Log-concavity and log-convexity of moments of averages of i.i.d. random variables

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Abstract

We show that the sequence of moments of order less than 1 of averages of i.i.d. positive random variables is log-concave. For moments of order at least 1, we conjecture that the sequence is log-convex and show that this holds eventually for integer moments (after neglecting the first \( p^2 \) terms of the sequence).

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1 Introduction

Suppose \( X_1, X_2, \ldots \) are i.i.d. copies of a positive random variable and \( f \) is a nonnegative function. This article is concerned with certain combinatorial properties of the sequence

\[
a_n = E f \left( \frac{X_1 + \cdots + X_n}{n} \right), \quad n = 1, 2, \ldots
\]

For instance, \( f(x) = x^p \) is a fairly natural choice leading to the sequence of moments of averages of the \( X_i \). Since we have the identity

\[
\sum_{i=1}^{n+1} x_i = \sum_{i=1}^{n+1} \frac{\sum_{j: j \neq i} x_j}{n},
\]

we conclude that the sequence \( (a_n)_{n=1}^{\infty} \) is nondecreasing when \( f \) is convex. What about inequalities involving more than two terms?

Such inequalities have been studied to some extent. One fairly general result is due to Boland, Proschan and Tong from [1] (with applications in reliability theory). It asserts in particular that

\[
E \phi (X_1 + \cdots + X_n, X_{n+1} + \cdots + X_{2n}) \leq E \phi (X_1 + \cdots + X_{n-1}, X_n + \cdots + X_{2n})
\]

for a symmetric (invariant under permuting coordinates) continuous random vector \( X = (X_1, \ldots, X_{2n}) \) with nonnegative components and a symmetric convex function \( \phi : [0, +\infty)^2 \rightarrow \mathbb{R} \).

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We obtain a satisfactory answer to a natural question of log-convexity/concavity of sequences \((a_n)\) for completely monotone functions, also providing insights into the case of power functions.

2 Results

Recall that a nonnegative sequence \((x_n)_{n=1}^{\infty}\) supported on a set of contiguous integers is called log-convex (resp. log-concave) if \(x_n^2 \leq x_{n-1}x_{n+1}\) (resp. \(x_n^2 \geq x_{n-1}x_{n+1}\)) for all \(n \geq 2\) (for background on log-convex/concave sequences, see for instance \([4, 6]\)). One of the crucial properties of log-convex sequences is that log-convexity is preserved by taking sums (which follows from the Cauchy-Schwarz inequality, see for instance \([4]\)).

Recall that an infinitely differentiable function function \(f : (0, \infty) \rightarrow (0, \infty)\) is called completely monotone if we have \((-1)^nf^{(n)}(x) \geq 0\) for all positive \(x\) and \(n = 1, 2, \ldots\); equivalently, by Bernstein’s theorem (see for instance \([3]\)), the function \(f\) is the Laplace transform of a nonnegative Borel measure \(\mu\) on \([0, +\infty)\), that is
\[
f(x) = \int_0^\infty e^{-tx}d\mu(t). \tag{2}
\]
For example, when \(p < 0\), the function \(f(x) = x^p\) is completely monotone. Such integral representations are at the heart of our first two results.

**Theorem 1.** Let \(f : (0, \infty) \rightarrow (0, \infty)\) be a completely monotone function. Let \(X_1, X_2, \ldots\) be i.i.d. positive random variables. Then the sequence \((a_n)_{n=1}^{\infty}\) defined by (1) is log-convex.

**Theorem 2.** Let \(f : [0, \infty) \rightarrow [0, \infty)\) be such that \(f(0) = 0\) and its derivative \(f'\) is completely monotone. Let \(X_1, X_2, \ldots\) be i.i.d. nonnegative random variables. Then the sequence \((a_n)_{n=1}^{\infty}\) defined by (1) is log-concave.

In particular, applying these to the functions \(f(x) = x^p\) with \(p < 0\) and \(0 < p < 1\) respectively, we obtain the following corollary.

**Corollary 3.** Let \(X_1, X_2, \ldots\) be i.i.d. positive random variables. The sequence
\[
b_n = E\left(\frac{X_1 + \cdots + X_n}{n}\right)^p, \quad n = 1, 2, \ldots \tag{3}
\]
is log-convex when \(p < 0\) and log-concave when \(0 < p < 1\).

For \(p > 1\), we pose the following conjecture.

**Conjecture 1.** Let \(p > 1\). Let \(X_1, X_2, \ldots\) be i.i.d. nonnegative random variables. Then the sequence \((b_n)\) defined in (3) is log-convex.

We offer a partial result supporting this conjecture.

