Injective Tauberian operators on $L_1$ and operators with dense range on $\ell_\infty$

William B. Johnson†, Amir Bahman Nasseri, Gideon Schechtman‡ and Tomasz Tkocz§

Abstract

There exist injective Tauberian operators on $L_1(0,1)$ that have dense, non closed range. This gives injective, non surjective operators on $\ell_\infty$ that have dense range. Consequently, there are two quasi-complementary, non complementary subspaces of $\ell_\infty$ that are isometric to $\ell_\infty$.

1 Introduction

A (bounded, linear) operator $T$ from a Banach space $X$ into a Banach space $Y$ is called Tauberian provided $T^{**-1}Y = X$. The structure of Tauberian operators when the domain is an $L_1$ space is well understood and exposed in González and Martínez-Abejón’s book [5, Chapter 4]. (For convenience they only consider $L_1(\mu)$ when $\mu$ is finite and purely nonatomic, but their proofs for the results we mention work for general $L_1$ spaces.) In particular, [5, Theorem 4.1.3] implies that when $X$ is an $L_1$ space, an operator $T : X \to Y$ is Tauberian iff whenever $(x_n)$ is a sequence of disjoint unit vectors, there is an

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so that the restriction of $T$ to $[x_N]_{n=N}^\infty$ is an isomorphism (and, moreover, the norm of the inverse of the restricted operator is bounded independently of the disjoint sequence). From this it follows that an injective operator $T : X \to Y$ is Tauberian iff it isomorphically preserves isometric copies of $\ell_1$ in the sense that the restriction of $T$ to any subspace of $X$ that is isometrically isomorphic to $\ell_1$ is an isomorphism. (Recall that a subspace of an $L_1$ space is isometrically isomorphic to $\ell_1$ iff it is the closed linear span of a sequence of non zero disjoint vectors [11, Chapter 14.5].) Since $Tu$ is Tauberian if $T$ is Tauberian and $u$ is an isomorphism, one deduces that an injective Tauberian operator from an $L_1$ space isomorphically preserves isometric copies of $\ell_1$ in the sense that the restriction of $T$ to any subspace of $X$ that is isomorphic to $\ell_1$ is an isomorphism. Thus injective Tauberian operators from an $L_1$ space are opposite to $\ell_1$-singular operators; i.e., operators whose restriction to every subspace isomorphic to $\ell_1$ is not an isomorphism.

The main result in this paper is a negative solution to [5, Problem 1]: Suppose $T$ is a Tauberian operator on an $L_1$ space. Must $T$ be upper semi-Fredholm; i.e., must the range $\mathcal{R}(T)$ of $T$ be closed and the null space $\mathcal{N}(T)$ of $T$ be finite dimensional? The basic example is a Tauberian operator on $L_1(0, 1)$ that has infinite dimensional null space. This is rather striking because the Tauberian condition is equivalent to the statement that there is $c > 0$ so that the restriction of the operator to $L_1(A)$ is an isomorphism whenever the subset $A$ of $[0, 1]$ has Lebesgue measure at most $c$.

In fact, we show that there is an injective, dense range, non surjective Tauberian operator on $L_1(0, 1)$. Since $T$ is Tauberian, $T^{**}$ is also injective, so $\mathcal{R}(T^*)$ is dense and proper, and $T^*$ is injective because $\mathcal{R}(T)$ is dense. This solves a problem [10] the second author raised on MathOverFlow.net that led to the collaboration of the authors.

2 The examples

We begin with a lemma that is an easy consequence of characterizations of Tauberian operators on $L_1$ spaces.

**Lemma 1** Let $X$ be an $L_1$ space and $T$ an operator from $X$ to a Banach space $Y$. The operator $T$ is Tauberian if and only if there is $r > 0$ and a natural number $N$ so that if $(x_n)_{n=1}^N$ are disjoint unit vectors in $X$, then $\max_{1 \leq n \leq N} \|Tx_n\| \geq r$. 

