Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

14/01/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 1

A few baby inequalities

Question 1. Prove that for real numbers $x_1, \ldots, x_n$ we have

$$|x_1 + \ldots + x_n| \leq |x_1| + \ldots + |x_n|.$$

Question 2. Given a natural number $n \geq 1$ find minimum value of $\sum_{i=1}^{n} \sqrt{a_i^2 + (2i-1)^2}$ subject to positive numbers a_i satisfying $\sum_{i=1}^{n} a_i = n^2$.

Question 3. Prove that for $x \geq 1$ and $h > 0$ we have

$$\sqrt{x+h} - \sqrt{x} \leq h/2.$$

Does this hold for all $x > 0$?

Question 4. Prove that for positive numbers $x_1, \ldots, x_n$ the arithmetic mean is greater or equal that the geometric mean,

$$\sqrt[n]{x_1 \cdot \ldots \cdot x_n} \leq \frac{x_1 + \ldots + x_n}{n}.$$

Question 5. Prove that for numbers $x_1, \ldots, x_n > -1$ that have the same sign we have

$$(1+x_1) \cdot \ldots \cdot (1+x_n) \geq 1 + x_1 \ldots + x_n.$$

Question 6 (Bernoulli's inequality)**.** Prove that for a real number $x > -1$ and a natural number $n \geq 1$ we have

$$(1+x)^n \geq 1 + nx.$$

Question 7 (The Cauchy-Schwarz inequality)**.** Prove that for real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ we have

$$\sum_{i=1}^{n} x_i y_i \leq \sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}.$$

Question 8 (Hölder's inequality)**.** Prove that for real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ and $p, q \geq 1$ which satisfy $1/p + 1/q = 1$ we have

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

Question 9. Prove that for $x \in (0, \pi/2)$ we have

$$\sin x < x < \tan x.$$

Question 10. Prove that for $x \in \mathbb{R}$ we have

$$1 + x \leq e^x.$$

Question 11. Prove that for $x > 0$ we have

$$\frac{x}{x+1} \leq \ln(1+x) \leq x.$$

A FEW ESTIMATES INVOLVING FACTORIALS

Question 12. Prove that

$$\left(\frac{n}{e}\right)^n \leq n! \leq 3\frac{n^{n+1}}{e^n}.$$

Conclude that $\sqrt[n]{n!}/n \longrightarrow 1/e$.

Question 13. Prove that

$$n^{n/2} \leq n! \leq \left(\frac{n+1}{2}\right)^n.$$

Question 14. Prove that

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k.$$

Question 15. Prove that

$$\frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq 4^n.$$

Question 16. Give a natural number $n \geq 1$ prove that the sequence $a_k = \binom{n}{k}$ is log-concave, i.e. $a_k^2 \geq a_{k-1}a_{k+1}$ for $k = 2, \ldots, n-1$.

Question 17 (†)**.** Prove that for a natural number $n \geq 1$ we have

$$\frac{e}{2n+2} < e - \left(1 + \frac{1}{n}\right)^n < \frac{e}{2n+1}.$$

Conclude that

$$\left(\frac{(1+1/n)^n}{e}\right)^n \xrightarrow[n\to\infty]{} 1/\sqrt{e}.$$

Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

21/01/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 2

Question 1 (♡)**.** Let $f(x) = \sum_{k=-2013}^{2013} |x-k|$. Given $c \in \mathbb{R}$ and $\epsilon > 0$ find $\delta > 0$ such that

$$\forall\, x \in \mathbb{R} \quad |x-c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

Hint: You may want to prove that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq 4027|x-y|.$$

Question 2 (♡)**.** Find all $a \in \mathbb{R}$ such that the function $f_a \colon \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f_a(x) = \begin{cases} \min\{1/|x|, a\}, & x \neq 0 \\ a^2 - 1, & x = 0 \end{cases}.$$

is continuous.

Question 3 (♡)**.** Let $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions. Prove that the functions $M(x) = \max\{f(x), g(x)\}$, $m(x) = \min\{f(x), g(x)\}$ are also continuous.

