Problem solving seminar
IMC Preparation, Set IV

Instructions

1. Work independently.
2. Do not use any books, notes, nor calculators.
3. Please write down your solutions for each problem on **individual** sheets
4. Please submit your work via pigeonholes opposite room B1.38 or email (Problems 1, 2, 3, 4 & 5 to TT) **by Friday, 16 May, 11:59 AM**
5. The ranking is available online at `http://www.mimuw.edu.pl/~tkocz/teaching_1314.php`. The winner will get a prize (a box of chocolates)!

Good luck!
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Problems

1. Consider the function f defined for positive real numbers x, y, z,

$$f(x,y,z) = \frac{(x+y+z)(xy+yz+zx)}{(x+y)(y+z)(z+x)}.$$

What is the image of f?

2. Let A, $B \in M_{2\times 2}(\mathbb{R})$ be such that $A^2 + B^2 = AB$. Show that $(AB - BA)^2 = 0$.

3. Let $n \geq 3$. Let $A_1A_2\ldots A_n$ be a regular n-gon inscribed in a circle with radius 1. Prove that

$$\prod_{k=1}^{n-1}(5 - |A_1A_{k+1}|^2) = F_n^2,$$

where the sequence (F_n) is defined recursively as $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$ (the Fibonacci sequence).

4. Fix $1 \leq k \leq n$. Let $A_1, \ldots, A_m$ be distinct subsets of the set $\{1, \ldots, n\}$ such that $|A_i \cap A_j| = k$ for all $i \neq j$. Prove that $m \leq n$.

Here $|A|$ denotes the cardinality of A.

5. Let $v_0, v_1, \ldots, v_n \in \mathbb{R}^n$ be vectors of length 1 such that $|v_i - v_j| > \sqrt{2}$ for all $i \neq j$. Prove that any n of them are linearly independent.