**Theorem 4.** Let \(X_1, X_2, \ldots\) be i.i.d. nonnegative random variables, let \(p\) be a positive integer and let \(b_n\) be defined by (3). Then for every \(n \geq p^2\), we have \(b_n^2 \leq b_{n-1}b_{n+1}\).
Remark 5. When \( p = 2 \), we have \( b_n = \frac{n \mathbb{E} X_1^2 + n(n-1)\mathbb{E} X_1^2}{n^2} = (\mathbb{E} X_1)^2 + n^{-1} \text{Var}(X_1) \), which is clearly a log-convex sequence (as a sum of two log-convex sequences). The following argument for \( p = 3 \) was kindly communicated to us by Krzysztof Oleszkiewicz: when \( p = 3 \), we can write

\[
b_n = (\mathbb{E} X_1)^3 + (\mathbb{E} X_1)^3 - 2(\mathbb{E} X_1^2)(\mathbb{E} X_1) n^{-2} + (\mathbb{E} X_1) \text{Var}(X_1)(3n^{-1} - n^{-2}).
\]

The sequences \((n^{-2})\) and \((3n^{-1} - n^{-2})\) are log-convex. By the Cauchy-Schwarz inequality, the factor at \( n^{-2} \) is nonnegative,

\[
\mathbb{E} X_1^2 \leq \sqrt{\mathbb{E} X_1^3} \sqrt{\mathbb{E} X_1} \leq \frac{\mathbb{E} X_1^3}{2\mathbb{E} X_1} + \frac{(\mathbb{E} X_1)^2}{2},
\]

so again \((b_n)\) is log-convex as a sum of three log-convex sequences.

It remains elusive how to group terms and proceed along these lines in general. Our proof of Theorem 4 relies on this idea, but uses a straightforward way of rearranging terms.

The rest of this paper is occupied with the proofs of Theorems 1, 2, 4 (in their order of statement) and then we conclude with additional remarks and conjectures.

3 Proofs

3.1 Proof of Theorem 1

Suppose that \( f \) is completely monotone. Using (2) and independence, we have

\[
a_n = \mathbb{E} f \left( \frac{X_1 + \ldots + X_n}{n} \right) = \int_0^\infty \left[ \mathbb{E} e^{-tX_1/n} \right]^n d\mu(t).
\]

Let \( u_n(t) = \left[ \mathbb{E} e^{-tX_1/n} \right]^n \). It suffices to show that for every positive \( t \), the sequence \((u_n(t))\) is log-convex (because sums/integrals of log-convex sequences are log-convex: the Cauchy-Schwarz inequality applied to the measure \( \mu \) yields

\[
\left( \int \sqrt{u_{n-1}(t)u_{n+1}(t)} d\mu(t) \right)^2 \leq \left( \int u_{n-1}(t)d\mu(t) \right) \left( \int u_{n+1}(t)d\mu(t) \right),
\]

which combined with \( u_n(t) \leq \sqrt{u_{n-1}(t)u_{n+1}(t)} \), gives \( a_n^2 \leq a_{n-1}a_{n+1} \). The log-convexity of \((u_n(t))\) follows from Hölder’s inequality,

\[
\mathbb{E} e^{-tX_1/n} = \mathbb{E} e^{-\frac{tX_1}{n+1}} e^{-\frac{tX_1}{n+1}} \leq \left( \mathbb{E} e^{-\frac{tX_1}{n+1}} \right)^{\frac{n+1}{2n}} \left( \mathbb{E} e^{-\frac{tX_1}{n+1}} \right)^{\frac{n+1}{2n}},
\]

which finishes the proof.
3.2 Proof of Theorem 2

Suppose now that \( f(0) = 0 \) and \( f' \) is completely monotone, say \( f'(x) = \int_0^\infty e^{-tx}d\mu(t) \) for some nonnegative Borel measure \( \mu \) on \((0, \infty)\) (by (2)). Introducing a new measure \( d\nu(t) = \frac{1}{t}d\mu(t) \) we can write

\[
 f(y) = f(y) - f(0) = \int_0^y f'(x)dx = \int_0^\infty \int_0^\infty te^{-tx}1_{(0<x<y)}dxd\nu(t). 
\]

Integrating against \( dx \) gives

\[
 f(y) = \int_0^\infty [1 - e^{-ty}]d\nu(t). 
\]

Let \( F \) be the Laplace transform of \( X_1 \), that is \( F(t) = \mathbb{E}e^{-tX_1}, \quad t > 0 \).

