21-737 Probabilistic Combinatorics
Homework IV: Azuma-Hoeffding, Dynamic Concentration
Due: Friday, March 31

1. A sequence $X_0, X_1, \ldots$ of random variables is a **supermartingale** if
$$E[X_{i+1}|X_0, \ldots, X_i] \le X_i \qquad \text{for } i = 0, 1 \ldots$$
Suppose $0 \equiv X_0, X_1, \ldots$ is a supermartingale for which there are constants η, N such that $10\eta < N$ and
$$X_i - \eta \le X_{i+1} \le X_i + N.$$
Prove that for any $0 < \alpha < t\eta$ we have
$$Pr(X_t \ge \alpha) \le \exp\left\{\frac{-\alpha^2}{3t\eta N}\right\}.$$

2. The *Hajós* number of a graph G is the maximum number k such that there are k vertices in G with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number?

3. Let G be the graph whose vertices are all 7^n vectors of length n over $[7]$, in which two vertices are adjacent if they differ in precisely one coordinate. Let $U \subseteq [7]^n$ be a set of 7^{n-1} vertices, and let W be the set of vertices in G whose distance from U exceeds $(2 + c)\sqrt{n}$, where $c > 0$ is a constant. Prove that $|W| \le 7^n e^{-c^2/2}$.

4. Let S_n be the collection of all permutations of $[n]$. For a permutation $\pi = (\pi(1), \ldots, \pi(n))$ in S_n, let $X(\pi)$ be the length of a longest increasing sequence (i.e. a sequence $i_1 < i_2 < \cdots < i_k$ such that $\pi(i_1) < \pi(i_2) < \cdots < \pi(i_k)$). Show that if π is chosen uniformly at random from $S(n)$ then $Pr(|X - E[X]| > \alpha\sqrt{n})$ decays exponentially in α.

5. Consider the following process. We have a collection of n bins, and a sequence of balls arrive one at a time. When each ball arrives, 2 bins are chosen at random, and the ball is placed in the bin that contains fewer balls (ties are broken arbitrarily). Let the stopping time T be the step at which $n^{4/5}$ bins are empty. Determine an explicit function $f(n)$ such that $T = f(n)(1 \pm o(1))$ whp.

6. The following process is known as *random greedy triangle packing.* We begin with $G(0)$, which is the complete graph on n vertices. At step $i \ge 1$ let xyz be a triangle chosen uniformly at random from the collection of all triangles $G(i)$ and set
$$G(i + 1) = G(i) - \{xy, yz, xz\}.$$
The process continues until there are no triangles remaining in the graph.
Let $Q(i)$ be the number of triangles in $G(i)$, and set $p = p(i) = 1 - 6i/n^2$. (We make this choice so that $p\binom{n}{2}$ is approximately the number of edges in $G(i)$.) Prove
$$Q(i) < \frac{p^3n^3}{6} + \frac{pn^2}{3} \qquad \text{for } i = 1, \ldots, \frac{n^2}{6} - n^{3/4}.$$
You may assume that for every pair of vertex x, y the number of common neighbors of x and y is $np^2 \pm O(n^{2/3})$ for all i.