$$

Week 5

Class	[3, 5, 7, 8 (HW01) optional] 6, 7, 10 (HW02)
Wishes	?

Extra Question. Prove the identity

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x(x-1)\dots(x-k+1) = x^n.$$

Week 6

Class	1, 2, 4 (HW 03)
Wishes	?

Extra Question. Decide whether it is possible for a spider to visit every vertex of the web shown in the picture below exactly once and return to the starting point. In other words, is this graph Hamiltonian?

Week 7

Class	6, 7, 9 (HW 03)
Wishes	?

Extra Question. Let L be the Laplacian of a graph G. Prove that the multiplicity of the zero eigenvalue of L equals the number of the connected components of G.

Extra Question. Let $G = (V, E)$ be a tree. Prove that the number of leaves in G equals

$$2 + \sum_{\substack{v \in V \\ \deg(v) \geq 3}} (\deg(v) - 2).$$

Conclude that if a tree has the maximum degree Δ, then it has at least Δ leaves.

Week 8

Class	6, 7, 8 (HW 04)
Wishes	?

Extra Question. A standard 52-card deck is divided into 13 piles of 4 cards each. Prove that it is possible to get all 13 ranks by selecting one card from every pile.

Extra Question. A *Latin rectangle* is an $r \times n$ matrix that has the numbers $1, 2, \ldots, n$ as its entries with no number occurring more than once in any row or column where $r \leq n$. An $n \times n$ Latin rectangle is called a *Latin square*.

Prove that if $r < n$, then it is possible to append $n - r$ rows to an $r \times n$ Latin rectangle to form a Latin square.

Week 9

Class	3, 8, 9 (HW 04)
Wishes	?

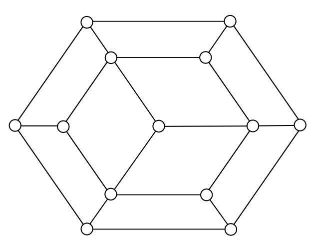

Figure 1: Is this web Hamiltonian?

Extra Question. Prove that if every vertex of a graph has degree at most k, then it is possible to colour its vertices with $k+1$ colours so that adjacent vertices have different colours.

Extra Question. Let us split the first n natural numbers into k classes. Prove that if $n \geq k!e$, then one of the classes constains three integers x, y, z with $x + y = z$.

Week 10

Class	Art gallery theorem
Wishes	?

Extra Question. In a few steps we shall prove the following result known as *the art gallery theorem*:

If S is a simple polygon with n vertices, then there is a set T of at most $n/3$ points of S such that for any point p of S there is a point q of T with the segment pq lying entirely in S.

A *simple polygon* is a plane figure that is bounded by a chain of nonintersecting straight line segments joined pairwise to form a closed path. A simple polygon with n vertices is sometimes called a simple n-gon. The measure of an *exterior angle* of a vertex A of a simple polygon is equal $180^\circ - \angle A$, where $\angle A$ is the measure of the interior angle at A. Mind that it can be negative! A vertex is called *convex* if the measure of its exterior angle is positive.

Step I. By imagining yourself as a turtle walking along the boundary of a simple n-gon show that the sum of its exterior angles equals 360°, thus the sum of its interior angles is $(n-2) \cdot 180^\circ$. Conclude that a simple n-gon has at least 3 convex vertices.

Step II. Prove inductively on n that a simple n-gon can be triangulated by drawing $n-3$ noncrossing diagonals.

Step III. Prove inductively on n that a triangulation of a simple n-gon is a 3-colourable graph meaning that it is possible to colour the vertices with 3 colours so that none adjacent vertices share have the same colour. Note that the vertices of each triangle will get different colours.

Step IV. Fix a colouring of a triangulation of a simple n-gon S. Choose the colour which appears the least number of times. The vertices coloured with it form the set T with $\#T \leq n/3$. Show that T is the desired set.

Extra Question (The discrete cube). Let $\Sigma_n = \{-1, 1\}^n$. This set equipped with the so-called *Hamming distance* $d_H(x, y) = \#\{i = 1, \dots, n,\ x_i \neq y_i\}$ is a metric space. Let Q_n be the graph with vertex set Σ_n and two vertices x, y connected with an edge if and only if they are 1 apart, $d_H(x, y) = 1$. The graph Q_n is sometimes called the *discrete cube* (its vertices are the corners of the usual cube $[-1, 1]^n$ connected with the usual edges).

The Laplacian L_n of Q_n is a $2^n \times 2^n$ matrix. It acts on $\mathbb{R}^{2^n}$ which can be viewed as the linear space of all real-valued functions defined on Σ_n For a subset $S \subset \{1, \dots, n\}$ consider the function $w_S(x_1, \dots, x_n) = \prod_{i \in S} x_i$. Show that w_S is an eigenvector of L_n. Find its corresponding eigenvalue. Find the number of spanning trees of Q_n.

Rules

1. There will be extra questions each week.
2. You can submit your solution to any extra question at any time during the term (no deadlines ;)).
3. Please submit your work by email or into my pigeon hole which is located on the first floor (opposite B1.38).
4. The author of the first correct solution of each question will receive a small prize (e.g. a chocolate bar, subject to my limited resources).
5. By Friday in Week n you can email your wishes for the Monday class in Week $n+1$, $n \in \{2, 3, \ldots, 9\}$.