Question 4 (♡)**.** Suppose that for some $c \in \mathbb{R}$ a function $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following property

$$\forall (x_n)_{n=1}^{\infty} \quad x_n \to c \implies f(x_n) \to f(c).$$

Prove that f is continuous at c.

Remark. Cf. Exercise 6 from Assignment 1.

Question 5 (♠)**.** Let $f, g \colon \mathbb{R} \longrightarrow \mathbb{R}$ be bounded below, say $f(x), g(x) \geq 0$ for all $x \in \mathbb{R}$. Suppose that $|g(x) - g(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that the function $f \Box g$ defined by

$$(f \Box g)(x) = \inf_{t \in \mathbb{R}} \{f(x) + g(x-t)\}.$$

is continuous.

Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

28/01/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 3

Question 1 (♡)**.** Let $f\colon [0,+\infty) \longrightarrow [0,+\infty)$ satisfy for all $x, y \geq 0$

$$|f(x) - f(y)| \leq q|x-y|,$$

with some constant $q \in (0,1)$. Fix $x_0 \geq 0$ and define recursively the sequence $(x_n)_{n\geq 0}$ by $x_{n+1} = f(x_n)$, $n \geq 0$. Prove that it converges. What can be said about the limit?

Question 2 (♡)**.** Let $f(x) = \sqrt{1+x}$ for $x \geq 0$. Prove that for any $x_0 \geq 0$ the sequence $(x_n)_{n\geq 0}$ defined recursively by $x_{n+1} = f(x_n)$ for $n \geq 0$ converges and compute the limit.

Question 3 (♡)**.** Define the function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ |\sin x| & \text{if } x \notin \mathbb{Q}. \end{cases}$$

At which points is f continuous?

Definition. We say that a function $f\colon (A,B) \longrightarrow \mathbb{R}$ possesses *the Intermediate Value Property* if for any $a < b$ in the domain such that $f(a) \neq f(b)$, and any z between $f(a)$ and $f(b)$ there is some $c \in (a,b)$ between a and b with $f(c) = z$.

Question 4 (♠)**.** Give an example of a function which is *not* continuous, and yet has got the Intermediate Value Property.

Question 5 (♡)**.** Prove that the equation $(1-x)\cos x = \sin x$ has a solution in the interval $(0,1)$.

Question 6 (♡)**.** Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a T - periodic continuous function, i.e. $f(x+T) = f(x)$ for all $x \in \mathbb{R}$, where $T > 0$ is the period. Prove that there exists x_0 such that

$$f(x_0 + T/2) = f(x_0).$$

Question 7 (♡/♠)**.** Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be *additive*, i.e. satisfy for all $x, y \in \mathbb{R}$ *Cauchy's functional equation*

$$f(x+y) = f(x) + f(y).$$

Prove that

(a) $f(0) = 0$

(b) $f(-x) = f(x)$

(c) For any $x_1, \ldots, x_n \in \mathbb{R}$ we have $f(x_1 + \ldots + x_n) = f(x_1) + \ldots + f(x_n)$

(d) For an integer k and a real number x we have $f(kx) = kf(x)$

(e) For a rational number q we have $f(q) = qf(1)$

(f) In addition, if f satisfies one of these assumptions:

   (i) f is continuous
   (ii) f is continuous at one point
   (iii) f is monotone
   (iv) f is bounded above/below

   then $f(x) = xf(1)$ for all $x \in \mathbb{R}$.

Questions difficulty legend: $\heartsuit$ - easy $\spadesuit$ - medium $\bigstar$ - hard $\dagger$ - very hard

04/02/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 4

Question 1 ($\heartsuit/\spadesuit$)**.** Compute the following limit (if it exists)

(a) $\lim_{x\to 0} x\cos\frac{1}{x}$

(b) $\lim_{x\to+\infty} x\left(\sqrt{x^2+1}-\sqrt[3]{x^3+1}\right)$

(c) $\lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2}\cos x\right)}{\sin(\sin x)}$.

Question 2 ($\heartsuit$)**.** Prove that for any $x\in\mathbb{R}$ we have $x-1<\lfloor x\rfloor\le x$.

Question 3 ($\heartsuit$)**.** Compute the following limit (if it exists)

(a) $\lim_{x\to 0} x\left\lfloor\frac{1}{x}\right\rfloor$

(b) $\lim_{x\to 0}\frac{\lfloor x\rfloor}{x}$.

