21-737 Probabilistic Combinatorics
Homework III: Poisson Approximation
Due: Friday, March 3

1. Find a threshold function for the following property: $G_{n,p}$ contains a collection of at least $n/6$ pairwise vertex disjoint triangles. (I.e. find a function $f(n)$ such that $p = o(f(n))$ implies that the probability of the existence of such a collection goes to 0 while $p = \omega(f(n))$ implies that this probability goes to 1.)

2. Suppose we have a sequence of probability spaces $\Omega_1, \Omega_2, \ldots, \Omega_n, \ldots$, an integer $m = m(n)$ and events $A_1, \ldots, A_m$ in Ω_n with corresponding indicator random variables $X_1, X_2, \ldots, X_m$. Set
$$X = \sum_{i=1}^{m} X_i$$
and suppose further that there exists a constant μ such that $E[X] \to \mu$. Prove that if
$$\sum_{I \in \binom{[m]}{k}} Pr\left(\bigwedge_{i \in I} A_i \right) \to \frac{\mu^k}{k!}$$
for all fixed k then
$$Pr(X = 1) \to \mu e^{-\mu}$$

3. **True or False.** Janson's inequality holds when we replace
$$\Delta = \sum_{i \sim j} Pr(B_i \wedge B_j)$$
with
$$\Delta' = \sum_i \sum_j Cov(Y_i, Y_j).$$
Of course, you should justify your answer.

4. The following model is known as the **configuration model** for generating a random regular graph. Let d be a fixed constant. We begin with vertex set $[n]$. Associated with each vertex we introduce a set of d **configuration points**. There are nd configuration points in all. (We assume that this is an even number.) Now, we choose a perfect matching on the set of configuration points uniformly at random. (Such a matching is sometimes called a configuration.) For each such matching we get a d-regular graph (possibly with loops and multiple edges) on vertex set $[n]$ by placing the edge $\{u, v\}$ in the graph for each edge in the configuration that joins a configuration point that corresponds to u and a configuration point that corresponds to v. In other words, we project the matching onto the graph by identifying each configuration point with its corresponding vertex.

   Let M_d be the random (multi-)graph so produced. Let $\mathcal{S}$ be the event that the graph produced by the configuration model has neither loops nor multiple edges.

(a) Prove that there exists a constant $c > 0$ such that $Pr(\mathcal{S}) > c$ for n sufficiently large.

(b) Let G_1 and G_2 be fixed d-regular simple graphs on vertex set $[n]$. Prove

$$Pr(M_d = G_1 \mid \mathcal{S}) = Pr(M_d = G_2 \mid \mathcal{S}).$$

(c) Let C_k denote the number of k-cycles in M_d for $k \geq 1$ (where 1-cycles are loops and 2-cycles are 2 parallel edges). Set $c_k = \frac{1}{2k}(d-1)^k$ and prove that

$$\mathbb{P}(C_k = 1) \to c_k e^{-c_k} \text{ as } n \to \infty.$$