1. Consider the random graph \( G_{n,1/2} \). Show that we have
\[
\chi(G_{n,1/2}) < (1 + o(1)) \frac{n}{\log_2 n}
\]
with high probability (meaning that the probability of this event tends to 1 as \( n \) tends to infinity). Recall that \( \chi \) is the chromatic number, the minimum number of colors in a proper vertex coloring of the graph.

*Hint: ‘Reveal’ \( G_{n,1/2} \) in \( n \) steps by revealing the edges between \( i \) and \( \{1, \ldots, i - 1\} \) at step \( i \). Then use conditional probabilities.*

2. Show that there is a constant \( C \) such that: if \( H \) is a \( t \)-uniform, \( t \)-regular hypergraph on \( V = [n] \), then there exists \( \sigma : V \to \{\pm 1\} \) such that
\[
|\sigma(H)| \leq C \sqrt{t \log t} \quad \forall H \in \mathcal{H}
\]
and
\[
|\sigma(V)| < nt^{-10},
\]
where define \( \sigma(X) = \sum_{x \in X} \sigma(x) \).

3. Let \( Y_1, Y_2, \ldots, Y_s \) be chosen uniformly and independently at random from \( [m] := \{1, 2, \ldots, m\} \) and set \( Y = \{Y_1, \ldots, Y_s\} \).

(a) Show that for any \( \emptyset \neq A \subseteq [m] \) and \( i \in [m] \setminus A \) we have
\[
Pr(i \in Y | A \subseteq Y) \leq Pr(i \in Y)
\]
(b) Show
\[
Pr(Y = [m]) < [1 - (1 - 1/m)^s]^m.
\]

4. For a graph \( G = (V, E) \) and sets of colors \( \{S(v) : v \in V\} \), with each \( S(v) \) a subset of some universal set of colors \( \Gamma \), a coloring \( \sigma : V \to \Gamma \) is \( \Gamma \)-legal if it is a proper coloring (i.e. adjacent vertices get different colors) and \( \sigma(v) \in S(v) \) for all \( v \in V \).

The **list-chromatic number** of \( G \), denoted \( \chi_l(G) \), is the smallest \( t \) such that for every choice of \( \{S(v) : v \in V\} \) such that \( |S(v)| = t \quad \forall v \) there exists an \( \Gamma \)-legal coloring.

Show that for a bipartite graph \( G \) of maximum degree \( \Delta \) we have
\[
\chi_l(G) = O(\Delta/(\log \Delta)).
\]

5. Let \( k \geq 1 \) be fixed. Let \( G = (V, E) \) be a simple graph, and let \( S(v) \) be a set of at least \( 10k \) colors for each \( v \in V \). Assume that for each \( v \in V \) and each color \( \gamma \) we have
\[
|\{w : w \sim v \text{ and } \gamma \in S(w)\}| \leq k.
\]
Prove that \( G \) has an \( S \)-legal coloring.
6. Suppose $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ satisfy

$$A \not\subseteq B \not\subseteq A \quad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$ 

Prove that

$$|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leq 2^{n/2}.$$