A Note on $G$-intersecting Families

Tom Bohman*   Ryan R. Martin†

Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213

August 26, 2002

Abstract

Consider a graph $G$ and a $k$-uniform hypergraph $\mathcal{H}$ on common vertex set $[n]$. We say that $\mathcal{H}$ is $G$-intersecting if for every pair of edges in $X,Y \in \mathcal{H}$ there are vertices $x \in X$ and $y \in Y$ such that $x = y$ or $x$ and $y$ are joined by an edge in $G$. This notion was introduced by Bohman, Frieze, Ruszinkó and Thoma who proved a natural generalization of the Erdős-Ko-Rado Theorem for $G$-intersecting $k$-uniform hypergraphs for $G$ sparse and $k = O(n^{1/4})$. In this note, we extend this result to $k = O(\sqrt{n})$.

1 Introduction

A hypergraph is said to be intersecting if every pair of edges has a nonempty intersection. The well-known theorem of Erdős, Ko and Rado [3] details the extremal $k$-uniform intersecting hypergraph on $n$ vertices.

**Theorem 1 (Erdős-Ko-Rado).** Let $k \leq n/2$ and $\mathcal{H}$ be a $k$-uniform, intersecting hypergraph on vertex set $[n]$. We have $|\mathcal{H}| \leq \binom{n-1}{k-1}$. Furthermore, $|\mathcal{H}| = \binom{n-1}{k-1}$ if and only if there exists $v \in [n]$ such that $\mathcal{H} = \{e \in \binom{[n]}{k} : v \in e\}$.

Of course, for $k > n/2$ the hypergraph consisting of all $k$-sets is intersecting. So, extremal $k$-intersecting hypergraphs come in one of two forms, depending on the value of $k$.

Bohman, Frieze, Ruszinkó and Thoma [1] introduced a generalization of the notion of an intersecting hypergraph. Let $G$ be a graph on a vertex set $[n]$ and $\mathcal{H}$ be a hypergraph, also on vertex set $[n]$. We say $\mathcal{H}$ is $G$-intersecting if for any $e,f \in \mathcal{H}$, we have $e \cap f \neq \emptyset$ or there are vertices $v,w$ with $v \in e$, $w \in f$ and $v \sim_G w$. We are interested in the size and structure of maximum $G$-intersecting hypergraphs; in particular, we investigate

$$N(G,k) = \max \left\{ |\mathcal{H}| : \mathcal{H} \subseteq \binom{[n]}{k} \text{ and } \mathcal{H} \text{ is } G\text{-intersecting} \right\}.$$

*Supported in part by NSF grant DMS-0100400.
†Supported in part by NSF VIGRE Grant DMS-9819950
Clearly, Erdős-Ko-Rado gives the value of $N(E_n, k)$ where $E_n$ is the empty graph on vertex set $[n]$. For a discussion of $N(G, k)$ for some other specific graphs see [1].

In this note we restrict our attention to sparse graphs: those graphs for which $n$ is large and the maximum degree of $G$, $\Delta(G)$, is a constant in $n$. What form can a maximum $G$-intersecting family take? If $K$ is a maximum clique in $G$ then a candidate for a maximum $G$-intersecting family is

$$\mathcal{H}_K := \left\{ X \in \binom{[n]}{k} : X \cap K \neq \emptyset \right\}.$$ 

Note that such a hypergraph can be viewed as a natural generalization of the maximum intersecting hypergraphs given by Erdős-Ko-Rado. However, for many graphs and maximum cliques $K$ one can add hyperedges to $\mathcal{H}_K$ to obtain a larger $G$-intersecting hypergraph.