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Proof: The condition in the lemma clearly implies that if \((x_n)\) is a disjoint sequence of unit vectors in \(X\), then \(\liminf_n \|Tx_n\| > 0\), which is one of the equivalent conditions for \(T\) to be Tauberian [5, Theorem 4.1.3]. On the other hand, suppose that there are disjoint collections \((x^n_k)_{k=1}^{n}\), \(n = 1, 2, \ldots\) with \(\max_{1 \leq k \leq n} \|Tx^n_k\| \to 0\) as \(n \to \infty\). Then the closed sublattice generated by \(\bigcup_{n=1}^{\infty} (x^n_k)_{k=1}^{n}\) is a separable abstract \(L_1\) space (meaning that it is a Banach lattice such that \(\|x+y\| = \|x\| + \|y\|\) whenever \(|x| \lor |y| = 0\)) and hence is order isometric to \(L_1(\mu)\) for some probability \(\mu\) by Kakutani’s theorem (see e.g. [7, Theorem 1.b.2]). Choose \(1 \leq k(n) \leq n\) so that the support of \(x^n_{k(n)}\) in \(L_1(\mu)\) has measure at most \(1/n\). Since \(T\) is Tauberian, by [5, Proposition 4.1.8] necessarily \(\liminf_n \|Tx^n_{k(n)}\| > 0\), which is a contradiction.

The reason that Lemma 1 is useful for us is that the condition in the Lemma is stable under ultraproducts. Call an operator that satisfies the condition in Lemma 1 \((r,N)\)-Tauberian. For background on ultraproducts of Banach spaces and of operators, see [4, Chapter 8]. We use the fact that the ultraproduct of \(L_1\) spaces is an abstract \(L_1\) space and hence is order isometric to \(L_1(\mu)\) for some measure \(\mu\).

Lemma 2 Let \((X_k)\) be a sequence of \(L_1\) spaces, and for each \(k\) let \(T_k\) be a norm one linear operator from \(X_k\) into a Banach space \(Y_k\). Assume that there is \(r > 0\) and a natural number \(N\) so that each operator \(T_k\) is \((r,N)\)-Tauberian. Let \(U\) be a free ultrafilter on the natural numbers. Then \((T_k)_{\mathcal{U}} : (X_k)_{\mathcal{U}} \to (Y_k)_{\mathcal{U}}\) is \((r,N)\)-Tauberian.

Proof: The vectors \((x_k)\) and \((y_k)\) are disjoint in the abstract \(L_1\) space \((X_k)_{\mathcal{U}}\) iff \(\lim_{\mathcal{U}} \|x_k \land y_k\| = 0\), so it is only a matter of proving that if \(T\) is \((r,N)\)-Tauberian from some \(L_1\) space \(X\), then for each \(\varepsilon > 0\) there is \(\delta > 0\) so that if \(x_1, \ldots, x_N\) are unit vectors in \(X\) and \(\|x_n \land x_m\| < \delta\) for \(1 \leq n < m \leq N\), then \(\max_{1 \leq n \leq N} \|Tx_n\| > r - \varepsilon\). But if \(x_1, \ldots, x_N\) are unit vectors that are \(\varepsilon\)-disjoint as above, and \(y_1, \ldots, y_n\) are defined by

\[
y_n := \|x_n\| - (\|x_n\| \land (\lor \{|x_m| : m \neq n\}) \right) \text{sign}(x_n),
\]

then...
then the $y_n$ are disjoint and all have norm at least $1 - N\delta$. Normalize the $y_n$ and apply the $(r, N)$-Tauberian condition to this normalized disjoint sequence to see that $\max_{1 \leq n \leq N} \|Tx_n\| > r - \varepsilon$ if $\delta = \delta(\varepsilon, N)$ is sufficiently small.

An example that answers [5, Problem 1] is the restriction of an ultraproduct of operators on finite dimensional $L_1$ spaces constructed in [3].

**Theorem 1** There is a Tauberian operator $T$ on $L_1(0, 1)$ that has an infinite dimensional null space. Consequently, $T$ is not upper semi-Fredholm.

