A Note on G-intersecting Families

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August 26, 2002

Abstract

Consider a graph G and a k-uniform hypergraph $\mathcal{H}$ on common vertex set $[n]$. We say that $\mathcal{H}$ is *G-intersecting* if for every pair of edges in $X, Y \in \mathcal{H}$ there are vertices $x \in X$ and $y \in Y$ such that $x = y$ or x and y are joined by an edge in G. This notion was introduced by Bohman, Frieze, Ruszinkó and Thoma who proved a natural generalization of the Erdős-Ko-Rado Theorem for G-intersecting k-uniform hypergraphs for G sparse and $k = O(n^{1/4})$. In this note, we extend this result to $k = O\left(\sqrt{n}\right)$.

1 Introduction

A hypergraph is said to be *intersecting* if every pair of edges has a nonempty intersection. The well-known theorem of Erdős, Ko and Rado [3] details the extremal k-uniform intersecting hypergraph on n vertices.

Theorem 1 (Erdős-Ko-Rado). *Let $k \le n/2$ and $\mathcal{H}$ be a k-uniform, intersecting hypergraph on vertex set $[n]$. We have $|\mathcal{H}| \le \binom{n-1}{k-1}$. Furthermore, $|\mathcal{H}| = \binom{n-1}{k-1}$ if and only if there exists $v \in [n]$ such that $\mathcal{H} = \{e \in \binom{n}{k} : v \in e\}$.*

Of course, for $k > n/2$ the hypergraph consisting of all k-sets is intersecting. So, extremal k-intersecting hypergraphs come in one of two forms, depending on the value of k.

Bohman, Frieze, Ruszinkó and Thoma [1] introduced a generalization of the notion of an intersecting hypergraph. Let G be a graph on a vertex set $[n]$ and $\mathcal{H}$ be a hypergraph, also on vertex set $[n]$. We say $\mathcal{H}$ *is G-intersecting* if for any $e, f \in \mathcal{H}$, we have $e \cap f \neq \emptyset$ or there are vertices v, w with $v \in e$, $w \in f$ and $v \sim_G w$. We are intersected in the size and structure of maximum G-intersecting hypergraphs; in particular, we investigate

$$N(G,k) = \max\left\{|\mathcal{H}| : \mathcal{H} \subseteq \binom{[n]}{k} \text{ and } \mathcal{H} \text{ is } G\text{-intersecting}\right\}.$$

*Supported in part by NSF grant DMS-0100400.
†Supported in part by NSF VIGRE Grant DMS-9819950

Clearly, Erdős-Ko-Rado gives the value of $N(E_n, k)$ where E_n is the empty graph on vertex set $[n]$. For a discusssion of $N(G, k)$ for some other specific graphs see [1].

In this note we restrict our attention to sparse graphs: those graphs for which n is large and the maximum degree of G, $\Delta(G)$, is a constant in n. What form can a maximum G-intersecting family take? If K is a maximum clique in G then a candidate for a maximum G-intersecting family is

$$\mathcal{H}_K := \left\{ X \in \binom{[n]}{k} : X \cap K \neq \emptyset \right\}.$$

Note that such a hypergraph can be viewed as a natural generalization of the maximum intersecting hypergraphs given by Erdős-Ko-Rado. However, for many graphs and maximum cliques K one can add hyperedges to $\mathcal{H}_K$ to obtain a larger G-intersecting hypergraph.

Consider, for example, C_n, the cycle on vertex set $[n]$ (i.e. the graph on $[n]$ in which u and v are adjacent iff $u - v \in \{1, n-1\} \bmod n$). The set $\{2, 3\}$ is a maximum clique in C_n and the set

$$\mathcal{H}_{\{2,3\}} \cup \left\{ X \in \binom{[n]}{k} : \{1, 4\} \subseteq X \right\} \tag{1}$$

is G intersecting. Bohman, Frieze, Ruszinkó and Thoma showed that

$$N(C_n, k) = \binom{n}{k} - \binom{n-2}{k} + \binom{n-4}{k-2} \tag{2}$$

(i.e. the hypergraph given in (1) is maximum) for k less than a certain constant times $n^{1/4}$. In fact, they showed that for arbitrary sparse graphs and k small, $N(G, k)$ is given by a hypergraph that consists of $\mathcal{H}_K$ for some clique K together with a number of 'extra' hyperedges that cover the clique K in G (see Theorem 1 of [1]). In this note we extend this result to larger values of k.

Theorem 2. *Let G be a graph on n vertices with maximum degree Δ and clique number ω. There exists a constant C (depending only on Δ and ω) such that if $\mathcal{H}$ is a G-intersecting k-uniform hypergraph and $k < Cn^{1/2}$ then*

$$|\mathcal{H}| \leq \binom{n}{k} - \binom{n-\omega}{k} + \binom{\omega(\Delta - \omega + 1)}{2}\binom{n-\omega-2}{k-2}.$$

Furthermore, if $\mathcal{H}$ is a G-intersecting family of maximum cardinality then there exists a maximum clique K in G such that $\mathcal{H}$ contains all k-sets that intersect K.