Consider, for example, $C_n$, the cycle on vertex set $[n]$ (i.e. the graph on $[n]$ in which $u$ and $v$ are adjacent iff $u - v \in \{1, n - 1\}$ mod $n$). The set $\{2, 3\}$ is a maximum clique in $C_n$ and the set

$$\mathcal{H}_{\{2,3\}} \cup \left\{ X \in \binom{[n]}{k} : \{1, 4\} \subseteq X \right\} \quad \text{(1)}$$

is $G$ intersecting. Bohman, Frieze, Ruszinkó and Thoma showed that

$$N(C_n, k) = \binom{n}{k} - \binom{n-2}{k} + \binom{n-4}{k-2} \quad \text{(2)}$$

(i.e. the hypergraph given in (1) is maximum) for $k$ less than a certain constant times $n^{1/4}$. In fact, they showed that for arbitrary sparse graphs and $k$ small, $N(G, k)$ is given by a hypergraph that consists of $\mathcal{H}_K$ for some clique $K$ together with a number of ‘extra’ hyperedges that cover the clique $K$ in $G$ (see Theorem 1 of [1]). In this note we extend this result to larger values of $k$.

**Theorem 2.** Let $G$ be a graph on $n$ vertices with maximum degree $\Delta$ and clique number $\omega$. There exists a constant $C$ (depending only on $\Delta$ and $\omega$) such that if $\mathcal{H}$ is a $G$-intersecting $k$-uniform hypergraph and $k < Cn^{1/2}$ then

$$|\mathcal{H}| \leq \binom{n}{k} - \binom{n-\omega}{k} + \binom{\omega(\Delta-\omega+1)}{2} \binom{n-\omega-2}{k-2}.$$ 

Furthermore, if $\mathcal{H}$ is a $G$-intersecting family of maximum cardinality then there exists a maximum clique $K$ in $G$ such that $\mathcal{H}$ contains all $k$-sets that intersect $K$.

An immediate corollary of this Theorem is that (2) holds for $k < C\sqrt{n}$.

Of course, a maximum $G$-intersecting hypergraph will not be of the form ‘$\mathcal{H}_K$ together with some extra hyperedges’ if $k$ is too large. Even for sparse graphs, when $k$ is large enough, there are hypergraphs that consist of nearly all of $\binom{[n]}{k}$ that are $G$-intersecting. In particular, Bohman, Frieze, Ruszinkó and Thoma showed that if $G$ is a sparse graph with
minimum degree \( \delta \), \( c \) is a constant such that \( c - (1 - c)^{\delta + 1} > 0 \) and \( k > cn \), then the size of the largest \( G \)-intersecting, \( k \)-uniform hypergraph is at least \( (1 - e^{-\Omega(n)}) \binom{n}{k} \) (see Theorem 7 of [1]). In some sense, this generalizes the trivial observation that \( \binom{\lfloor n \rfloor}{k} \) is intersecting for \( k > n/2 \).

There is a considerable gap between the values of \( k \) for which we have established these two types of behavior for maximum \( G \)-intersecting families. For example, for \( C_n \) we have (2) for \( k < C \sqrt{n} \) while we have \( N(C_n, k) > (1 - o(1)) \binom{n}{k} \) for \( k \) greater than roughly \( .32n \). What happens for other values of \( k \)? Are there other forms that a maximum \( G \)-intersecting family can take? Bohman, Frieze, Ruszinkó and Thoma conjecture that this is not the case, at least for the cycle.

**Conjecture 1.** There exists a constant \( c \) such that for any fixed \( \epsilon > 0 \)

\[
k \leq (c - \epsilon) n \quad \Rightarrow \quad N(C_n, k) = \binom{n}{k} - \binom{n - 2}{k} + \binom{n - 4}{k - 2}
\]

\[
k \geq (c + \epsilon) n \quad \Rightarrow \quad N(C_n, k) = (1 - o(1)) \binom{n}{k}
\]

The remainder of this note consists of the proof of Theorem 2.

## 2 Utilizing \( \tau \)

Let \( \mathcal{H} \) be a hypergraph and \( G \) be a graph on vertex set \([n]\). For \( X \subseteq [n] \), we define

\[
N(X) := \{ v \in V(G) : v \sim_G w \text{ for some } w \in X \} \cup X.
\]

For \( x \in [n] \) we write \( N(x) \) for \( N(\{x\}) \). We will define the hypergraph \( \mathcal{F} \) by setting \( f \in \mathcal{F} \) if and only if \( f = N(h) \) for some \( h \in \mathcal{H} \). Note that if \( \mathcal{H} \) is \( G \)-intersecting, then

\[
h \in \mathcal{H}, f \in \mathcal{F} \Rightarrow h \cap f \neq \emptyset.
\]

The quantity \( \tau(\mathcal{F}) \) is the cover number of \( \mathcal{F} \).

