21-737 Probabilistic Combinatorics
Homework IV: Hints
Follow-up solutions due Wednesday, April 12

1. Following the proof of Azuma-Hoeffding given in lecture we should arrive at
$$Pr(X_t \geq \alpha) \leq e^{-\lambda\alpha}\left[\frac{\eta}{N+\eta}e^{N\lambda} + \frac{N}{N+\eta}e^{-\eta\lambda}\right]^t.$$
This can be written as
$$Pr(X_t \geq \alpha) \leq e^{-\lambda\alpha-\lambda\eta t}\left[\frac{\eta}{N+\eta}e^{N\lambda+\eta\lambda} + 1 - \frac{\eta}{N+\eta}\right]^t.$$
Now set
$$\lambda = \frac{1}{N+\eta}\log\left(1+\frac{\alpha(N+\eta)}{t\eta N}\right)$$
and use $1+x \leq e^x$ and $\log(1+x) \geq x - x^2/2$.

2. Consider $G_{n,1/2}$. Show that every suitably large set has many non-edges. And use pigeonhole.

3. Consider the probability space given by the uniform distribution on the set $[7]^n$. Let the random variable X be the distance to the set U and form a Doob martingale by 'exposing' the coordinates one at a time (this fits into the general framework we set up in lecture). Use Azuma-Hoeffding twice; one of your applications should take into account that fact that we know $Pr(X = 0)$.

4. Generate a random permutation of $[n]$ in a sequence of steps in such a way that we can understand the change in the conditional expectation of the length of the longest increasing sequence as we go along.

   Here is on way to do this. Consider a random sequence $Y_1, \ldots, Y_n$ where Y_i is chosen u.a.r. from $[i]$. We then get a permuation π by letting Y_i be the position of $\pi(i)$ among $\{\pi(1), \pi(2), \ldots, \pi(i)\}$.

5. Prove dynamic concentration of the number of empty bins. Scale this number linearly. Also scale time linearly. In order to get to the desired number of empty bins remaining we will have to let the ending time go to infinity with n.

6. **Note.** *There was an error in the statement of the problem. We want to prove the desired bound on $Q(i)$ for*
$$i = 1, \ldots, \frac{n^2}{6} - n^{7/4}.$$
So, we stop making this claim when the graph of edges unsaturated by the collection of selected triangles has $\Theta(n^{7/4})$ edges.

   - For each edge xy denote the co-degree of xy by Y_{xy}. The bounds on Y_{xy} that the problem allows us to assume are only used to bound the one-step change in the super-martingale we eventually define.

- When defining our martingale note that we can write $E[Q(i+1) - Q(i) \mid \mathcal{F}_i]$ as a function of

$$\sum_{xy \in E(i)} Y_{xy}^2$$

and we can bound this sum in terms of $Q(i)$ using Cauchy-Schwartz (or Jensen).
- Introduce the *critical interval*

$$\left(\frac{n^3p^3}{6} + \frac{n^2p}{4} \, , \, \frac{n^3p^3}{6} + \frac{n^2p}{3} \right).$$

Only keep track when $Q(i)$ is in this critical interval.