An immediate corollary of this Theorem is that (2) holds for $k < C\sqrt{n}$.

Of course, a maximum G-intersecting hypergraph will not be of the form '$\mathcal{H}_K$ together with some extra hyperedges' if k is too large. Even for sparse graphs, when k is large enough, there are hypergraphs that consist of nearly all of $\binom{[n]}{k}$ that are G-intersecting. In particular, Bohman, Frieze, Ruszinkó and Thoma showed that if G is a sparse graph with minimum degree δ, c is a constant such that $c-(1-c)^{\delta+1}>0$ and $k>cn$, then the size of the largest G-intersecting, k-uniform hypergraph is at least $(1-e^{-\Omega(n)})\binom{n}{k}$ (see Theorem 7 of [1]). In some sense, this generalizes the trivial observation that $\binom{[n]}{k}$ is intersecting for $k>n/2$.

There is a considerable gap between the values of k for which we have established these two types of behavior for maximum G-intersecting families. For example, for C_n we have (2) for $k<C\sqrt{n}$ while we have $N(C_n,k)>(1-o(1))\binom{n}{k}$ for k greater than roughly $.32n$. What happens for other values of k? Are there other forms that a maximum G-intersecting family can take? Bohman, Frieze, Ruszinkó and Thoma conjecture that this is not the case, at least for the cycle.

Conjecture 1. *There exists a constant c such that for any fixed $\epsilon>0$*

$$k\le(c-\epsilon)n \quad\Rightarrow\quad N(C_n,k)=\binom{n}{k}-\binom{n-2}{k}+\binom{n-4}{k-2}$$

$$k\ge(c+\epsilon)n \quad\Rightarrow\quad N(C_n,k)=(1-o(1))\binom{n}{k}$$

The remainder of this note consists of the proof of Theorem 2.

2 Utilizing τ

Let $\mathcal{H}$ be a hypergraph and G be a graph on vertex set $[n]$. For $X\subseteq[n]$, we define

$$N(X):=\{v\in V(G): v\sim_G w \text{ for some } w\in X\}\cup X.$$

For $x\in[n]$ we write $N(x)$ for $N(\{x\})$. We will define the hypergraph $\mathcal{F}$ by setting $f\in\mathcal{F}$ if and only if $f=N(h)$ for some $h\in\mathcal{H}$. Note that if $\mathcal{H}$ is G-intersecting, then

$$h\in\mathcal{H}, f\in\mathcal{F}\Rightarrow h\cap f\neq\emptyset. \tag{3}$$

The quantity $\tau(\mathcal{F})$ is the cover number of $\mathcal{F}$.

The proof of Theorem 2 follows immediately from Lemma 1, which deals with the case where $\tau(\mathcal{F})\ge 2$ and Lemma 2, which deals with the case where $\tau(\mathcal{F})=1$.

Lemma 1. *Let G be a graph on n vertices with maximum degree Δ and clique number ω, both constants. If $k<\sqrt{\frac{\omega n}{2(\Delta+1)^2}}$, $\mathcal{H}$ is a k-uniform, G-intersecting hypergraph on n vertices and n is sufficiently large, then $\tau(\mathcal{F})=1$ or*

$$|\mathcal{H}|<\binom{n}{k}-\binom{n-\omega}{k}. \tag{4}$$

Proof.

Suppose, by way of contradiction, that $\tau = \tau(\mathcal{F}) \geq 2$ and (4) does not hold. For $v \in [n]$ set $\mathcal{H}_v = \{f \in \mathcal{F} : u \in f\}$, and for $Y \subseteq [n]$ set $\mathcal{H}_Y = \{f \in \mathcal{F} : Y \subseteq f\}$. Let $\mathcal{F}_u$ and $\mathcal{F}_Y$ be defined analogously

We first use $\tau > 1$ to get an upper bound $|\mathcal{H}_u|$ for an arbitrary $u \in [n]$. First note that, since $\tau > 1$, there exists $X_1 \in \mathcal{F}$ such that $u \notin X_1$. It follows from (3) that each $f \in \mathcal{F}_u$ must intersect X_1. In other words, we have

$$\mathcal{F}_u = \bigcup_{u_1 \in X_1} \mathcal{F}_{\{u,u_1\}}.$$

This observation can be iterated: if $i < \tau$ and $Y = \{u = u_0, u_1, \ldots, u_{i-1}\}$ then there exists $X_i \in \mathcal{F}$ such that $X_i \cap Y = \emptyset$, and we have

$$\mathcal{F}_Y = \bigcup_{u_i \in X_i} \mathcal{F}_{Y \cup \{u_i\}}.$$

Since $|f| \leq (\Delta + 1)k$ for all $f \in \mathcal{F}$, it follows that we have

$$|\mathcal{H}_u| \leq ((\Delta + 1)k)^{\tau - 1} \binom{n - \tau}{k - \tau}. \tag{5}$$

