Problem solving seminar

Tomasz Tkocz

Linear algebra

Remark. $M_{n\times n}(\mathbb{R})$ denotes the set of all $n \times n$ real matrices. I is the identity matrix.

Question 1. Let $A, B \in M_{n\times n}(\mathbb{R})$ satisfy $AB - BA = A$. Prove that $\det A = 0$.

Question 2. Let $A, B \in M_{2\times 2}(\mathbb{R})$ be such that for some positive integer n we have $(AB-BA)^n = I$. Prove that

(a) n is even,

(b) $(AB - BA)^4 = I$.

Definition. An $n \times n$ matrix A is called *nilpotent* if $A^m = 0$ for some positive integer m.

Question 3. Prove that if $A \in M_{n\times n}(\mathbb{R})$ is nilpotent, then $A^n = 0$.

Question 4. Let $A, B \in M_{n\times n}(\mathbb{R})$ be such that the matrices

$$A + t_1 B, A + t_2 B, \ldots, A + t_{n+1} B$$

are nilpotent, where $t_1, \ldots, t_{n+1}$ are some distinct real numbers. Prove that A and B are also nilpotent.

Question 5 (†)**.** Prove that $A \in M_{n\times n}(\mathbb{R})$ is nilpotent if and only if $\mathrm{tr} A^k = 0$ for $k = 1, \ldots, n$.

Question 6. Given matrices $A, B \in M_{n\times n}(\mathbb{R})$ prove that

(a) $\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}$,

(b) $\operatorname{rank} AB \geq \operatorname{rank} A + \operatorname{rank} B - n$.

Question 7. Let $A_1, \ldots, A_k \in M_{n\times n}(\mathbb{R})$ be rank $n - 1$ matrices. Prove that $k < n$ implies

$$A_1 \cdot \ldots \cdot A_k \neq 0.$$

Question 8. Let $A, B \in M_{n\times n}(\mathbb{R})$ satisfy $AB - BA = \alpha A$ for some real number α. Prove that

(a) $A^k B - BA^k = \alpha k A^k$ for $k \geq 1$,

(b) A is nilpotent.

Question 9. There are n people sitting at the round table. Each person has got a sticky note on which a prime is written down. Every minute a certain person modifies her or his number by multiplying it by a number of a neighbour. Is it possible that at some point there are two people who have got the same number?

Remark. † questions may be slightly harder.