21-737 Probabilistic Combinatorics

Homework IV: Poisson Approximations, Hoeffding-Azuma

Due: Friday, November 7

1. Prove that for every $\epsilon > 0$ there is some $n_0 = n_0(\epsilon)$ so that for every $n > n_0$ and $k \leq (2 - \epsilon) \log_2 n$ there is a graph on n vertices containing *every* graph on at most k vertices as an induced subgraph.

Hint: Use the Extended Janson's Inequality. See Section 8.1 in Alon and Spencer.

2. Let

$$p = p(n) = \frac{\log n + c}{n}.$$

where c is a constant. Let G be a random subgraph of $K_{n,n}$ given by taking each edge with probability p (where these choices are made independently). Show

$$Pr(G \text{ has a perfect matching}) \rightarrow e^{-2e^{-c}}$$

as $n \to \infty$.

Hint: First consider the event that there is an isolated vertex. On the rest of the probability space use Hall's Theorem.

3. Suppose we have a sequence of probability spaces $\Omega_1, \Omega_2, \ldots, \Omega_n, \ldots$, an integer m = m(n) and events A_1, \ldots, A_m in Ω_n with corresponding indicator random variables X_1, X_2, \ldots, X_m . Set

$$X = \sum_{i=1}^{m} X_i$$

and suppose further that there exists a constant μ such that $E[X] \to \mu$. Prove that if

$$\sum_{I \in \binom{[m]}{b}} Pr\left(\bigwedge_{i \in I} A_i\right) \to \frac{\mu^k}{k!}$$

for all fixed k then

$$Pr(X=1) \to \mu e^{-\mu}$$

Hint: Adapt the application of the Bonferonni inequalities that we used in class

- 4. The $Haj\acute{o}s$ number of a graph G is the maximum number k such that there are k vertices in G with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Haj\acute{o}s number?
- 5. Let G be the graph whose vertices are all 7^n vectors of length n over [7], in which two vertices are adjacent if they differ in precisely one coordinate. Let $U \subseteq [7]^n$ be a set of 7^{n-1} vertices, and let W be the set of vertices in G whose distance from U exceeds $(2+c)\sqrt{n}$, where c>0 is a constant. Prove that $|W| \leq 7^n e^{-c^2/2}$.