An Upper Bound for Spherical Caps

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Abstract

We prove an useful upper bound for the measure of spherical caps.

Consider the uniformly distributed measure $\sigma_{n-1}$ on the Euclidean unit sphere $S^{n-1} \subset \mathbb{R}^n$. On the sphere, as among only a handful other spaces, the isoperimetric problem is completely solved. This goes back to Lévy [Lé] and Schmidt [Sch] and states that caps have the minimal measure of a boundary among all sets with a fixed mass. For $\varepsilon \in [0,1)$ and $\theta \in S^{n-1}$ the cap $C(\varepsilon, \theta)$, or shortly $C(\varepsilon)$, is a set of points $x \in S^{n-1}$ for which $x \cdot \theta \geq \varepsilon$, where $\cdot$ stands for the standard scalar product in $\mathbb{R}^n$. See figure 1.

A few striking properties of the high-dimensional sphere are presented in [Ba, Lecture 1, 8]. In such considerations, we often need a good estimation of the measure of a cap. Following the method used in [Ba, Lemma 2.2], we extend its proof to the skipped case of large $\varepsilon$ and get in an elementary way the desired bound.

**Theorem.** For any $\varepsilon \in [0,1)$

$$\sigma_{n-1} (C(\varepsilon)) \leq e^{-n\varepsilon^2/2}.$$  

*Proof.* In the case of small $\varepsilon$, for convenience, we repeat a beautiful argument used by Ball. Namely, for $\varepsilon \in [0,1/\sqrt{2}]$ we have (see Figure 2)

$$\sigma_{n-1} (C(\varepsilon)) = \frac{\text{vol}_n (\text{Cone} \cap B^n(0,1))}{\text{vol}_n (B^n(0,1))} \leq \frac{\text{vol}_n (B^n(P, \sqrt{1-\varepsilon^2}))}{\text{vol}_n (B^n(0,1))} = \sqrt{1-\varepsilon^2} \leq e^{-n\varepsilon^2/2}.$$  

For $\varepsilon \in [1/\sqrt{2}, 1)$, it is enough to consider a different auxiliary ball which includes the set Cone $\cap B^n(0,1)$, see Figure 3. We obtain

$$\sigma_{n-1} (C(\varepsilon)) \leq \frac{\text{vol}_n (B^n(Q,r))}{\text{vol}_n (B^n(0,1))} = r^n = \left(\frac{1}{2\varepsilon}\right)^n \leq e^{-n\varepsilon^2/2},$$  

where the last inequality follows from the estimate

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Figure 1: A cap $C(\varepsilon, \theta)$.

Figure 2: Small $\varepsilon$.

Figure 3: Large $\varepsilon$. By the congruence $\frac{1}{\varepsilon} = \frac{1}{r}$.  

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Due to convexity, this is only to be checked at the boundary of our interval $[1/\sqrt{2}, 1]$, which reduces for both endpoints to the evident inequality $\sqrt{e} < 2$.

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References

