21-737: Final Exam Sample Problems

Closed book. 4 pages of notes allowed.

1. Let $x_1, \ldots, x_n$ be a collection of boolean variables (i.e. each of these variables take values in the set {true, false}.) Recall that an instance of the 3-SAT problem is a formula of the form
$$\bigwedge_{i=1}^{m} (a_i \vee b_i \vee c_i).$$
where a_i, b_i, c_i are literals, i.e. a variable or its negation ($\wedge$ means 'and' and $\vee$ means 'or'). An instance is satisfiable if there exist values of the variables for which the formula the true.

   We choose a random 3-SAT instance $F(n, m)$ by choosing each of the m clauses uniformly at random from the collection of all $\binom{n}{3}2^3$ possible clauses. Let $c > 0$ be a constant. Determine a condition on c that would imply
$$\lim_{n\to\infty} Pr\left(F(n, cn) \text{ is satisfiable}\right) = 0.$$

2. Let $t \geq 2$ be fixed while $n \to \infty$. Prove that there exists a $K_{t,t}$-free graph with n vertices and $\Omega(n^{2-\frac{2}{t+1}})$ edges. Here $K_{t,t}$ is the complete bipartite graph with both parts having t vertices.

3. Let P be the transition matrix of a Markov chain on state space Ω with stationary distribution π. Recall that
$$d(t) = \max_{x\in\Omega} \|1_x P^t - \pi\|_{TV}.$$
Prove that for any $t \geq 0$ we have $d(t+1) \leq d(t)$.

4. We choose a set of integers $A \subseteq [n]$ uniformly at random. Let the random variable X be the number of arithmetic progressions of length $(\log_2 n)/10$ in A. Prove that there is a sequence of integers $f(n)$ such that $f(n) \to \infty$ and $X = (1 + o(1))f(n)$ with high probability. (Of course, you should try to make the $o(1)$ term as small as possible.)

5. The *Hajós* number of a graph G is the maximum number k such that there are k vertices in G with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number?

6. Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size $10d$ where $d \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its list $S(v)$.

7. Consider the following process. We have a collection of n bins, and a sequence of balls arrive one at a time. When each ball arrives, 2 bins are chosen at random, and the ball is placed in the bin that contains fewer balls (ties are broken arbitrarily). Let $X(i)$ be

the number of empty bins after i balls have arrived. Find a function $x(t)$ of a continuous variable t such that

$$X(cn) \approx x(c)n$$

for any constant c.