1. Consider the random graph $G_{n,1/2}$. Show that we have
\[ \chi(G_{n,1/2}) < (1 + o(1)) \frac{n}{\log_2 n} \]
with high probability (meaning that the probability of this event tends to 1 as $n$ tends to infinity). Recall that $\chi$ is the chromatic number, the minimum number of colors in a proper vertex coloring of the graph.

**Hint:** ‘Reveal’ $G_{n,1/2}$ in $n$ steps by revealing the edges between $i$ and $\{1, \ldots, i-1\}$ at step $i$. Then use conditional probabilities.

2. Show that there is a constant $C$ such that: if $H$ is a $t$-uniform, $t$-regular hypergraph on $V = [n]$, then there exists $\sigma : V \to \{\pm 1\}$ such that
\[ |\sigma(H)| \leq C \sqrt{t \log t} \quad \forall H \in \mathcal{H} \]
and
\[ |\sigma(V)| < nt^{-10}, \]
where define $\sigma(X) = \sum_{x \in X} \sigma(x)$.

**Hint:** You might make use of the following:

**Chernoff Bound.** If $X_1, X_2, \ldots, X_n$ are i.i.d. variables with $\Pr(X_i = -1) = \Pr(X_i = 1) = 1/2$ then
\[ \Pr\left( \sum_{i=1}^{n} X_i > \lambda \sqrt{n} \right) \leq e^{-\lambda^2/2}. \]

3. Let $Y_1, Y_2, \ldots, Y_s$ be chosen uniformly and independently at random from $[m] := \{1, 2, \ldots, m\}$ and set $Y = \{Y_1, \ldots, Y_s\}$.

   (a) Show that for any $\emptyset \neq A \subseteq [m]$ and $i \in [m] \setminus A$ we have
   \[ \Pr(i \in Y | A \subseteq Y) \leq \Pr(i \in Y) \]
   (b) Show
   \[ \Pr(Y = [m]) < \left[1 - (1 - 1/m)^s\right]^m. \]

**Hint:** Use a coupling.

4. For a graph $G = (V, E)$ and sets of colors $(S(v) : v \in V)$, with each $S(v)$ a subset of some universal set of colors $\Gamma$, a coloring $\sigma : V \to \Gamma$ is $S$-legal if it is a proper coloring (i.e. adjacent vertices get different colors) and $\sigma(v) \in S(v)$ for all $v \in V$. 
The list-chromatic number of $G$, denoted $\chi_\ell(G)$, is the smallest $t$ such that for every choice of $\{S(v) : v \in V\}$ such that $|S(v)| = t \ \forall v$ there exists an $S$-legal coloring.

Show that for a bipartite graph $G$ of maximum degree $\Delta$ we have

$$\chi_\ell(G) = O(\Delta/(\log \Delta)).$$

5. Let $k \geq 1$ be fixed. Let $G = (V, E)$ be a simple graph, and let $S(v)$ be a set of at least $10k$ colors for each $v \in V$. Assume that for each $v \in V$ and each color $\gamma$ we have

$$|\{w : w \sim v \text{ and } \gamma \in S(w)\}| \leq k.$$

Prove that $G$ has an $S$-legal coloring.