21-737: Midterm Exam Review Sheet

Format.

- The test will be during the normal class time on Friday, October 24.
- The test is closed book

Topics Covered.

- Basic Probabilistic Method (including alterations)
- Second Moment Method
- Lovász Local Lemma (including the Moser-Tardos algorithmic local lemma)
- Correlations inequalities (Harris, FKG, 4-functions theorem)
- Poisson Paradigm (Janson's inequality and Brun's sieve)

Sample Questions.

1. Let $C_1, C_2, \ldots, C_m$ be a collection of k-SAT clauses with the property that every variable appears in at most r of the clauses. Prove that if $r < 2^{k-2}/k$ then then there is an assignment of values to the underlying variables that satisfies clauses $C_1, C_2, \ldots, C_m$.
2. Let $t \geq 2$ be fixed while $n \to \infty$. Prove that there exists a $K_{t,t}$-free graph with n vertices and $\Omega(n^{2-\frac{2}{t+1}})$ edges. Here $K_{t,t}$ is the complete bipartite graph with both parts having t vertices.
3. We choose a set of integers $A \subseteq [n]$ uniformly at random. Let the random variable X be the number of arithmetic progressions of length $(\log_2 n)/10$ in A. Prove that there is a sequence of integers $f(n)$ such that $f(n) \to \infty$ and $X = (1 + o(1))f(n)$ with high probability.
4. Recall that if G is a graph then $\Delta(G)$ is the maximum degree of G. Prove
$$Pr\left(\Delta\left(G_{2n,1/2}\right) \leq n-1\right) \geq \frac{1}{4^n}.$$
5. Recall that a threshold function for an increasing graph property $\mathcal{A}$ is a function $f(n)$ such that
$$\begin{aligned} p(n) = o(f(n)) \quad &\Rightarrow \quad Pr(G_{n,p} \in \mathcal{A}) \to 0 \\ p(n) = \omega(f(n)) \quad &\Rightarrow \quad Pr(G_{n,p} \in \mathcal{A}) \to 1 \end{aligned}$$
   (a) Let the random variable X be the number of vertices of degree at most 1 in $G_{n,p}$. Determine $E[X]$.
   (b) Determine a threshold function for the graph property $\mathcal{A} = \{G : \delta(G) \geq 2\}$.