The proof of Theorem 2 follows immediately from Lemma 1, which deals with the case where \( \tau(\mathcal{F}) \geq 2 \) and Lemma 2, which deals with the case where \( \tau(\mathcal{F}) = 1 \).

**Lemma 1.** Let \( G \) be a graph on \( n \) vertices with maximum degree \( \Delta \) and clique number \( \omega \), both constants. If \( k < \sqrt{\frac{\omega n}{(2(\Delta + 1)^2} \), \( \mathcal{H} \) is a \( k \)-uniform, \( G \)-intersecting hypergraph on \( n \) vertices and \( n \) is sufficiently large, then \( \tau(\mathcal{F}) = 1 \) or

\[
|\mathcal{H}| < \binom{n}{k} - \frac{n - \omega}{k}.
\]
Proof.

Suppose, by way of contradiction, that \( \tau = \tau(\mathcal{F}) \geq 2 \) and (4) does not hold. For \( v \in [n] \) set \( \mathcal{H}_v = \{ f \in \mathcal{F} : u \in f \} \), and for \( Y \subseteq [n] \) set \( \mathcal{H}_Y = \{ f \in \mathcal{F} : Y \subseteq f \} \). Let \( \mathcal{F}_u \) and \( \mathcal{F}_Y \) be defined analogously.

We first use \( \tau > 1 \) to get an upper bound \( |\mathcal{H}_u| \) for an arbitrary \( u \in [n] \). First note that, since \( \tau > 1 \), there exists \( X_1 \in \mathcal{F} \) such that \( u \notin X_1 \). It follows from (3) that each \( f \in \mathcal{F}_u \) must intersect \( X_1 \). In other words, we have

\[
\mathcal{F}_u = \bigcup_{u_1 \in X_1} \mathcal{F}_{\{u, u_1\}}.
\]

This observation can be iterated; if \( i < \tau \) and \( Y = \{ u = u_0, u_1, \ldots, u_{i-1} \} \) then there exists \( X_i \in \mathcal{F} \) such that \( X_i \cap Y = \emptyset \), and we have

\[
\mathcal{F}_Y = \bigcup_{u_i \in X_i} \mathcal{F}_{Y \cup \{u_i\}}.
\]

Since \( |f| \leq (\Delta + 1)k \) for all \( f \in \mathcal{F} \), it follows that we have

\[
|\mathcal{H}_u| \leq ((\Delta + 1)k)^{\tau - 1} \binom{n-\tau}{k-\tau}.
\]

(5)

On the other hand, by the definition of \( \tau \), there exists \( v \in [n] \) for which

\[
\frac{1}{\tau} \left[ \binom{n}{k} - \binom{n - \omega(G)}{k} \right] \leq |\mathcal{F}_v|.
\]

It follows that there exists \( u \in [n] \) such that

\[
\frac{1}{\tau(\Delta + 1)} \left[ \binom{n}{k} - \binom{n - \omega(G)}{k} \right] \leq |\mathcal{H}_u|.
\]

Applying (5) to this vertex we have

\[
\binom{n}{k} - \binom{n - \omega(G)}{k} \leq \tau(\Delta + 1)^{\tau - 1} \binom{n-\tau}{k-\tau}.
\]

In order to show that this is a contradiction, we first note that \( \tau(\Delta + 1)^{\tau - 1} \binom{n-\tau}{k-\tau} \) is a function that is decreasing in \( \tau \). Indeed, for \( \tau \geq 2 \) we have

\[
\frac{n-\tau}{k-\tau} \geq \frac{n-2}{k-2} \geq \frac{3}{2}(\Delta + 1)k \geq \frac{\tau + 1}{\tau}(\Delta + 1)k
\]

