21-737 Probabilistic Combinatorics
Homework V: Giant Component, Markov Chains
Due: Friday, April 28

1. Let M be a large constant. Let G be a graph on a vertex set V of size n such that the susceptibility of G is Mn and the number of vertices in the largest component of G is at most M^2. Let F be a collection of random edges where each potential edge is included, independently, with probability $20/(Mn)$. Prove that with high probability $G + F$ has a connected component with a linear number of vertices.

2. Prove the following: For any Markov Chain, states x, y, and positive integer n we have

$$\sum_{k=0}^{n} P_{x,y}^{k} \leq \sum_{k=0}^{n} P_{y,y}^{k}.$$

3. Let $n \geq 3$ be an integer and let $0 \leq a_1 < a_2 < \cdots < a_k < n$. We view $a_1, \dots, a_k$ as elements of $\mathbb{Z}_n$. Let

$$u_i = \begin{cases} a_1 + n - a_k & \text{if } i = 1 \\ a_i - a_{i-1} & \text{for } i = 2, \dots, k \end{cases}.$$

   Consider the following process on the cycle C_n (with vertices labeled with the elements of $\mathbb{Z}_n$). We begin with one token at each a_i. In each time step we choose some vertex containing one or more tokens, and move all tokens on that vertex to one of its neighbors, each with probability $1/2$.

   Let T be the time it takes for all tokens to occupy a single vertex. Find an expression for $E[T]$ in terms of the u_i's. Use this expression to compute the expected cover time for the random walk on the cycle C_k.

4. Let (X_t, Y_t) be a coupling of Markov chains such that for some $\alpha < 1$ and some $t_0 > 0$ the coupling time

$$\tau_{\text{couple}} = \min\{t \geq 0 : X_t = Y_t\}$$

   satisfies $Pr(\tau_{\text{couple}} \leq t_0) \geq \alpha$ for all pairs of initial states x, y. Prove

$$E[\tau_{\text{couple}}] \leq \frac{t_0}{\alpha}.$$

5. Let G be a d-regular graph. Let $w \in V(G)$ and N be the neighborhood of w in G. Let $v \in V(G)$ and let $v = Y_0, Y_1, \dots$ be the random walk on G starting at vertex v. For each non-negative integer m define the random variable m to be

$$X_m = |\{k \in [m] : Y_k = w\}| - \frac{1}{d}\left|\{k \in [m] : Y_{k-1} \in N\}\right|.$$

   Show that for any $\lambda > 0$ we have

$$Pr(|X_m - E[X_m]| > \lambda) < 2e^{-\lambda^2/(2m)}.$$