21-737: Midterm Exam Sample Problems

Closed book. 2 pages of notes allowed.

1. Let $x_1, \ldots, x_n$ be a collection of boolean variables (i.e. each of these variables take values in the set {true, false}.) Recall that an instance of the 3-SAT problem is a formula of the form
$$\bigwedge_{i=1}^{m} (a_i \vee b_i \vee c_i).$$
where a_i, b_i, c_i are literals, i.e. a variable or its negation ($\wedge$ means 'and' and $\vee$ means 'or'). An instance is satisfiable if there exist values of the variables for which the formula the true.

   We choose a random 3-SAT instance $F(n, m)$ by choosing each of the m clauses uniformly at random from the collection of all $\binom{n}{3}2^3$ possible clauses. Let $c > 0$ be a constant. Determine a condition on c that would imply
$$\lim_{n\to\infty} Pr\left(F(n, cn) \text{ is satisfiable}\right) = 0.$$

2. Let $C_1, C_2, \ldots, C_m$ be a collection of k-SAT clauses with the property that every variable appears in at most r of the clauses. Prove that if $r < 2^{k-2}/k$ then then there is an assignment of values to the underlying variables that satisfies clauses $C_1, C_2, \ldots, C_m$.

3. Let $t \geq 2$ be fixed while $n \to \infty$. Prove that there exists a $K_{t,t}$-free graph with n vertices and $\Omega(n^{2-\frac{2}{t+1}})$ edges. Here $K_{t,t}$ is the complete bipartite graph with both parts having t vertices.

4. We choose a set of integers $A \subseteq [n]$ uniformly at random. Let the random variable X be the number of arithmetic progressions of length $(\log_2 n)/10$ in A. Prove that there is a sequence of integers $f(n)$ such that $f(n) \to \infty$ and $X = (1 + o(1))f(n)$ with high probability.

5. Recall the following model, known as the **configuration model**, for generating a random regular graph. Let d be a fixed constant. We begin with vertex set $[n]$. Associated with each vertex we introduce a set of d **clones** . (The clones are sometimes called *configuration points* or *half edges*). There are nd clones in all. (We assume that this is an even number.) We choose a perfect matching on the set of clones uniformly at random. (Such a matching is sometimes called a configuration.) For each such matching we get a d-regular graph (possibly with loops and multiple edges) on vertex set $[n]$ by placing the edge $\{u, v\}$ in the graph for each edge in the configuration that joins a clone of u and a clone of v. In other words, we project the matching onto the graph by identifying each clone with its corresponding vertex. Let G_d be the random (multi-)graph so produced.

   (a) Let M denote the number of pairs of parallel edges in G_3. Prove
$$\mathbb{P}(M = 0) \to \frac{1}{e} \text{ as } n \to \infty.$$

(b) Let C_k denote the number of k-cycles in G_d for $k \geq 1$ (where 1-cycles are loops and 2-cycles are 2 parallel edges). What is

$$\lim_{n \to \infty} \mathbb{P}(C_k = 0)?$$

You do not need to prove this, simply state what we anticipate this value to be. Also explain why we expect this to be the case.