21-737 Probabilistic Combinatorics
Homework III: Lovász Local Lemma, Correlation Inequalities
Due: Friday, October 10

The first two problems require the following definitions. For a graph $G = (V, E)$ and sets of colors $(S(v) : v \in V)$, with each $S(v)$ a subset of some universal set of colors Γ, a coloring $\sigma : V \to \Gamma$ is S-**legal** if it is a proper coloring (i.e. adjacent vertices get different colors) and $\sigma(v) \in S(v)$ for all $v \in V$.

The **list-chromatic number** of G, denoted $\chi_\ell(G)$, is the smallest t such that for *every* choice of $(S(v) : v \in V)$ such that $|S(v)| = t \ \forall v$ there exists an S-legal coloring.

1. Show that for a bipartite graph G of maximum degree Δ we have
$$\chi_\ell(G) = O(\Delta/(\log \Delta)).$$

2. Let $k \geq 1$ be fixed. Let $G = (V, E)$ be a simple graph, and let $(S(v) : v \in V)$ be sets of at least $10k$ colors for each vertex. Assume that for each $v \in V$ and each color γ we have
$$|\{w : w \sim v \text{ and } \gamma \in S(w)\}| \leq k.$$
Prove that G has an S-legal coloring.

3. Use the Local Lemma to show
$$R(3, t) = \Omega\left(\frac{t^2}{\log^2 t}\right).$$

4. A family of sets $\mathcal{F}$ is **intersecting** if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ are intersecting subset of a set X where $|X| = n$. Prove
$$\left|\bigcup_{i=1}^{k} \mathcal{F}_i\right| \leq 2^n - 2^{n-k}$$
Hint: Assume each $\mathcal{F}_i$ is a maximal intersecting family.

5. Suppose $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ satisfy
$$A \not\subseteq B \text{ and } B \not\subseteq A \qquad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$
Prove that
$$|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leq 2^{n/2}.$$