21-737 Probabilistic Combinatorics
Homework II: Basics, Local Lemma, Correlation Inequalities
Due: Friday, February 17

1. Consider the random graph $G_{n,1/2}$. Show that we have

$$\chi\left(G_{n,1/2}\right) < (1+o(1))\frac{n}{\log_2 n}$$

with high probability (meaning that the probability of this event tends to 1 as n tends to infinity). Recall that χ is the chromatic number, the minimum number of colors in a proper vertex coloring of the graph.

Hint: 'Reveal' $G_{n,1/2}$ in n steps by revealing the edges between i and $\{1,\ldots,i-1\}$ at step i. Then use conditional probabilities.

2. Show that there is a constant C such that: if $\mathcal{H}$ is a t-uniform, t-regular hypergraph on $V=[n]$, then there exists $\sigma : V \to \{\pm 1\}$ such that

$$|\sigma(H)| \le C\sqrt{t\log t} \quad \forall H \in \mathcal{H}$$

and

$$|\sigma(V)| < nt^{-10},$$

where define $\sigma(X) = \sum_{x\in X}\sigma(x)$.

3. Let $Y_1, Y_2, \ldots, Y_s$ be chosen uniformly and independently at random from $[m] := \{1,2,\ldots,m\}$ and set $Y = \{Y_1,\ldots,Y_s\}$.

   (a) Show that for any $\emptyset \ne A \subseteq [m]$ and $i \in [m]\setminus A$ we have

$$Pr(i\in Y | A\subseteq Y) \le Pr(i \in Y)$$

   (b) Show

$$Pr(Y=[m]) < [1-(1-1/m)^s]^m.$$

4. For a graph $G=(V,E)$ and sets of colors $(S(v) : v \in V)$, with each $S(v)$ a subset of some universal set of colors Γ, a coloring $\sigma : V \to \Gamma$ is *S-legal* if it is a proper coloring (i.e. adjacent vertices get different colors) and $\sigma(v)\in S(v)$ for all $v\in V$.

   The *list-chromatic number* of G, denoted $\chi_\ell(G)$, is the smallest t such that for *every* choice of $\{S(v) : v\in V\}$ such that $|S(v)| = t \ \ \forall v$ there exists an S-legal coloring.

   Show that for a bipartite graph G of maximum degree Δ we have

$$\chi_\ell(G) = O(\Delta/(\log\Delta)).$$

5. Let $k \ge 1$ be fixed. Let $G=(V,E)$ be a simple graph, and let $S(v)$ be a set of at least $10k$ colors for each $v\in V$. Assume that for each $v \in V$ and each color γ we have

$$|\{w : w \sim v \text{ and } \gamma\in S(w)\}| \le k.$$

   Prove that G has an S-legal coloring.

6. Suppose $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ satisfy

$$A \not\subseteq B \not\subseteq A \qquad \forall A \in \mathcal{A}, B \in \mathcal{B}.$$

Prove that

$$|\mathcal{A}|^{1/2} + |\mathcal{B}|^{1/2} \leq 2^{n/2}.$$