(c) $\lim_{x\to 0} x^2\left(1+2+\ldots+\left\lfloor\frac{1}{|x|}\right\rfloor\right)$.

Question 4 ($\spadesuit$)**.** Let $f\colon\mathbb{R}\longrightarrow\mathbb{R}$ be an increasing function such that $\lim_{x\to\infty}\frac{f(2x)}{f(x)}=1$. Prove that $\lim_{x\to\infty}\frac{f(cx)}{f(x)}=1$ for every $c>0$.

Question 5 ($\bigstar$)**.** Let $f\colon[0,\infty)\longrightarrow\mathbb{R}$ possess the property: *for every $a\ge 0$ the limit $\lim_{n\to\infty}f(a+n)$ exists and equals* 0. Does it imply that the limit $\lim_{x\to\infty}f(x)$ exists?

Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

11/02/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 5

Question 1 (♡). Prove that

(a) $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$, $n \geq 0$

(b) $\lim_{x\to\infty} x^n e^{-x} = 0$, $n \geq 0$

(c) $\lim_{x\to 0+} x^p \ln x = 0$, $p \in (0,1)$

(d) $\lim_{x\to\infty} \frac{\ln^n x}{x} = 0$, $n \geq 0$.

Question 2 (♠). Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be given by the formula

$$f(x) = \begin{cases} e^{-1/x^2}, & x > 0 \\ 0, & x \leq 0 \end{cases}.$$

Is f differentiable? What can you say about the second derivative? About the higer order derivatives?

Question 3 (♡). Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function with $\sup_{x\in\mathbb{R}} |f'(x)| = L < \infty$. Prove that f is L-*Lipschitz*, i.e.

$$|f(x) - f(y)| \leq L|x-y|, \qquad \text{for all reals } x, y.$$

Question 4 (★). Find all differentiable functions $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ satisfying for all reals $x \neq y$

$$\frac{f(y)-f(x)}{y-x} = f'\left(\frac{x+y}{2}\right).$$

What is the geometric interpretation of this equation?

Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

18/02/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 6

Question 1 (♠). Let $f\colon (a,b) \longrightarrow \mathbb{R}$ be differentiable. Prove that f' possesses the intermediate value property.

Question 2 (♠). Does there exist a differentiable function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f'(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0\,? \\ 1, & x > 0 \end{cases}$$

Question 3 (♡). Suppose that $f(0) = 0$ and that $f'(0)$ exists. Given a positive integer k compute

$$\lim_{x\to 0} \frac{1}{x}\left(f(x) + f\left(\frac{x}{2}\right) + \ldots + f\left(\frac{x}{k}\right)\right).$$

Question 4 (♠). Let $f(x) = a_1 \sin x + a_2 \sin(2x) + \ldots + a_n \sin(nx)$, where $a_1, a_2, \ldots, a_n$ are reals. Prove that if $|f(x)| \le |\sin x|$ for every $x \in \mathbb{R}$ then $|a_1 + 2a_2 + \ldots + na_n| \le 1$.

25/02/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 7

Question 1 (♡). Prove that if $|x| < 1/2$ then the approximate formula

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

gives the value of $\sqrt{1+x}$ with the error at most $\frac{1}{2}|x|^3$.

Observe that $\sqrt{1+1/8} = \frac{3\sqrt{2}}{4}$ and find an approximate value of $\sqrt{2}$. What is the error?

Question 2 (Bernoulli's inequality in full glory ♡). Let $x > -1$, $x \neq 0$. Prove that

(a) $(1+x)^\alpha > 1 + \alpha x$, if $\alpha > 1$ or $\alpha < 0$

(b) $(1+x)^\alpha < 1 + \alpha x$, if $0 < \alpha < 1$.