**Proof:** An immediate consequence of [3, Proposition 6 & Theorem 1] is that there is a norm one $(r, N)$-Tauberian operator $T_n$ from $\ell^1_n$ into itself with $\dim \mathcal{N}(T_n) > r n$. The ultraproduct $\bar{T} := (T_n)_U$ is then a norm one $(r, N)$-Tauberian operator on the gigantic $L_1$ space $X_1 := (\ell^1_n)_U$, and the null space of $\bar{T}$ is infinite dimensional. Take any separable infinite dimensional subspace $X_0$ of $\mathcal{N}(\bar{T})$ and let $X$ be the closed sublattice of $X_1$ generated by $X_0$. Let $Y$ be the sublattice of $X_1$ generated by $\bar{T}X$ and let $T$ be the restriction of $\bar{T}$ to $X$, considered as an operator into $Y$. So $X$ and $Y$ are separable $L_1$ spaces and by Lemmas 1 and 2 the operator $T$ is Tauberian. Of course, by construction $\mathcal{N}(T)$ is infinite dimensional and reflexive (because $T$ is Tauberian). Thus $X$ is not isomorphic to $\ell_1$ and hence is isomorphic to $L_1(0, 1)$. So is $Y$, but that does not matter: $Y$, being a separable $L_1$ space, embeds isometrically into $L_1(0, 1)$.

We want to “soup up” the operator $T$ in Theorem 1 to get an injective, non surjective, dense range Tauberian operator on $L_1(0, 1)$. We could quote a general result [6, Theorem 3.4] of González and Onieva to shorten the presentation, but we prefer to give a short direct proof.

We recall a simple known lemma:

**Lemma 3** Let $X$ and $Y$ be separable infinite dimensional Banach spaces and $\varepsilon > 0$. Let $Y_0$ be a countable dimensional dense subspace of $Y$. Then there is a nuclear operator $u : X \to Y$ so that $u$ is injective and $\|u\|_\Lambda < \varepsilon$ and $uX \supset Y_0$.

**Proof:** Recall that an $M$-basis for a Banach space $X$ is a biorthogonal system $(x_\alpha, x_\alpha^*) \subset X \times X^*$ such that the linear span of $(x_\alpha)$ is dense in $X$ and $\cap_\alpha \mathcal{N}(x_\alpha^*) = \{0\}$. Every separable Banach space $X$ has an $M$-basis.
[8]; moreover, the vectors \((x_\alpha)\) in the \(M\)-basis can span any given countable dimensional dense subspace of \(X\).

Take \(M\)-bases \((x_n, x_n^*)\) and \((y_n, y_n^*)\) for \(X\) and \(Y\), respectively, normalized so that \(\|x_n^*\| = 1 = \|y_n\|\) and such that the linear span of \((y_n)\) is \(Y_0\). Choose \(\lambda_n > 0\) so that \(\sum_n \lambda_n < \varepsilon\) and set \(u(x) = \sum_n \lambda_n \langle x_n^*, x \rangle y_n\).

**Theorem 2** There is an injective, non surjective, dense range Tauberian operator on \(L_1(0,1)\).

**Proof:** By Theorem 1 there is a Tauberian operator \(T\) on \(L_1(0,1)\) that has an infinite dimensional null space. By Lemma 3 there is a nuclear operator \(\tilde{v} : \mathcal{N}(T) \to L_1(0,1)\) that is injective and has dense range, and we can extend \(\tilde{v}\) to a nuclear operator \(v\) on \(L_1(0,1)\). We can choose \(\tilde{v}\) so that \(\tilde{v}(\mathcal{N}(T)) \cap TL_1(0,1)\) is infinite dimensional by the last statement in Lemma 3. This guarantees that the Tauberian operator \(T_1 := T + v\) has an infinite dimensional null space (this allows us to avoid breaking the following argument into cases).

