1. A sequence $X_0, X_1, \ldots$ of random variables is a **supermartingale** if
\[ E[X_{i+1}|X_0, \ldots, X_i] \leq X_i \quad \text{for } i = 0, 1 \ldots \]
Suppose $0 \equiv X_0, X_1, \ldots$ is a supermartingale for which there are constants $\eta, N$ such that $10\eta < N$ and
\[ X_i - \eta \leq X_{i+1} \leq X_i + N. \]
Prove that for any $0 < \alpha < t\eta$ we have
\[ \Pr(X_t \geq \alpha) \leq \exp \left\{ -\frac{\alpha^2}{3t\eta N} \right\}. \]

**Hint.** Following the proof of Hoeffding-Azuma given in lecture arrive at
\[ \Pr(X_t \geq \alpha) \leq e^{-\lambda \alpha - \lambda t} \left[ \frac{\eta}{N + \eta} e^{N\lambda + \eta} + 1 - \frac{\eta}{N + \eta} \right]^t. \]

Then consider
\[ \lambda = \frac{1}{N + \eta} \log \left( 1 + \frac{\alpha(N + \eta)}{t\eta N} \right) \]

2. (a) Suppose $X$ is a random variables and $\mathcal{F}$ is an algebra in the associated probability space. Define
\[ \Var[X \mid \mathcal{F}] = E[X^2 \mid \mathcal{F}] - E^2[X \mid \mathcal{F}]. \]
Note that $\Var[X \mid \mathcal{F}]$ is a random variable. Prove
\[ \Var[X] \geq E[\Var[X \mid \mathcal{F}]]. \]

(b) For a fixed graph $G$ let $\mathcal{M}$ be the collection of all matchings on $G$. Let $M$ be chosen uniformly at random from $\mathcal{M}$ at let $\eta = |M|$. Show that for any matching $N$ in $G$ we have
\[ \Var(\eta) \geq \frac{1}{2} E[|M \cap N|]. \]

3. The Hajós number of a graph $G$ is the maximum number $k$ such that there are $k$ vertices in $G$ with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number?

4. Let $G$ be the graph whose vertices are all $7^n$ vectors of length $n$ over $[7]$, in which two vertices are adjacent if they differ in precisely one coordinate. Let $U \subseteq [7]^n$ be a set of $7^{n-1}$ vertices, and let $W$ be the set of vertices in $G$ whose distance from $U$ exceeds $(2 + c)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \leq 7^ne^{-c^2/2}$. 

5. Let $S_n$ be the collection of all permutations of $[n]$. For a permutation $\pi = (\pi(1), \ldots, \pi(n))$ in $S_n$, let $X(\pi)$ be the length of a longest increasing sequence (i.e. a sequence $i_1 < i_2 < \cdots < i_k$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$). Show that if $\pi$ is chosen uniformly at random from $S(n)$ then $\Pr(|X - E[X]| > \alpha \sqrt{n})$ decays exponentially in $\alpha$.

6. Consider the following process. We have a collection of $n$ bins, and a sequence of balls arrive one at a time. When each ball arrives, 2 bins are chosen at random, and the ball is placed in the bin that contains fewer balls (ties are broken arbitrarily). Let the stopping time $T$ be the step at which $\frac{n^4}{5}$ bins are empty. Determine an explicit function $f(n)$ such that $T = f(n)(1 \pm o(1))$ whp.

**Bonus Question.** The following process is known as random greedy triangle packing. We begin with $G(0)$, which is the complete graph on $n$ vertices. At step $i \geq 1$ let $xyz$ be a triangle chosen uniformly at random from the collection of all triangles $G(i)$ and set

$$G(i + 1) = G(i) - \{xy, yz, xz\}.$$

The process continues until there are no triangles remaining in the graph.

Let $Q(i)$ be the number of triangles in $G(i)$ and let $Y_{xy}(i)$ be the co-degree of $x$ and $y$ in $G(i)$. Set $p = p(i) = 1 - \frac{6i}{n^2}$. (We make this choice so that $p(n^2)$ is approximately the number of edges in $G(i)$.) Throughout this exercise, you may assume $Y_{xy}(i) = np^2 \pm O(n^{2/3})$ for all $i$ and all pairs $x, y$.

(a) Prove

$$E[Q(i + 1) - Q(i) \mid F_i] \leq 2 - \frac{9Q(i)}{|E(i)|}.$$

(b) Prove

$$Q(i) < \frac{p^3 n^3}{6} + \frac{p n^2}{3} \quad \text{for } i = 1, \ldots, \frac{n^2}{6} - n^{7/4}.$$

*Hint. Introduce the critical interval

$$\left( \frac{n^3 p^3}{6} + \frac{n^2 p}{4}, \frac{n^3 p^3}{6} + \frac{n^2 p}{3} \right).$$

Only keep track when $Q(i)$ is in this critical interval.*