15/01/2014

Problem solving seminar
Homework I

Instructions

1. Work independently.
2. There is no time limit, so do not rush.
3. Do not use any books, notes, nor calculators.
4. Bring your solutions to the class on 22nd January.

Good luck!
Tomasz Tkocz

Problems

1. Let $n \geq 2$ and let $x_1, \ldots, x_n$ be vectors in $\mathbb{R}^d$. Prove that there exists a subset $I \subset \{1, \ldots, n\}$ such that

$$4\left(\sum_{i \in I} x_i\right) \cdot \left(\sum_{i \notin I} x_i\right) \geq \sum_{i \neq j} x_i \cdot x_j,$$

where $\cdot$ denotes the standard scalar product. We adopt the convention that $\sum_{i \in \varnothing} x_i = 0$.

2. Given positive numbers $t_1, \ldots, t_n$ let $a_{ij} = \min\{t_i, t_j\}$, $i, j = 1, \ldots, n$. Prove that for every real numbers $x_1, \ldots, x_n$ we have

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j \geq 0.$$

3. Let $r \in (0,1)$ and denote $C_r = (1+r)/(1-r)$. Prove that for any real numbers $x_0, \ldots, x_n$ which are not all equal to zero

$$C_r^{-1} \sum_{k=0}^{n} x_k^2 < \sum_{0 \leq k, l \leq n} x_k x_l r^{|k-l|} < C_r \sum_{k=0}^{n} x_k^2.$$