(note that the condition \( k < \sqrt[2]{\frac{\omega}{2(\Delta + 1)^2}} \) is used in the second inequality). It follows that we have

\[
\binom{n}{k} - \binom{n - \omega(G)}{k} \leq 2(\Delta + 1)^2k \binom{n-2}{k-2},
\]
which is not true if \( k < \sqrt{\frac{n\omega(G)}{2(\Delta+1)^2}} \) and \( n \) is large enough. 

\[ \]

**Lemma 2.** Let \( G \) be a graph on \([n]\) with maximum degree \( \Delta \), a constant. If \( \mathcal{H} \) is a \( k \)-uniform, \( G \)-intersecting hypergraph on \([n]\), \( k \leq \sqrt{\frac{n}{\Delta(\Delta+1)}} \), \( \tau(\mathcal{F}) = 1 \), \( n \) is sufficiently large and \( \mathcal{H} \) is of maximum size, then there exists a maximum-sized clique \( K \) in \( G \) such that \( \mathcal{H} \) contains every \( k \)-set that intersects \( K \).

**Proof.** Let us suppose \( \mathcal{H} \) is of maximum size and let \( u \) be a cover for \( \mathcal{F} \), the hypergraph defined above.

For \( v \in [n] \), let \( \mathcal{H}_v \) denote the members of \( \mathcal{H} \) that contain \( v \). Since \( \mathcal{H} \) is assumed to be extremal, we may assume that \( |\mathcal{H}_u| = \left(\begin{array}{c}n-1\end{array}\right)_k \). Let \( K \) be the set of \( v \in [n] \) such that \( |\mathcal{H}_v| = \left(\begin{array}{c}n-1\end{array}\right)_k \). If \( n > (\Delta + 2)k \) then \( K \) must be a clique in \( G \); otherwise, we could find two sets that are not \( G \)-intersecting in \( \mathcal{H} \).

We now show that the clique \( K \) is maximal. Assume for the sake of contradiction that \( v \) is adjacent to every element of \( K \) but \( v \notin K \) (i.e. \( |\mathcal{H}_v| < \left(\begin{array}{c}n-1\end{array}\right)_k \)). There exists \( h \in \mathcal{H} \) that \( h \) contains no member of \( N(v) \). It follows from (3) that we have

\[
|\mathcal{H}_v| < (\Delta + 1)k \left(\begin{array}{c}n-2\end{array}\right)_{k-2}.
\]

Since this bounds holds for all vertices in \( N(u) \setminus K \), if we have

\[
\Delta(\Delta + 1)k \left(\begin{array}{c}n-2\end{array}\right)_{k-2} < \left(\begin{array}{c}n-|K| - 1\end{array}\right)_{k-1}
\]

then the number of \( k \)-sets that contain \( v \) but do not intersect \( K \) outnumber those edges in \( \mathcal{H} \) that contain no member of \( K \). In other words, if (6) holds then we get a contradiction to the maximality of \( \mathcal{H} \). However, (6) holds for \( n \) sufficiently large (here we use \( k < \sqrt{\frac{n}{\Delta(\Delta+1)}} \)).

It remains to show that \( K \) is a maximum clique. Since \( K \) is maximal, it must be that any member of \( \mathcal{H} \) that does not contain a member of \( K \) must contain at least 2 members of \( N(K) \setminus K \). If

\[
\left(\begin{array}{c}n\end{array}\right) - \left(\begin{array}{c}n - \left|K\right|\end{array}\right)_{k} + \left(\begin{array}{c}|K|\left(\Delta - \left|K\right| + 1\right)\end{array}\right)_{k-2} < \left(\begin{array}{c}n\end{array}\right) - \left(\begin{array}{c}n - \left|K\right| - 1\end{array}\right)_{k}
\]

and there is some clique of size \( |K| + 1 \), then \( \mathcal{H} \) cannot be maximum-sized. But (7) holds for \( k = o(n) \). So the maximum-sized \( G \) intersecting family must contain all members of \( \bigcup_{v \in K} \mathcal{F}_v \) for some \( K \) with \( \left|K\right| = \omega(G) \). 

\[ \]
References