Now \(\mathcal{N}(T_1) \cap \mathcal{N}(T) = \{0\}\), so again by Lemma 3 and the extension property of nuclear operators there is a nuclear operator \(u : L_1(0,1)/\mathcal{N}(T) \to \ell_1\) so that the restriction of \(u\) to \(Q_{\mathcal{N}(T)}\mathcal{N}(T_1)\) is injective and has dense range (here for a subspace \(E\) of \(X\), the operator \(Q_E\) is the quotient mapping from \(X\) onto \(X/E\)). Finally, define \(T_2 : L_1(0,1) \to L_1(0,1) \oplus_1 \ell_1\) by \(T_2x := T_1x \oplus uQ_{\mathcal{N}(T)}x\). Then \(T_2\) is an injective Tauberian operator with dense range. \(T_2\) is not surjective because \(P_{\ell_1}T_2\) is nuclear by construction, where \(P_{\ell_1}\) is the projection of \(L_1(0,1) \oplus_1 \ell_1\) onto \(\{0\} \oplus_1 \ell_1\). Since \(L_1(0,1) \oplus_1 \ell_1\) is isomorphic to \(L_1(0,1)\), this completes the proof.

**Corollary 1** There is an injective, dense range, non surjective operator on \(\ell_\infty\). Consequently, there is a quasi-complementary, non complementary decomposition of \(\ell_\infty\) into two subspaces each of which is isometrically isomorphic to \(\ell_\infty\).

**Proof:** Let \(T\) be an injective, dense range, non surjective Tauberian operator on \(L_1(0,1)\) (Theorem 2). Since \(T\) is Tauberian, \(T^{**}\) is also injective, so \(T^*\) has dense range but \(T^*\) is not surjective because its range is not closed, and \(T^*\) is injective because \(T\) has dense range. The operator \(T^*\) translates to an operator on \(\ell_\infty\) that has the same properties because \(L_\infty\) is isomorphic to \(\ell_\infty\) by an old result due to Pelczyński (see, e.g., [1, Theorem 4.3.10]) (notice however that, unlike \(T^*\), the operator on \(\ell_\infty\) cannot be weak* continuous).
For the “consequently” statement, let $S$ be any norm one injective, dense range, non surjective operator on $\ell_\infty$. In the space $\ell_\infty \oplus \ell_\infty$, which is isometric to $\ell_\infty$, define $X := \ell_\infty \oplus \{0\}$ and $Y := \{(x, Sx) : x \in \ell_\infty\}$. Obviously $X$ and $Y$ are isometric to $\ell_\infty$ and $X + Y = \ell_\infty \oplus S\ell_\infty$, which is a dense proper subspace of $\ell_\infty \oplus \ell_\infty$. Finally, $X \cap Y = \{0\}$ since $S$ is injective, so $X$ and $Y$ are quasi-complementary, non complemental subspaces of $\ell_\infty \oplus \ell_\infty$. □

Theorem 2 and the MathOverFlow question [10] suggest the following problem: Suppose $X$ is a separable Banach space (so that $X^*$ is isometric to a weak* closed subspace of $\ell_\infty$) and $X^*$ is non separable. Is there a dense range operator on $X^*$ that is not surjective? The answer is “no”: Argyros, Arvanitakis, and Tolias [2] constructed a separable space $X$ so that $X^*$ is non separable, hereditarily indecomposable (HI), and every strictly singular operator on $X^*$ is weakly compact. Since $X^*$ is HI, every operator on $X^*$ is of the form $\lambda I + S$ with $S$ strictly singular. If $\lambda \neq 0$, then $\lambda I + S$ is Fredholm of index zero by Kato’s classical perturbation theory. On the other hand, since every weakly compact subset of the dual to a separable space is norm separable, every strictly singular operator on $X^*$ has separable range. (Thanks to Spiros Argyros for bringing this example to our attention.)

Any operator $T$ on $l^\infty$ that has dense range but is not surjective has the property that 0 is an interior point of $\sigma(T)$. This follows from Thm 2.6 in [9], where it is shown that $\partial \sigma(T) \subset \sigma_p(T^*)$ for any operator $T$ acting on a $C(K)$ space which has the Grothendieck property.

References


[9] A. B. Nasseri, The spectrum of operators on $C(K)$ with the Grothendieck property and characterization of $J$-Class Operators which are adjoints.


W. B. Johnson  
Department of Mathematics  
Texas A&M University  
College Station, TX 77843 U.S.A.  
johnson@math.tamu.edu

A. B. Nasseri  
Fakultät für Mathematik  
Technische Universität Dortmund  
D-44221 Dortmund, Germany  
amirbahman@hotmail.de