Then

\[
\mathbb{E}f\left(\frac{X_1 + \ldots + X_n}{n}\right) = \int_0^\infty \left[1 - F(t/n)^n\right]d\nu(t) = \int_0^\infty G(n,t)d\nu(t),
\]

where, to shorten the notation, we introduce the following nonnegative function

\[
 G(\alpha,t) = 1 - F(t/\alpha)^\alpha, \quad \alpha,t > 0.
\]

To show the inequality

\[
 \left[\mathbb{E}f\left(\frac{X_1 + \ldots + X_n}{n}\right)\right]^2 \geq \mathbb{E}f\left(\frac{X_1 + \ldots + X_{n-1}}{n-1}\right) \cdot \mathbb{E}f\left(\frac{X_1 + \ldots + X_{n+1}}{n+1}\right) 
\]

it suffices to show that pointwise

\[
 G(n,s)G(n,t) \geq \frac{1}{2}G(n-1,s)G(n+1,t) + \frac{1}{2}G(n+1,s)G(n-1,t),
\]

for all \( s, t > 0 \). This follows from two properties of the function \( G \):

1) for every fixed \( t > 0 \) the function \( \alpha \mapsto G(\alpha,t) \) is nondecreasing,

2) the function \( G(\alpha,t) \) is concave on \((0, \infty) \times (0, \infty)\).

Indeed, by 2) we have

\[
 G(n,s)G(n,t) \geq \frac{G(n-1,s) + G(n+1,s)}{2} \cdot \frac{G(n-1,t) + G(n+1,t)}{2},
\]

so it suffices to prove that

\[
 G(n-1,s)G(n-1,t) + G(n+1,s)G(n+1,t) \\
 - G(n-1,s)G(n+1,t) - G(n+1,s)G(n-1,t) \\
 = \left[G(n-1,s) - G(n+1,s)\right] \cdot \left[G(n-1,t) - G(n+1,t)\right]
\]
is nonnegative, which follows by 1).

It remains to prove 1) and 2). To prove the former notice that $F(t/\alpha)^\alpha = \left(\mathbb{E}e^{-tX/\alpha}\right)^\alpha$ is the $1/\alpha$-moment of $e^{-tX}$. To prove the latter notice that by Hölder’s inequality the function $t \mapsto \ln F(t)$ is convex. Therefore its perspective function $H(\alpha, t) = \alpha \ln F(t/\alpha)$ is convex (see, e.g. Ch. 3.2.6 in [2]), which implies that $F(t/\alpha)^\alpha = e^{H(\alpha, t)}$ is also convex.

### 3.3 Proof of Theorem 4

We recall a standard combinatorial formula: first by the multinomial theorem and independence, we have

$$
\mathbb{E}(X_1 + \cdots + X_n)^p = \sum_{m=1}^{n-1} \frac{p!}{q_1! \cdots q_m!} \mathbb{E}(X_1^{q_1} \cdots X_n^{q_m}) = \sum_{m=1}^{n-1} \frac{p!}{q_1! \cdots q_m!} \mu_{q_1} \cdots \mu_{q_m},
$$

where the sum is over all sequences $(q_1, \ldots, q_m)$ of nonnegative integers such that $n \geq m$ and we denote $\mu_k = \mathbb{E}X_1^k$, $k \geq 0$. Now we partition the summation according to the number $m$ of positive terms in the sequence $(q_1, \ldots, q_m)$: if $Q_m$ is the set of all sequences $q = (q_1, \ldots, q_m)$ of length $m$ of positive integers with $q_1 + \cdots + q_m = p$, we can write

$$
\mathbb{E}(X_1 + \cdots + X_n)^p = \sum_{m=q \in Q_m} \frac{p!}{q_1! \cdots q_m!} \mu_{q_1} \cdots \mu_{q_m},
$$

where $\alpha(q) = l_1! \cdots l_h!$ for $q = (q_1, \ldots, q_m)$ with $h$ distinct terms such that there are $l_1$ terms of type 1, $l_2$ terms of type 2, etc. (so $l_1 + \cdots + l_h = m$). The factor $\frac{n!}{\alpha(q)(n-m)!}$ arises because given a sequence $q \in Q_m$, there are exactly

$$
\binom{n}{l_1} \binom{n-l_1}{l_2} \binom{n-l_1-l_2}{l_3} \cdots \binom{n-l_1-\cdots-l_{h-1}}{l_h} = \frac{n!}{l_1! \cdots l_h! (n-l_1-\cdots-l_h)!} = \frac{n!}{\alpha(q)(n-m)!}
$$

many nonnegative integer sequences $(p_1, \ldots, p_n)$ such that $\mu_{p_1} \cdots \mu_{p_n} = \mu_{q_1} \cdots \mu_{q_m}$ (equivalently, $\{p_1, \ldots, p_n\} = \{q_1, \ldots, q_m\}$, as sets).

