1. A sequence $X_0, X_1, \ldots$ of random variables is a supermartingale if
   \[ E[X_{i+1}|X_0, \ldots, X_i] \leq X_i \quad \text{for } i = 0, 1, \ldots \]
   Suppose \( 0 \equiv X_0, X_1, \ldots \) is a supermartingale for which there are constants \( \eta, N \) such that \( 10\eta < N \) and
   \[ X_i - \eta \leq X_{i+1} \leq X_i + N. \]
   Prove that for any \( 0 < \alpha < t\eta \) we have
   \[ \Pr(X_t \geq \alpha) \leq \exp\left\{ -\frac{\alpha^2}{3t\eta N} \right\}. \]

2. The Hajós number of a graph \( G \) is the maximum number \( k \) such that there are \( k \) vertices in \( G \) with a path between each pair so that all the \( \binom{k}{2} \) paths are internally pairwise disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number?

3. Let \( G \) be the graph whose vertices are all \( 7^n \) vectors of length \( n \) over \([7]\), in which two vertices are adjacent if they differ in precisely one coordinate. Let \( U \subseteq [7]^n \) be a set of \( 7^n-1 \) vertices, and let \( W \) be the set of vertices in \( G \) whose distance from \( U \) exceeds \((2 + c)\sqrt{n}\), where \( c > 0 \) is a constant. Prove that \( |W| \leq 7^n e^{-c^2/2} \).

4. Let \( S_n \) be the collection of all permutations of \([n]\). For a permutation \( \pi = (\pi(1), \ldots, \pi(n)) \) in \( S_n \), let \( X(\pi) \) be the length of a longest increasing sequence (i.e. a sequence \( i_1 < i_2 < \cdots < i_k \) such that \( \pi(i_1) < \pi(i_2) < \cdots < \pi(i_k) \)). Show that if \( \pi \) is chosen uniformly at random from \( S(n) \) then \( \Pr(|X - E[X]| > \alpha\sqrt{n}) \) decays exponentially in \( \alpha \).

5. Consider the following process. We have a collection of \( n \) bins, and a sequence of balls arrive one at a time. When each ball arrives, 2 bins are chosen at random, and the ball is placed in the bin that contains fewer balls (ties are broken arbitrarily). Let the stopping time \( T \) be the step at which \( n^{4/5} \) bins are empty. Determine an explicit function \( f(n) \) such that \( T = f(n)(1 + o(1)) \) whp.

6. The following process is known as random greedy triangle packing. We begin with \( G(0) \), which is the complete graph on \( n \) vertices. At step \( i \geq 1 \) let \( xyz \) be a triangle chosen uniformly at random from the collection of all triangles \( G(i) \) and set
   \[ G(i+1) = G(i) - \{xy, yz, xz\}. \]
   The process continues until there are no triangles remaining in the graph. Let \( Q(i) \) be the number of triangles in \( G(i) \), and set \( p = p(i) = 1 - 6i/n^2 \). (We make this choice so that \( p(n^3) \) is approximately the number of edges in \( G(n) \).) Prove
   \[ Q(i) < \frac{p^3n^3}{6} + \frac{pn^2}{3} \]
   for \( i = 1, \ldots, n^2/6 - n^{3/4} \).
   You may assume that for every pair of vertex \( x, y \) the number of common neighbors of \( x \) and \( y \) is \( np^2 + O(n^{2/3}) \) for all \( i \).