21-737 Probabilistic Combinatorics
Homework I: Basics, Second Moment
Due Friday February 3

1. Let X, Y be disjoint sets such that $|X| = |Y| = n$. Define the *random bipartite graph* $B = B_{n,p}$ to have vertex set $X \cup Y$ and (random) edge set E given by

$$Pr(\{x, y\} \in E) = p \qquad \forall x \in X, y \in Y,$$

with these events being mutually independent. Let the random variable M be the number of perfect matchings in B.

   (a) Find the asymptotic formula for that $p = p(n)$ for which $E[M] = 1$ (I.e. find a function $g(n)$ such that $g(n) \sim p(n)$. Try to choose $g(n)$ to be as simple as possible.)

   (b) Show that for p as in (a), $Pr(B$ has a perfect matching $) \to 0$ as n goes to ∞.

2. Let $g(k)$ be the smallest number of edges in a k-uniform hypergraph $\mathcal{H}$ on an even number of vertices with the property that every set of half of the vertices of $\mathcal{H}$ contains an edge of $\mathcal{H}$. Show $g(k) > \Omega(k2^k)$.

3. Let H be a graph and $n > |V(H)|$ be an integer. Suppose there is a graph on n vertices and t edges containing no copy of H. Show that if $kt > n^2 \log n$ then there exists a coloring of the edges of the complete graph K_n with k colors that has no monochromatic copy of H.

4. Let $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$ be n two-dimensional vectors where each x_i and y_i is an integer whose absolute value does not exceed $\frac{2^{n/2}}{100\sqrt{n}}$. Show that there exist disjoint sets $I, J \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

5. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a collection of events in a probability space. Let $\mu = \sum Pr(A_i)$ be the expected number of events from $\mathcal{A}$ that occur. Given a fixed integer ℓ, let Q be the event that some set of ℓ independent events from $\mathcal{A}$ occur. (In other words, Q is the event that, among the events in $\mathcal{A}$ that occur, there are ℓ that are mutually independent). Show

$$Pr(Q) \leq \frac{\mu^\ell}{\ell!}.$$

6. Fix $c > 0$ and let X denote the number of isolated edges in $G_{n,p}$, where $p = c/n$. Find precise asymptotic formulae for $E[X]$ and $Var[X]$. What can we conclude about the distribution of X?