On the other hand, by the definition of τ, there exists $v \in [n]$ for which

$$\frac{1}{\tau}\left[\binom{n}{k} - \binom{n - \omega(G)}{k}\right] \leq |\mathcal{F}_v|.$$

It follows that there exists $u \in [n]$ such that

$$\frac{1}{\tau(\Delta + 1)}\left[\binom{n}{k} - \binom{n - \omega(G)}{k}\right] \leq |\mathcal{H}_u|.$$

Applying (5) to this vertex we have

$$\binom{n}{k} - \binom{n - \omega(G)}{k} \leq \tau(\Delta + 1)^{\tau} k^{\tau - 1} \binom{n - \tau}{k - \tau}.$$

In order to show that this is a contradiction, we first note that $\tau(\Delta + 1)^{\tau} k^{\tau-1}\binom{n-\tau}{k-\tau}$ is a function that is decreasing in τ. Indeed, for $\tau \geq 2$ we have

$$\frac{n - \tau}{k - \tau} \geq \frac{n - 2}{k - 2} \geq \frac{3}{2}(\Delta + 1)k \geq \frac{\tau + 1}{\tau}(\Delta + 1)k$$

(note that the condition $k < \sqrt{\frac{\omega n}{2(\Delta+1)^2}}$ is used in the second inequality). It follows that we have

$$\binom{n}{k} - \binom{n - \omega(G)}{k} \leq 2(\Delta + 1)^2 k \binom{n - 2}{k - 2},$$

which is not true if $k < \sqrt{\frac{n\omega(G)}{2(\Delta+1)^2}}$ and n is large enough. $\square$

Lemma 2. *Let G be a graph on $[n]$ with maximum degree Δ, a constant. If $\mathcal{H}$ is a k-uniform, G-intersecting hypergraph on $[n]$, $k \leq \sqrt{\frac{n}{\Delta(\Delta+1)}}$, $\tau(\mathcal{F}) = 1$, n is sufficiently large and $\mathcal{H}$ is of maximum size, then there exists a maximum-sized clique K in G such that $\mathcal{H}$ contains every k-set that intersects K.*

Proof. Let us suppose $\mathcal{H}$ is of maximum size and let u be a cover for $\mathcal{F}$, the hypergraph defined above.

For $v \in [n]$, let $\mathcal{H}_v$ denote the members of $\mathcal{H}$ that contain v. Since $\mathcal{H}$ is assumed to be extremal, we may assume that $|\mathcal{H}_u| = \binom{n-1}{k-1}$. Let K be the set of $v \in [n]$ such that $|\mathcal{H}_v| = \binom{n-1}{k-1}$. If $n > (\Delta+2)k$ then K must be a clique in G; otherwise, we could find two sets that are not G-intersecting in $\mathcal{H}$.

We now show that the clique K is maximal. Assume for the sake of contradiction that v is adjacent to every element of K but $v \notin K$ (i.e. $|\mathcal{H}_v| < \binom{n-1}{k-1}$). There exists $h \in \mathcal{H}$ that h contains no member of $N(v)$. It follows from (3) that we have

$$|\mathcal{H}_v| < (\Delta+1)k\binom{n-2}{k-2}.$$

Since this bounds holds for all vertices in $N(u) \setminus K$, if we have

$$\Delta(\Delta+1)k\binom{n-2}{k-2} < \binom{n-|K|-1}{k-1} \tag{6}$$

then the number of k-sets that contain v but do not intersect K outnumber those edges in $\mathcal{H}$ that contain no member of K. In other words, if (6) holds then we get a contradiction to the maximality of $\mathcal{H}$. However, (6) holds for n sufficiently large (here we use $k < \sqrt{\frac{n}{\Delta(\Delta+1)}}$).

It remains to show that K is a maximum clique. Since K is maximal, it must be that any member of $\mathcal{H}$ that does not contain a member of K must contain at least 2 members of $N(K) \setminus K$. If

$$\binom{n}{k} - \binom{n-|K|}{k} + \binom{|K|(\Delta-|K|+1)}{2}\binom{n-|K|-2}{k-2} < \binom{n}{k} - \binom{n-|K|-1}{k} \tag{7}$$

and there is some clique of size $|K|+1$, then $\mathcal{H}$ cannot be maximum-sized. But (7) holds for $k = o(n)$. So the maximum-sized G intersecting family must contain all members of $\bigcup_{v\in K}\mathcal{F}_v$ for some K with $|K| = \omega(G)$. $\square$

References

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