21-737 Probabilistic Combinatorics Homework II: Basics, Second Moment Due Friday, September 26

- 1. (a) Suppose $X = X_1 + X_2 + \cdots + X_n$ where X_1, \dots, X_n are i.i.d. random variables with $Pr(X_i = 1) = Pr(X_i = -1) = 1/2$. Find $E[X^2]$ and $E[X^4]$.
 - (b) Suppose M is a uniform random $n \times n \{\pm 1\}$ matrix (note that we can view the entries in M as i.i.d. random variables). Find the mean and variance of $\det(M)$.
 - (c) Prove the following: If $v_1, \ldots, v_n \in \mathbb{R}^d$ and X_1, \ldots, X_n are i.i.d. random variables with $Pr(X_i = 1) = Pr(X_i = -1) = 1/2$ then

$$E\left[\left\|\sum_{i=1}^{n} X_{i} v_{i}\right\|^{2}\right] = \sum_{i=1}^{n} \|v_{i}\|^{2}$$

where $\|\cdot\|$ denotes the usual Euclidean norm.

2. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n two-dimensional vectors where each x_i and y_i is an integer whose absolute value does not exceed $\frac{2^{n/2}}{100\sqrt{n}}$. Show that there exist disjoint sets $I, J \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

Hint: See 4.6 Distinct Sums in Alon and Spencer.

- 3. Consider the random graph $G_{n,p}$. Let \mathcal{I} be the event that $G_{n,p}$ has at least one isolated vertex. Prove the following.
 - If $p = (\log(n) + f(n))/n$ where $f(n) = \omega(1)$ then $\mathbb{P}(\mathcal{I}) = o(1)$.
 - If $p = (\log(n) f(n))/n$ where $f(n) = \omega(1)$ then $\mathbb{P}(\mathcal{I}) = 1 o(1)$.
- 4. Consider the random graph $G_{n,1/2}$. Show that we have

$$\chi\left(G_{n,1/2}\right) < (1 + o(1)) \frac{n}{\log_2 n}$$

with high probability (meaning that the probability of this event tends to 1 as n tends to infinity). Recall that χ is the chromatic number, the minimum number of colors in a proper vertex coloring of the graph.

Hint: 'Reveal' $G_{n,1/2}$ in n steps by revealing the edges between i and $\{1,\ldots,i-1\}$ at step. Then use conditional probabilities.

Question 5 requires the following definitions and notation, which first appeared on Homework 1. Let G be a triangle-free graph on n vertices with maximum degree Δ , and let \mathcal{I} be the collection of all independent sets in G. We define the *independence polynomial* of G as

$$P_G(x) = \sum_{I \in \mathcal{I}} x^{|I|}.$$

We are interested in the following probability distribution on \mathcal{I} . Let $\gamma > 0$ and for each $I \in \mathcal{I}$ set

$$\mathbb{P}(I) = \mathbb{P}_{\gamma}(I) = \frac{\gamma^{|I|}}{P_G(\gamma)}.$$

(Note that this is indeed a probability distribution.) We futher define the random variable X to be the cardinality of an independent set chosen at random according to this distribution.

We say that a vertex v is **covered** by the independent set I if some neighbor of v is in I. We let v be a vertex of G chosen uniformly at random, and let the random variable Y be the number of neighbors of v that are not covered by the random independent set I. (Note that Y is defined on a probability space on the set $\mathcal{I} \times V$ with $\mathbb{P}((I,x)) = \frac{\gamma^{|I|}}{P_G(\gamma)n}$. Note further that if $x \in I$ then we have Y = 0.)

In the previous homework we proved the following:

$$\frac{E[X]}{n} = \frac{\gamma}{\gamma + 1} E\left[(1 + \gamma)^{-Y} \right]$$

and

$$\frac{E[X]}{n} \ge \frac{\gamma}{\gamma + 1} \cdot \frac{E[Y]}{\Delta}$$

5. (a) For z > 0 let W(z) denote the unique positive real such that $W(z)e^{W(z)} = z$. Use the above observations to prove

$$\frac{E[X]}{n} \ge \frac{\gamma}{1+\gamma} \cdot \frac{W(\Delta \log(\gamma+1))}{\Delta \log(\gamma+1)}.$$

(b) Show that E[X] is increasing as a function of γ and conclude

$$\frac{1}{|\mathcal{I}|} \sum_{I \in \mathcal{I}} |I| \ge (1 + o_{\Delta}(1)) \frac{n \log \Delta}{\Delta}$$