We have obtained

$$
\mathbb{E} \left( X_1 + \cdots + X_n \right)^p = \sum_{m=1}^{n-1} \frac{n!}{p!(n-m)!} \sum_{q \in Q_m} \beta(q) \mu_{q_1} \cdots \mu_{q_m},
$$

where $\beta(q) = \frac{p!}{\alpha(q) q_1! \cdots q_m!}$ and $\mu(q) = \mu_{q_1} \cdots \mu_{q_m}$. By homogeneity, we can assume that $\mu_1 = \mathbb{E}X_1 = 1$. Note that when $X_1$ is constant, we get from (4) that

$$
1 = \sum_{m=1}^{p} \frac{n!}{p!(n-m)!} \sum_{q \in Q_m} \beta(q).
$$

Since $Q_p = \{(1, \ldots, 1)\}$ and $\mu((1, \ldots, 1)) = 1$, when we subtract the two equations, the
terms corresponding to \( m = p \) cancel and we get

\[
b_n - 1 = \sum_{m=1}^{p-1} \frac{n!}{n^p(n-m)!} \sum_{q \in Q_m} \beta(q)(\mu(q) - 1).
\]

By the monotonicity of moments, \( \mu(q) \geq 1 \) for every \( q \), so \( (b_n) \) is a sum of the constant sequence \((1, 1, \ldots)\) and the sequences \( (u_n^{(m)}) = \left( \frac{n!}{n^p(n-m)!} \right) \), \( m = 1, \ldots, p-1 \), multiplied respectively by the nonnegative factors \( \sum_{q \in Q_m} \beta(q)(\mu(q) - 1) \). Since sums of log-convex sequences are log-convex, it remains to verify that for each \( 1 \leq m \leq p-1 \), we have \( (u_n^{(m)})^2 \leq u_{n-1}^{(m)}u_n^{(m)} \), \( n \geq p^2 \). The following lemma finishes the proof.

**Lemma 6.** Let \( p \geq 2 \), \( 1 \leq m \leq p-1 \) be integers. Then the function

\[
f(x) = \log \frac{x(x-1) \cdots (x-m+1)}{x^p}
\]

is convex on \([p^2-1, \infty)\).

**Proof.** The statement is clear for \( m = 1 \). Let \( 2 \leq m \leq p-1 \) and \( p \geq 3 \). We have

\[
x^2f''(x) = p - 1 - x^2 \sum_{k=1}^{m-1} \frac{1}{(x-k)^2}.
\]

To see that this is positive for every \( x \geq p^2-1 \) and \( 2 \leq m \leq p-1 \), it suffices to consider \( m = p-1 \) and \( x = p^2-1 \) (writing \( \frac{x}{x-k} = 1 + \frac{k}{x-k} \), we see that the right hand side is increasing in \( x \)). Since

\[
\sum_{k=1}^{p-2} \frac{1}{(p^2-1-k)^2} = \sum_{k=p^2-p+1}^{p^2-2} \frac{1}{k^2} \leq \sum_{k=p^2-p+1}^{p^2-2} \left( \frac{1}{k-1} - \frac{1}{k} \right) = \frac{1}{p^2-p} - \frac{1}{p^2-2},
\]

we have

\[
x^2f''(x) \geq p - 1 - (p^2-1)^2 \left( \frac{1}{p^2-p} - \frac{1}{p^2-2} \right) = \frac{(p-1)(p+2)}{p(p^2-2)},
\]

which is clearly positive. \( \square \)

### 4 Final remarks

**Remark 7.** Using majorization type arguments (see, e.g., [5]), Conjecture 1 can be verified in a rather standard but lengthy way for every \( p > 1 \) and \( n = 2 \). The idea is to establish a pointwise inequality: we conjecture that for nonnegative numbers \( x_1, \ldots, x_{2n} \) and a convex function \( \phi : [0, \infty) \to [0, \infty) \) we have

\[
\frac{1}{(2n)} \sum_{|I|=n} \phi \left( \frac{x_I x_{I^c}}{n^2} \right) \leq \frac{1}{(2n+1)} \sum_{|I|=n+1} \phi \left( \frac{x_I x_{I^c}}{n^2-1} \right),
\]
where for a subset $I$ of the set $\{1, \ldots, 2n\}$ we denote $x_I = \sum_{i \in I} x_i$. We checked that this holds for $n = 2$. Taking the expectation on both sides for $\phi(x) = x^p$ gives the desired result that $b_n^2 \leq b_{n-1} b_{n+1}$.

Remark 8. It is tempting to ask for generalisations of Conjecture 1 beyond the power functions, say to ask whether the sequence $(a_n)$ defined in (1) is log-convex for every convex function $f$. This is false, as can be seen by taking the function $f$ of the form $f(x) = \max\{x - a, 0\}$ and the $X_i$ to be i.i.d Bernoulli random variables.

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