Extra questions for first year maths students, Term 1 2012/2013

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Question 1. Prove that for any real number x the following inequality holds

$$|x+1| + |x+2| + \ldots + |x+2012| \geq 1006^2.$$

When does the equality hold?

Question 2. Find all differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying for all $x \in \mathbb{R}$ the following inequality

$$f'(x) \geq f(x).$$

Question 3. Let φ be Euler's totien function, i.e. for a positive integer n we define $\varphi(n)$ to be the number of positive integers less than or equal to n that are relatively prime to n. Prove that for any positive integer n

$$\sum_{d|n} \varphi(d) = n,$$

where the sum is over all positive divisors of n.

Question 4. Examine the convergence of the sequence

$$a_n = \sum_{k=1}^{n} \frac{k}{n^2+k} = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \ldots \frac{n}{n^2+n}.$$

Question 5. Given an irrational number α prove that the set $\{\{k\alpha\} : \ k \in \mathbb{Z}\}$ is a dense subset of the interval $[0,1]$, i.e. prove that for any numbers $0 < a < b < 1$ there exists an integer k such that $\{k\alpha\} \in (a,b)$.

Remark. The fractional part, denoted by $\{x\}$ for real x, is defined by the formula

$$\{x\} = x - \lfloor x \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the usual floor function.

Question 6. Given a parameter $\beta \in (0,1)$ prove that

$$\prod_{k=2}^{n} \left(1 - \frac{\beta}{k}\right) \xrightarrow[n\to\infty]{} 0.$$

Question 7. Let $u\colon (0,1) \longrightarrow \mathbb{R}$ be a differentiable function which satisfies

$$u'(t) \leq cu(t), \qquad \text{for all } t \in (0,1),$$

where c is some constant. Prove that

$$u(t) \leq u(0)e^{ct}, \qquad \text{for all } t \in (0,1).$$

Question 8. Find all integers a, b and c satisfying the equation

$$a^{2012} + b^{2012} - 8c^{1006} = 6.$$

Question 9. Does there exist a non-abelian group with less than 6 elements?

Question 10. Does the series $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$ converge?