Question 3 (♡). Prove that if $f''(x)$ exists then

$$f''(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}.$$

Question 4 (♠). Let $f \colon \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and

$$M_k = \sup\{|f^{(k)}(x)|; x \in \mathbb{R}\} < \infty, \qquad k = 0, 1, 2.$$

Prove that

$$M_1 \leq \sqrt{2M_0M_2}.$$

Question 5 (♡). Find

$$\lim_{x \to 0} \frac{e^{x^2/2} - 1}{\cosh x - 1}.$$

Question 6 (♠). Prove that

(a) $\cosh x \leq e^{x^2/2}$ for $x \in \mathbb{R}$

(b) $\cos x \leq e^{-x^2/2}$ for $x \in [0, \pi/2]$.

Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

04/03/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 8

Question 1 (♡/♠). Evaluate

(a) $\lim_{x\to 0} \frac{\ln(1+ex)}{x}$.

(b) $\lim_{x\to 1} \frac{\arctan\left(\frac{x^2-1}{x^2+1}\right)}{x-1}$.

(c) $\lim_{x\to\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right)$.

(d) $\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x}$.

(e) $\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x^2}$.

Question 2 (♡). Determine the interval of convergence for the series

(a) $\sum_{n=1}^\infty 2^{n^2} x^{n!}$.

(b) $\sum_{n=1}^\infty \left(1+\frac{1}{n}\right)^{(-1)^n n^2} x^n$.

Question 3 (♡). Recall the definition of $\underline{\lim}_{n\to\infty} a_n$ and $\overline{\lim}_{n\to\infty} a_n$. Prove that

(a) if $a_n \le b_n$ eventually, then $\overline{\lim}\, a_n \le \overline{\lim}\, b_n$.

(b) $\overline{\lim}(a_n+b_n) \le \overline{\lim}\, a_n + \overline{\lim}\, b_n$.

(c) $\overline{\lim}\, |a_n b_n| \le \overline{\lim}\, |a_n| \cdot \overline{\lim}\, |b_n|$. Show that the inequality can be strict.

(d) $\underline{\lim}\, \min\{a_n, b_n\} = \min\{\underline{\lim}\, a_n, \underline{\lim}\, b_n\}$.

Question 4 (♠). Suppose that $f\colon (-1,1) \longrightarrow \mathbb{R}$ is a function of class C^2 such that $f(0) = 0$. Compute

$$\lim_{x\to 0+} \sum_{k=1}^{\lfloor 1/\sqrt{x}\rfloor} f(kx).$$

Questions difficulty legend: ♡ - easy ♠ - medium ★ - hard † - very hard

11/03/2013

Analysis II, Term 2 2012/2013

Tomasz Tkocz

Support class 9

Question 1 (♡). Evaluate

(a) $\lim_{x\to 5}(6-x)^{1/(x-5)}$.

(b) $\lim_{x\to\infty}\frac{x-\sin x}{2x+\sin x}$.

Question 2 (♡/♠). Prove that for $x\in(0,\pi/2)$ and a positive integer n we have

(a) $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+\frac{x^{4n-3}}{(4n-3)!}-\frac{x^{4n-1}}{(4n-1)!}<\sin x<x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+\frac{x^{4n-3}}{(4n-3)!}-\frac{x^{4n-1}}{(4n-1)!}+\frac{x^{4n+1}}{(4n+1)!}$.

(b) $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\frac{x^{4n-4}}{(4n-4)!}-\frac{x^{4n-2}}{(4n-2)!}<\cos x<1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots+\frac{x^{4n-4}}{(4n-4)!}-\frac{x^{4n-2}}{(4n-2)!}+\frac{x^{4n}}{(4n)!}$.

Do these inequalities hold for $x\geq\pi/2$ as well?

Question 3 (♡). Prove that $e^x\geq 1+x$ for $x\in\mathbb{R}$ and then derive the inequality between means

$$\frac{x_1+\ldots+x_n}{n}\geq\sqrt[n]{x_1\cdot\ldots\cdot x_n}.$$

Question 4 (♡). Prove the inequality

(a) $1-1/x\leq\ln x\leq x/e$ for $x>0$.

(b) $2\tan x>\sinh x$ for $x\in(0,\pi/2)$.

Remark. Combining the inequalities $\cosh x\leq e^{x^2/2}$ and $\cos x\leq e^{-x^2/2}$ (see Support class 7, Question 6), one can actually show $c\tan x>\sinh x$ with $c=1$ which is sharp.