Problem solving seminar
IMC Preparation, Set III

Instructions

1. Work independently.
2. Do not use any books, notes, nor calculators.
3. Please write down your solutions for each problem on **individual** sheets
4. Please submit your work via pigeonholes opposite room B1.38 or email (Problems 1, 2 & 3 to RT, 4 & 5 to TT) **by Friday, 9 May, 11:59 AM**
5. The ranking is available online at `http://www.mimuw.edu.pl/~tkocz/teaching_1314.php`. The winner will get a prize (a box of chocolates)!

Good luck!
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Problems

1. Complex numbers a, b, c satisfy $a|bc| + b|ca| + c|ab| = 0$. Prove that

$$|(a-b)(b-c)(c-a)| \geq 3\sqrt{3}|abc|$$

2. Determine whether the series $\sum_{n=0}^{\infty} \arctan\left(\frac{1}{1+n+n^2}\right)$ converges. If so, compute its value.

3. The edges of a complete graph are painted with two colours, in such a way that for any four vertices there is a monochromatic triangle. Prove that it is possible to split the vertices into two groups such that each group is a complete monochromatic graph.

4. Let complex numbers $z_1, \ldots, z_n$, $w_1, \ldots, w_n$ be such that $z_k - w_l \neq 0$ for every k, l. Prove that

$$\det\left[\frac{1}{z_k - w_l}\right]_{k,l=1,\ldots,n} = \frac{\prod_{1\leq k<l\leq n}(z_l - z_k)(w_k - w_l)}{\prod_{1\leq k,l\leq n}(z_k - w_l)}.$$

5. Let $d \geq 2$ and let A be a bounded open subset of $\mathbb{R}^d$. Prove that there exist a finite or countable family $\mathcal{F}$ of pairwise disjoint closed balls such that $\bigcup_{B\in\mathcal{F}} B \subset A$ and $A \setminus \bigcup_{B\in\mathcal{F}} B$ is of measure zero.

A set $E \subset \mathbb{R}^d$ is of measure zero if for every $\epsilon > 0$ there are closed balls $B_1, B_2, \ldots$ such that $\bigcup_{i=1}^{\infty} B_i \supset E$ and $\sum_{i=1}^{\infty} |B_i| < \epsilon$, where $|B|$ denotes the volume of B.